Synchronisation and Chaos in Stochastic Hopf Bifurcation

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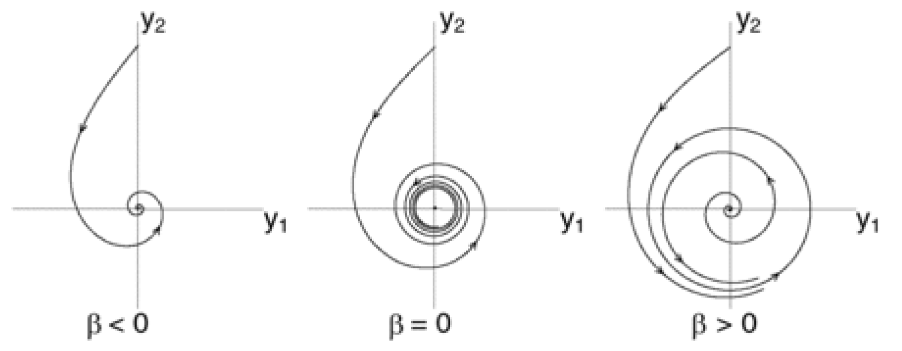
Introduction to Stochastic Hopf Bifurcation

We study the two-dimensional normal form of a Hopf bifurcation with additive white noise and phase-amplitude coupling:

$$\begin{aligned} \mathrm{d}y_1 &= (\beta y_1 - \omega y_2 - (a y_1 + b y_2)(y_1^2 + y_2^2))dt + \sigma dW_1(t), \\ dy_2 &= (\beta y_2 + \omega y_1 - (a y_1 - b y_2)(y_1^2 + y_2^2))dt + \sigma dW_2(t), \end{aligned} \tag{1}$$

where W_1, W_2 are independent Brownian motions and $\omega, a, \sigma, b > 0$. If $\sigma = 0$, such a system exhibits a supercritical Hopf bifurcation for bifurcation parameter $\beta \in \mathbb{R}$:



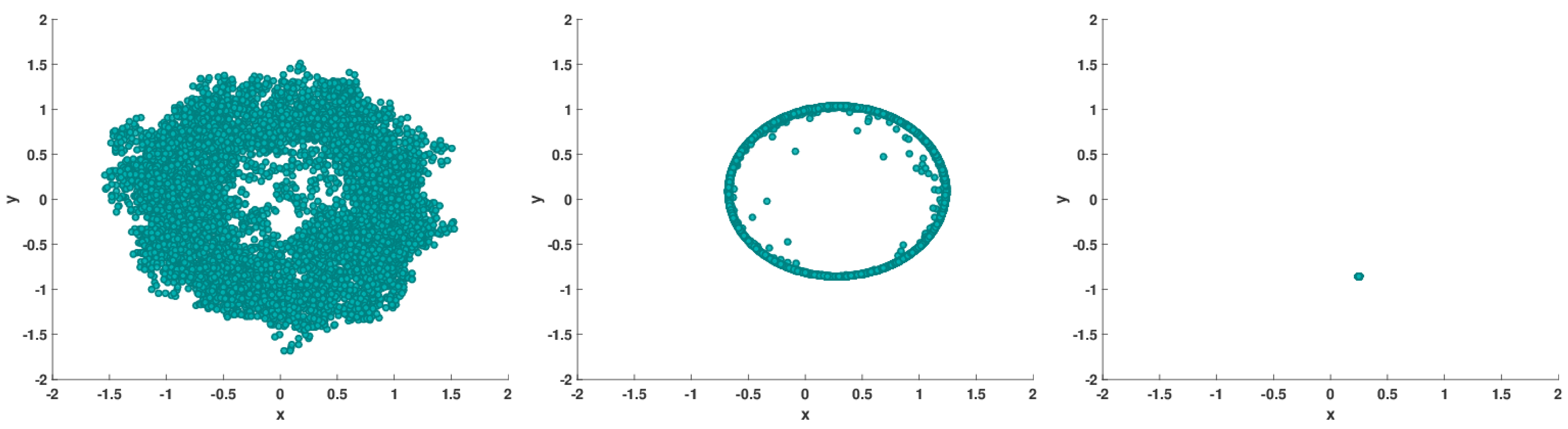
We investigate the system for $\sigma > 0$ and study the impact of the parameter b which represents shear via phase-amplitude coupling. This can be seen in polar coordinates

$$\begin{aligned} dr &= \left(\beta r - a r^3 + \frac{\sigma^2}{2r}\right) dt + \sigma dW_r(t), \\ d\phi &= [\omega + b r^2] dt + \frac{\sigma}{r} dW_\phi(t). \end{aligned}$$

Applications: In the Zebiak-Cane model, which describes the tropical Pacific annual mean climate state, a Hopf bifurcation occurs at a critical value of the ocean-atmosphere coupling strength β. Dijkstra et al. (2008) study the impact of noise on the Hopf bifurcation but don't provide a dynamical analysis and don't consider phase-amplitude coupling. We show transitions between ordered and chaotic behaviour depending on continuous time noise and shear, partially solving a long-standing theoretical problem posed by Lai-Sang Young and co-workers and also investigated by Wieczorek (2009) with respect to Laser models.

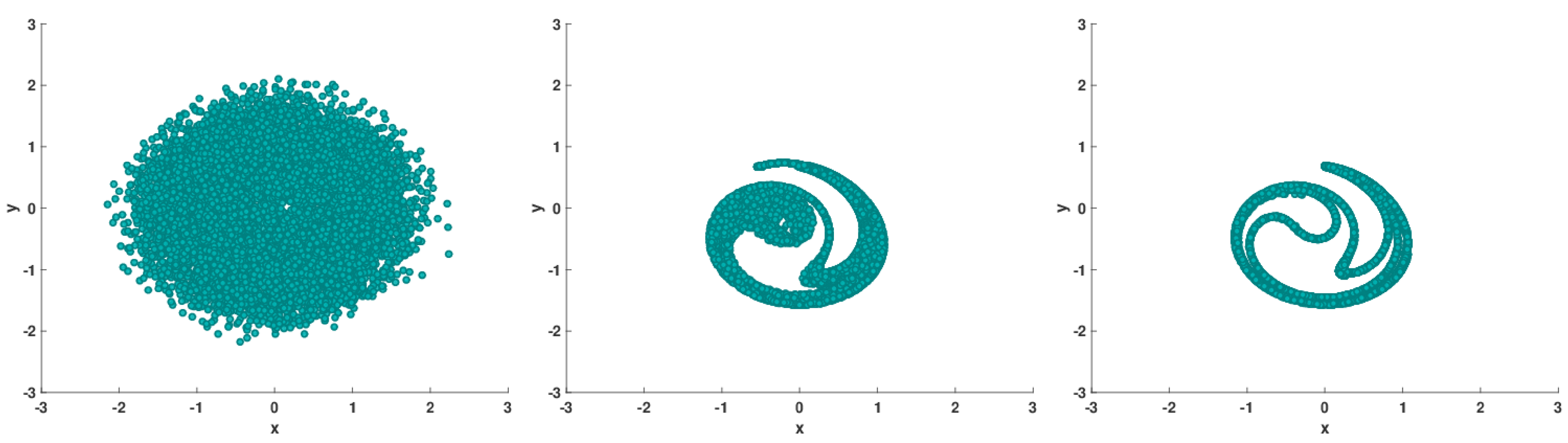
Numerical observations

We start simulations at times $t < 0$ and run the system until time 0. This allows to study fixed attracting objects. We make the following observations for $\beta > 0$, i.e. after the bifurcation. First we observe that, for small $b \geq 0$, trajectories with different initial conditions but exposed to the same noise realisations synchronise:



(a) $t = 0$ (b) $t = 3$ (c) $t = 20$

For $b \geq 8$, the pullback attractor seems to show chaotic behaviour:



(d) $t = 0$ (e) $t = 3$ (f) $t = 100$

Random Dynamical Systems induced by an SDE and invariant measures

We consider the problem of stochastic bifurcations within the framework of random dynamical systems: A **random dynamical system** (RDS) on the measurable space $(X, \mathcal{B})$ over a *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with time $\mathbb{T}$ is a $\mathcal{B}(\mathbb{T}) \bigotimes \mathcal{F} \bigotimes \mathcal{B}$-meas. map

$$\varphi : \mathbb{T} \times \Omega \times X \to \mathbb{R}^d, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x,$$

which satisfies for all $\omega \in \Omega$ and $t, s \in \mathbb{T}$ the *cocycle* property

$$\varphi(0, \omega) = \mathrm{id}, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega).$$

A **stochastic differential equation**(SDE) of the form

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t \quad X_0 = x, \quad \text{on } \mathbb{R}^d, \tag{2}$$

induces a continuous RDS (θ, φ) for time $\mathbb{T} = \mathbb{R}_+$ under typical Lipschitz and growth conditions. In this case $(\Omega, \mathbb{P})$ is the Wiener space and θ_t the ergodic shift map.

A **probability measure** μ on $\Omega \times \mathbb{R}^d$ is **invariant** for the RDS if for $\Theta_t : \Omega \times X \to \Omega \times X$ denoting the skew-product flow, i.e. $\Theta_t(\omega, x) = (\theta_t \omega, \varphi(t, \omega)x)$,

1. $\Theta_t \mu = \mu$ for all $t \in \mathbb{T}$,
2. the marginal of μ on Ω is $\mathbb{P}$, i.e. $\mu(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega)$.

Random attractors and Lyapunov exponents

Let the state space X be a Polish space (e.g. $\mathbb{R}^d$ as in our case): A **random attractor** $A : \Omega \to \mathcal{P}(X)$ of the RDS (θ, φ) is a $\mathbb{P}$-a.s. compact set valued mapping with

1. $\varphi(t, \omega)A(\omega) = A(\theta_t \omega)$ for all $t > 0$ and a.a. $\omega \in \Omega$,
2. $\lim_{t \to \infty} d(\varphi(t, \theta_{-t}\omega)B, A(\omega)) = 0$ $\mathbb{P}$-a.s. for every compact $B \subset X$.

If attraction in the limit just holds for all points $x \in X$, we call it a **random point attractor** which is the support for the disintegrations μ_ω of an invariant measure μ.

Consider the SDE (2): If $f \in C^{1,\delta}$ and $\sigma \in C^{2,\delta}$ for some $\delta > 0$, the induced RDS (θ, φ) is C^1. If it has an ergodic inv. measure μ and satisfies an integrability condition, there are real numbers $\lambda_1 > \cdots > \lambda_p$, the **Lyapunov exponents** of φ w.r.t. μ, s.t. for μ-a.e. (ω, x) and for all $0 \neq v \in \mathbb{R}^d$

$$\lim_{t \to \infty} \frac{1}{t} \log \| D\varphi(t, \omega, x)v \| \in \{\lambda_i\}_{i=1}^p.$$

Theorem (Synchronisation)

The random dynamcial system induced by the stochastic differential equation (1) possesses a random attractor $A(\omega)$ and exhibits synchronisation, i.e. $A(\omega)$ is a singleton, for any $\beta \in \mathbb{R}$ if $\lambda_1 < 0$. We know that $\lambda_1 < 0$ if

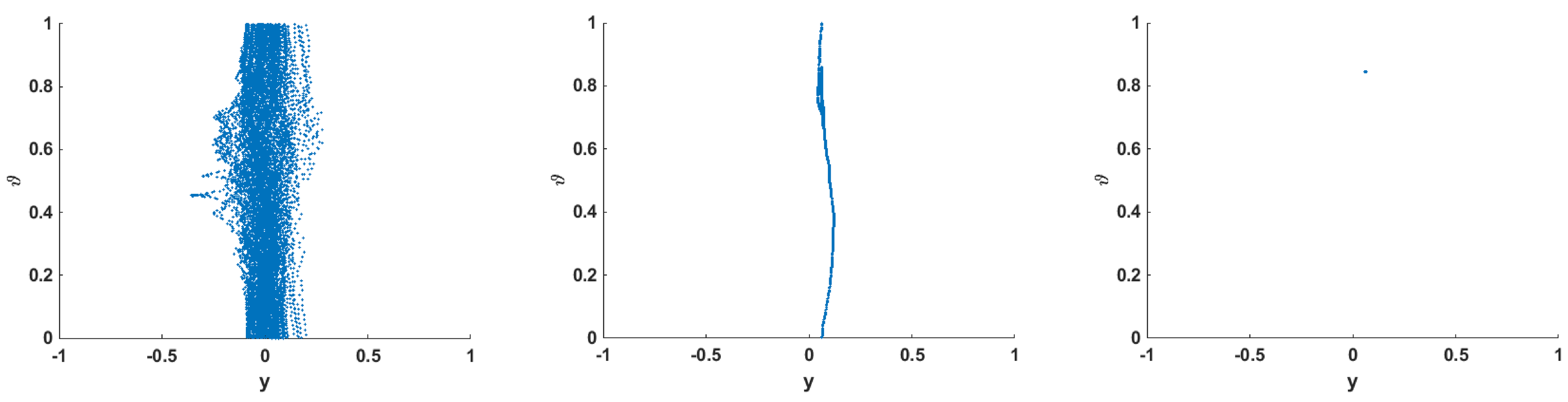
a) *$\beta \leq 0$ and $b < a$ (and/or σ small),*
b) *$\beta > 0$ and $0 \leq b \leq \frac{ac}{2(\alpha + c)} \leq \frac{1}{2}a$, where $c = \mathcal{O}(\sigma)$.*
c) *we fix $b \geq 0$, define $\varepsilon = \sigma^2 a^2 / \beta^2$ and let $\varepsilon \to 0$ ($\lambda_1 = C\varepsilon + \mathcal{O}(\varepsilon^2)$ with $C < 0$).*

Cylinder model

In the case of large shear we expect $\lambda_1 > 0$ which indicates the existence of a chaotic attractor. As this is difficult to show for (1), we consider the following simlified model of a stochastically driven limit cycle

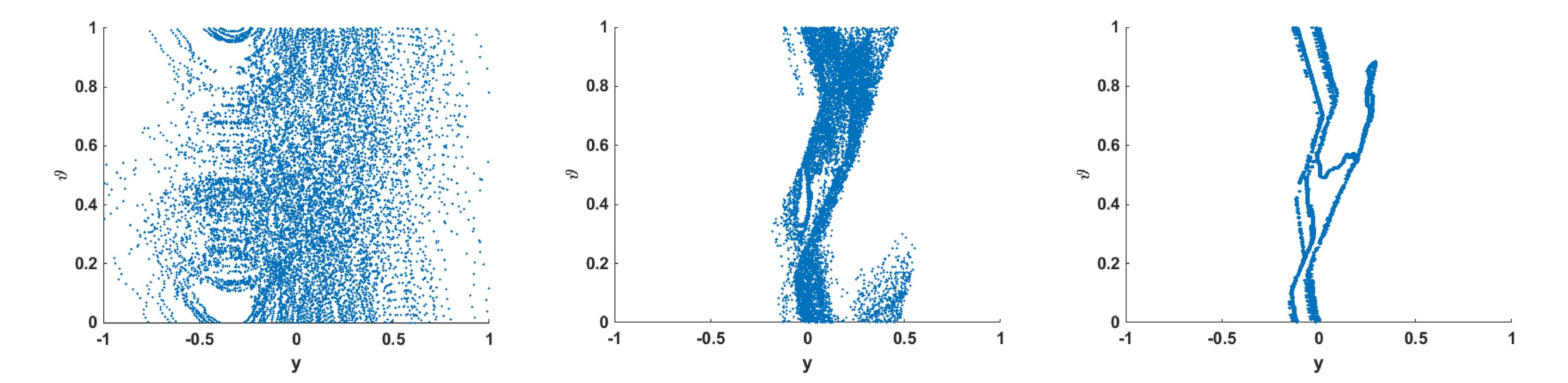
$$\begin{aligned} \mathrm{d}y &= -\alpha y \, \mathrm{d}t + \sigma f(\vartheta) \circ \mathrm{d}W_t^1 \,, \\ \mathrm{d}\vartheta &= (1 + by) \, \mathrm{d}t \,, \end{aligned} \tag{3}$$

where $(y, \vartheta) \in \mathbb{R} \times \mathbb{S}^1$ are cylindrical amplitude-phase coordinates, and W_t^1 denotes one-dimensional Brownian motion entering the equation as noise of Stratonovich type. For the parameter values $\sigma = 0.5, \alpha = 1.5, b = 3$, we observe synchronisation.



(g) $t = 0$ (h) $t = 3$ (i) $t = 20$

For parameter values $\sigma = 2, \alpha = 1.5, b = 3$, we observe chaos:



(j) $t = 0$ (k) $t = 3$ (l) $t = 100$

Theorem (Transition to chaos)

Consider the stochastic differential equation (3) where $f : \mathbb{S}^1 \simeq [0, 1) \to \mathbb{R}$ is continuous and piecewise linear with constant absolute value of the derivative almost everywhere. Then for all $\alpha > 0$ and $b \neq 0$, there exist $\sigma_-(\alpha, b) \leq \sigma_0(\alpha, b) \leq \sigma_+(\alpha, b)$ such that the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of the random attractor of (3) satisfies

$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_-(\alpha, b)\,, \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b)\,, \\ > 0 & \text{if } \sigma > \sigma_+(\alpha, b)\,. \end{cases}$$

This has the following implications: If $0 < \sigma < \sigma_-(\alpha, b)$, the random point attractor of (3) is an attracting random equilibrium. If $\sigma > \sigma_+(\alpha, b)$ the random point attractor of system (3) is a random strange attractor (and not an attracting random equilibrium).

References

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