
Collective dynamics in life sciences

Lecture 2: the Vicsek model

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1. The Vicsek model
2. Mean-Field model
3. Self-Organized Hydrodynamics (SOH)
4. Properties of the SOH model and extensions
5. Conclusion

1. The Vicsek model

Individual-Based (aka particle) model

self-propelled \Rightarrow all particles have same constant velocity a
align with their neighbours up to a certain noise

Time-discrete model

k -th particle position X_k^n , velocity direction V_k^n , at $t^n = n\Delta t$

$$X_k^{n+1} = X_k^n + aV_k^n \Delta t, \quad |V_k^n| = 1$$

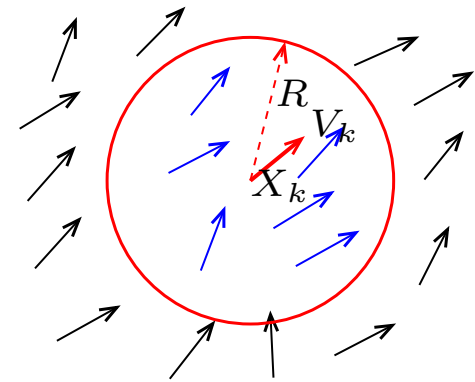
$$\mathcal{J}_k^n = \sum_{j, |X_j^n - X_k^n| \leq R} V_j^n, \quad \bar{V}_k^n = \frac{\mathcal{J}_k^n}{|\mathcal{J}_k^n|}$$

$$\arg(V_k^{n+1}) = \arg(\bar{V}_k^n + \tau_k^n)$$

τ_k^n drawn uniformly in $[-\tau, \tau]$; $R =$ interaction range

$\mathcal{J}_k^n =$ local particle flux in interaction disk

$\bar{V}_k^n =$ neighbors' average direction



2. Mean-Field model

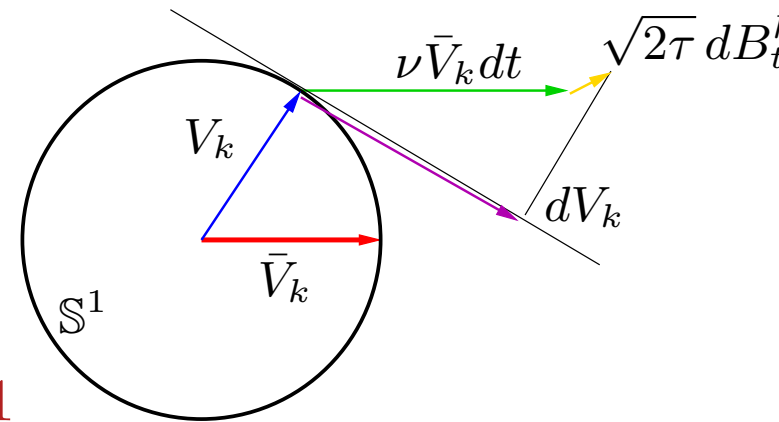
Passage to time continuous dynamics:

requires introduction of **new parameter**: **interaction frequency ν**

$$\dot{X}_k(t) = aV_k(t)$$

$$dV_k(t) = P_{V_k^\perp} \circ (\nu \bar{V}_k dt + \sqrt{2\tau} dB_t^k), \quad P_{V_k^\perp} = \text{Id} - V_k \otimes V_k$$

$$\mathcal{J}_k = \sum_{j, |X_j - X_k| \leq R} V_j, \quad \bar{V}_k = \frac{\mathcal{J}_k}{|\mathcal{J}_k|}$$



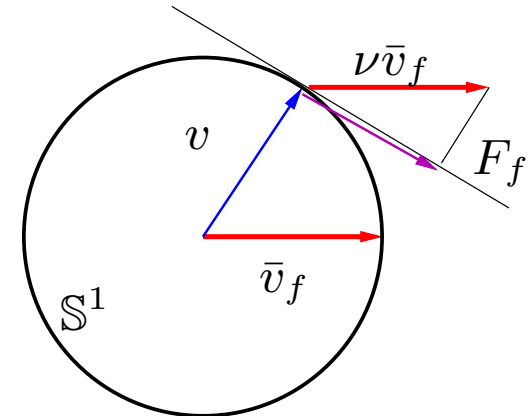
Recover original Vicsek by:

Time discretization Δt s.t. $\nu \Delta t = 1$

Gaussian noise \rightarrow uniform

Dimension $n = 2$; here $(X_k, V_k) \in \mathbb{R}^n \times \mathbb{R}^n, n \geq 2$

$f(x, v, t)$ = particle probability density
 satisfies a **Fokker-Planck** equation



$$\partial_t f + av \cdot \nabla_x f + \nabla_v \cdot (F_f f) = \tau \Delta_v f$$

$$F_f(x, v, t) = P_{v^\perp}(\nu \bar{v}_f(x, t)), \quad P_{v^\perp} = \text{Id} - v \otimes v$$

$$\bar{v}_f(x, t) = \frac{\mathcal{J}_f(x, t)}{|\mathcal{J}_f(x, t)|}, \quad \mathcal{J}_f(x, t) = \int_{|y-x| < R} \int_{\mathbb{S}^{n-1}} f(y, w, t) w \, dw \, dy$$

$\mathcal{J}_f(x, t)$ = particle flux in a neighborhood of x

$\bar{v}_f(x, t)$ = direction of this flux

$F_f(x, v, t)$ = projection of the flux direction on v^\perp

$(x, v) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$; $\nabla_{v \cdot}, \nabla_v$: div and grad on \mathbb{S}^{n-1}

Δ_v Laplace-Beltrami operator on the sphere

Highlights important **physical scales & small parameters**

Choose time scale t_0 , space scale $x_0 = at_0$

Set f scale $f_0 = 1/x_0^n$, F scale $F_0 = 1/t_0$

Introduce **dimensionless parameters** $\bar{\nu} = \nu t_0$, $\bar{\tau} = \tau t_0$, $\bar{R} = \frac{R}{x_0}$

Change variables $x = x_0 x'$, $t = t_0 t'$, $f = f_0 f'$, $F = F_0 F'$

Get the **scaled** Fokker-Planck system (omitting the primes):

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F_f f) = \bar{\tau} \Delta_v f$$

$$F_f(x, v, t) = P_{v^\perp}(\bar{\nu} \bar{v}_f(x, t)), \quad P_{v^\perp} = \text{Id} - v \otimes v$$

$$\bar{v}_f(x, t) = \frac{\mathcal{J}_f(x, t)}{|\mathcal{J}_f(x, t)|}, \quad \mathcal{J}_f(x, t) = \int_{|y-x| < \bar{R}} \int_{\mathbb{S}^{n-1}} f(y, w, t) w \, dw \, dy$$

Choice of t_0 such that $\bar{\tau} = \frac{1}{\varepsilon}$, $\varepsilon \ll 1$

Macroscopic scale:

there are many velocity diffusion events within one time unit

Assumption 1: $k := \frac{\bar{\nu}}{\bar{\tau}} = \mathcal{O}(1)$

Social interaction and diffusion act at the same scale

Implies $\bar{\nu}^{-1} = \mathcal{O}(\varepsilon)$, i.e. mean-free path is microscopic

Assumption 2: $\bar{R} = \varepsilon$

Interaction range is microscopic

and of the same order as mean-free path $\bar{\nu}^{-1}$

Possible variant: $\bar{R} = \mathcal{O}(\sqrt{\varepsilon})$: interaction range still small

but large compared to mean-free path. To be investigated later

With Assumption 2 ($\bar{R} = \mathcal{O}(\varepsilon)$)

Interaction is local at leading order: by Taylor expansion:

$$\mathcal{J}_f = J_f + \mathcal{O}(\varepsilon^2), \quad J_f(x, t) = \int_{\mathbb{S}^{n-1}} f(x, w, t) w \, dw$$

$J_f(x, t)$ = local particle flux. From now on, **neglect** $\mathcal{O}(\varepsilon^2)$ term

Fokker-Planck eq. in scaled variables

$$\varepsilon(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) + \nabla_v \cdot (F^\varepsilon f^\varepsilon) = \Delta_v f^\varepsilon$$

$$F^\varepsilon(x, v, t) = k P_{v \perp} u_{f^\varepsilon}(x, t)$$

$$u_{f^\varepsilon}(x, t) = \frac{J_{f^\varepsilon}}{|J_{f^\varepsilon}|}, \quad J_{f^\varepsilon}(x, t) = \int_{\mathbb{S}^{n-1}} f^\varepsilon(x, w, t) w \, dw$$

Hydrodynamic model is obtained in the **limit** $\varepsilon \rightarrow 0$

3. Self-Organized Hydrodynamics (SOH)

Model can be written

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

with **collision operator**

$$Q(f) = -\nabla_v \cdot (F_f f) + \Delta_v f$$

$$F_f = k P_{v^\perp} u_f$$

$$u_f = \frac{J_f}{|J_f|}, \quad J_f = \int_{\mathbb{S}^{n-1}} f(x, w, t) w \, dw$$

When $\varepsilon \rightarrow 0$, $f^\varepsilon \rightarrow f$ (formally) such that $Q(f) = 0$

\Rightarrow importance of the solutions of **$Q(f) = 0$ (equilibria)**

Q acts on v -variable only ((x, t) are just parameters)

Force F_f can be written: $F_f(v) = k \nabla_v(u_f \cdot v)$

Note u_f independent of v ((x, t) are fixed)

Rewrite:

$$\begin{aligned} Q(f)(v) &= \nabla_v \cdot [-f k \nabla_v(u_f \cdot v) + \nabla_v f] \\ &= \nabla_v \cdot [f \nabla_v(-k u_f \cdot v + \ln f)] \end{aligned}$$

Let $u \in \mathbb{S}^{n-1}$ be given: Solutions of $\nabla_v(-k u \cdot v + \ln f) = 0$ are proportional to :

$$f(v) = M_{ku}(v) := \frac{e^{ku \cdot v}}{\int_{\mathbb{S}^{n-1}} e^{ku \cdot v} dv}$$

von Mises-Fisher (VMF) distribution

Again:

$$M_{ku}(v) := \frac{e^{ku \cdot v}}{\int_{\mathbb{S}^{n-1}} e^{ku \cdot v} dv}$$

$k > 0$: concentration parameter; $u \in \mathbb{S}^{n-1}$: orientation

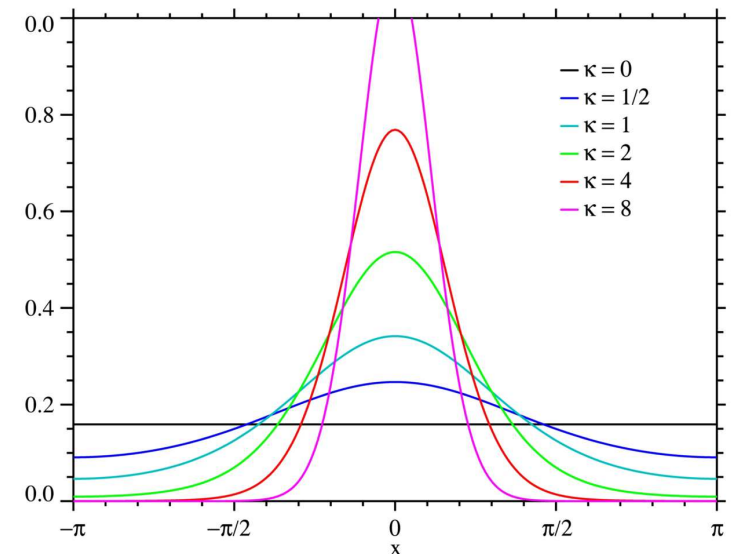
Order parameter: $c_1(k) = \int_{\mathbb{S}^{n-1}} M_{ku}(v) u \cdot v dv$

$$k \nearrow c_1(k), \quad 0 \leq c_1(k) \leq 1$$

Flux: $\int_{\mathbb{S}^{n-1}} M_{ku}(v) v dv = c_1(k)u$

Here:

concentration parameter k
and order parameter $c_1(k)$
are **constant**



Definition: equilibrium manifold $\mathcal{E} = \{f(v) \mid Q(f) = 0\}$

Theorem: $\mathcal{E} = \{\rho M_{ku} \text{ for arbitrary } \rho \in \mathbb{R}_+ \text{ and } u \in \mathbb{S}^{n-1}\}$

Note: $\dim \text{mediumblue } \mathcal{E} = n$

Proof: follows from entropy inequality:

$$H(f) = \int Q(f) \frac{f}{M_{ku_f}} dv = - \int M_{ku_f} \left| \nabla_v \left(\frac{f}{M_{ku_f}} \right) \right|^2 \leq 0$$

$$\text{follows from } Q(f) = \nabla_v \cdot \left[M_{ku_f} \nabla_v \left(\frac{f}{M_{ku_f}} \right) \right]$$

Then, $Q(f) = 0$ implies $H(f) = 0$ and $\frac{f}{M_{ku_f}} = \text{Constant}$
and f is of the form ρM_{ku}

Reciprocally, if $f = \rho M_{ku}$, then, $u_f = u$ and $Q(f) = 0$

$f^\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$ with $v \rightarrow f(x, v, t) \in \mathcal{E}$ for all (x, t)

Implies that $f(x, v, t) = \rho(x, t)M_{ku(x,t)}$

Need to specify the dependence of ρ and u on (x, t)

Requires n equations since $(\rho, u) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}$ are determined by n independent real quantities

f satisfies

$$\partial_t f + v \cdot \nabla_x f = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} Q(f^\varepsilon)$$

Problem: $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} Q(f^\varepsilon)$ is not known

Trick:

Collision invariant

is a function $\psi(v)$ such that $\int Q(f)\psi dv = 0, \quad \forall f$

Form a linear vector space \mathcal{C}

Multiply eq. by ψ : ε^{-1} term disappears

Find a conservation law:

$$\partial_t \left(\int_{\mathbb{S}^{n-1}} f(x, v, t) \psi(v) dv \right) + \nabla_x \cdot \left(\int_{\mathbb{S}^{n-1}} f(x, v, t) \psi(v) v dv \right) = 0$$

Have used that ∂_t or ∇_x and $\int \dots dv$ can be interchanged

Limit fully determined if $\dim \mathcal{C} = \dim \mathcal{E} = n$

$\mathcal{C} = \text{Span}\{1\}$. Interaction preserves mass but no other quantity

Due to self-propulsion, **no momentum conservation**

$\dim \mathcal{C} = 1 < \dim \mathcal{E} = n$. Is the limit problem ill-posed ?

Proof that $\psi(v) = 1$ is a CI ?

Obvious. $Q(f) = \nabla_v \cdot [\dots]$ is a divergence

By Stokes theorem on the sphere, $\int Q(f) dv = 0$

Use of the CI $\psi(v) = 1$: Get the **conservation law**

$$\partial_t \left(\int_{\mathbb{S}^{n-1}} f(x, v, t) dv \right) + \nabla_x \cdot \left(\int_{\mathbb{S}^{n-1}} f(x, v, t) v dv \right) = 0$$

With $f = \rho M_{ku}$ we have

$$\int f(x, v, t) dv = \rho(x, t), \quad \int f(x, v, t) v dv = \rho c_1 u$$

We end up with the **mass conservation eq.**

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho u) = 0$$

Given $u \in \mathbb{S}^{n-1}$, Define $\mathcal{Q}_u(f) = \nabla_v \cdot [M_{ku} \nabla_v (\frac{f}{M_{ku}})]$

Note $f \rightarrow \mathcal{Q}_u(f)$ is linear and $Q(f) = \mathcal{Q}_{u_f}(f)$

A function $\psi_u(v)$ is a **GCI associated to u** , iff

$$\int \mathcal{Q}_u(f) \psi_u dv = 0, \quad \forall f \text{ such that } u_f \parallel u$$

The set of GCI \mathcal{G}_u is a linear vector space

Theorem: Given $u \in \mathbb{S}^{n-1}$, \mathcal{G}_u is the n -dim vector space :

$$\mathcal{G}_u = \{v \mapsto C + h(u \cdot v) \beta \cdot v, \text{ with arbitrary } C \in \mathbb{R} \text{ and } \beta \in \mathbb{R}^n \text{ with } \beta \cdot u = 0\}.$$

Introduce $\cos \theta = u \cdot v$ and $h(\cos \theta) = g(\theta) / \sin \theta$

g is the unique solution in V of problem $Lg = \sin \theta$ with

$$Lg(\theta) = -\sin^{2-n} \theta e^{-k \cos \theta} \left(\sin^{n-2} \theta e^{k \cos \theta} g'(\theta) \right)' + (n-2) \sin^{-2} \theta g(\theta)$$

$$V = \{g \mid (n-2) (\sin \theta)^{\frac{n}{2}-2} g \in L^2(0, \pi), (\sin \theta)^{\frac{n}{2}-1} g \in H_0^1(0, \pi)\}$$

Use GCI $h(u \cdot v)\beta \cdot v$ for $\beta \in \mathbb{R}^n$ with $\beta \cdot u = 0$

Equivalently, use the vector valued function $\vec{\psi}_u(v) = h(u \cdot v)P_{u^\perp}v$

Multiply FP eq by GCI $\vec{\psi}_{u_{f^\varepsilon}}$: $O(\varepsilon^{-1})$ terms disappear

$$\int Q(f) \vec{\psi}_{u_f} dv = \int Q_{u_f}(f) \vec{\psi}_{u_f} dv = 0 \quad \text{by property of GCI}$$

Gives:
$$\int (\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) \vec{\psi}_{u_{f^\varepsilon}} dv = 0$$

As $\varepsilon \rightarrow 0$: $f^\varepsilon \rightarrow \rho M_{ku}$ and $\vec{\psi}_{u_{f^\varepsilon}} \rightarrow \vec{\psi}_u$ Leads to:

$$\int (\partial_t(\rho M_{ku}) + v \cdot \nabla_x(\rho M_{ku})) \vec{\psi}_u dv = 0$$

Not a conservation equation

because of dependence of $\vec{\psi}_u$ upon (x, t) through u

∂_t or ∇_x and $\int \dots dv$ cannot be interchanged

Velocity equation takes the form:

$$\rho(\partial_t u + c_2(u \cdot \nabla_x)u) + P_{u^\perp} \nabla_x \rho = 0$$

Computations are straightforward but tedious

Coefficient c_2 depends on GCI

$$c_2 = \frac{\int_0^\pi \cos \theta h(\cos \theta) e^{k \cos \theta} \sin^n \theta d\theta}{\int_0^\pi h(\cos \theta) e^{k \cos \theta} \sin^n \theta d\theta}$$

Self-Organized Hydrodynamics (SOH)

System for the density $\rho(x, t)$ and velocity direction $u(x, t)$:

$$\partial_t \rho + c_1 \nabla_x (\rho u) = 0$$

$$\rho (\partial_t u + c_2 (u \cdot \nabla_x) u) + P_{u^\perp} \nabla_x \rho = 0$$

$$|u| = 1$$

Rigorous limit $\varepsilon \rightarrow 0$

[N Jiang, L Xiong, T-F Zhang, arXiv:1508.04640]

4. Properties of the SOH model and extensions

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho u) = 0$$

$$\rho (\partial_t u + c_2 (u \cdot \nabla_x) u) + P_{u^\perp} \nabla_x \rho = 0, \quad |u| = 1$$

Similar to **Compressible Euler eqs.** of gas dynamics
System of **hyperbolic** eqs.

But **major differences**:

Geometric constraint $|u| = 1$

Preserved in time if satisfied by the initial condition
thanks to the projection operator P_{u^\perp}

But system **not in conservative form**

i.e. spatial derivatives not in divergence form

$c_2 \neq c_1$: **loss of Galilean invariance**

Vision anisotropy (or blind zone) reinforces this effect

[Frouvelle, M3AS 2012]

Local existence of smooth solutions

[PD Liu Motsch Panferov, MAA 20 (2013) 089]

in 2D and in 3D under the condition:

\exists a direction ω and $|u_0 \times \omega| \geq C > 0$ at $t = 0$

Both rely on **symmetrization and energy estimates**

Non-smooth solutions

Non-conservative model, no entropy

Shock relations unknown

SOH is **relaxation limit** $\zeta \rightarrow 0$ of:

$$\partial_t(\rho u) + c_2 \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho = -\frac{1}{\zeta} \rho (1 - |u|^2) u$$

But limit system not conservative:

Relaxation theory not applicable

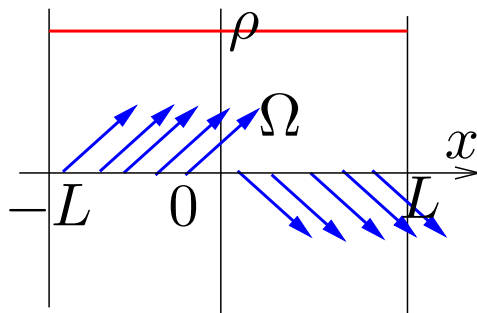
Selection principle: physically valid solutions =
consistent approximations of the Vicsek particle system

Numerical observation [S Motsch, L Navoret, MMS 9 (2011) 1253]

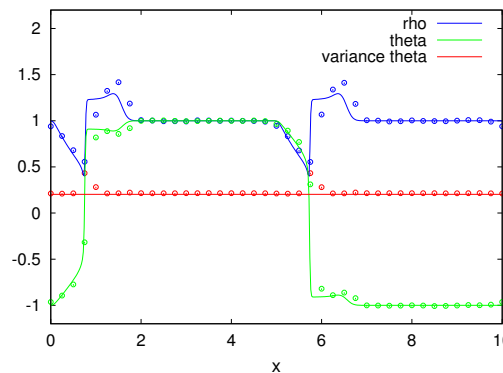
Relaxation based scheme → valid solutions

Standard shock capturing methods → not valid

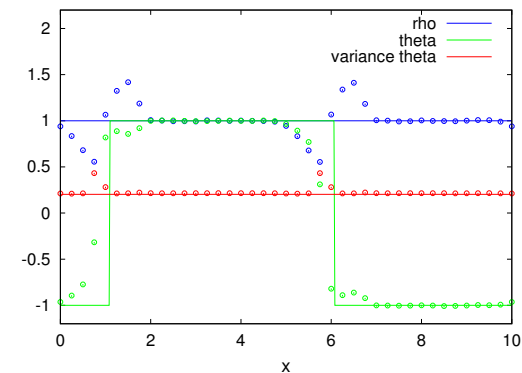
Initial conditions



Relaxation-based



Standard



Vicsek (dots), SOH (solid line), ρ (blue), θ (green), c_1 (red)

Mills: $\rho(r) = \rho_0 (r / r_0)^{c/d}$, $u = x^\perp / r$

are **stationary solutions**. Stability ?

Shape depends on noise level

small noise: $\rho(r)$ convex: sharp edged mills

large noise: $\rho(r)$ concave: fuzzy edges

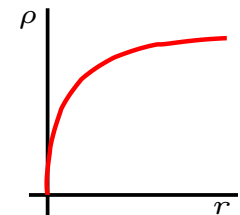
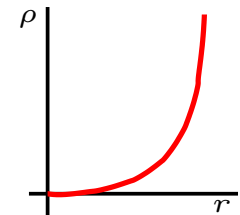
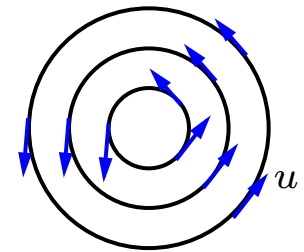
Previous models of active fluids

use **average velocity** (i.e. $c_1 u$)

[Toner, Tu & Ramaswamy, Annals of Physics 2005]

except e.g. [Baskaran & Marchetti, PRL 2008]

who use '**polarization vector**' ρu



So far: scaling of interaction range \bar{R} is such that $\bar{R} = \varepsilon$

\bar{R} is microscopic and of the same order as the mean-free path $\bar{\nu}^{-1}$

Different possibility is $\bar{R} = \sqrt{\varepsilon}$

\bar{R} is still microscopic

i.e. infinitesimally small at the macroscopic scale

but **much larger than the mean-free path $\bar{\nu}^{-1}$**

Interaction force must be Taylor expanded **at the next order**

$$F_f = kP_{v\perp} \left(u_f + \varepsilon \frac{H}{|J_f|} P_{u_f^\perp} \Delta_x J_f \right) + \mathcal{O}(\varepsilon^2)$$

H is a constant which only depends on the dimension

The $\mathcal{O}(\varepsilon)$ term comes into the FP eq

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{kH}{|J_{f^\varepsilon}|} \nabla_v \cdot (P_{v^\perp} P_{u_{f^\varepsilon}^\perp} \Delta_x J_{f^\varepsilon} f^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

Its contribution in the SOH model needs to be evaluated

The resulting model is:

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho u) = 0$$

$$\rho (\partial_t u + c_2 (u \cdot \nabla_x) u) + P_{u^\perp} \nabla_x \rho = c_3 P_{u^\perp} \Delta_x (\rho u), \quad |u| = 1$$

Viscous version of the SOH model

Similar to the compressible Navier-Stokes system

Scaling retains **non-local effects** via velocity diffusion

Local existence of smooth solutions in 2D. No result in 3D.

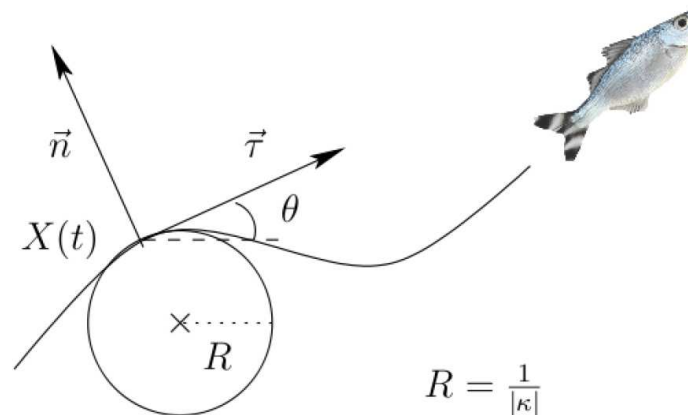
$$c_3 = kH((n-1) + c_2) > 0$$

Agents control **curvature** instead of direction

like driver with steering wheel

and try to **align with neighbors**

Persistent Turner [Gautrais et al, J. Math. Biol. 2009]



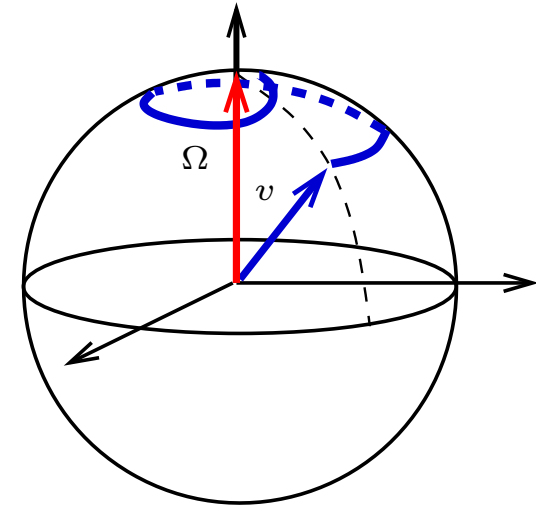
Macro model is **SOH**

Add **precession** (dimension = 3)

$$\varepsilon(\partial_t f + v \cdot \nabla_x f) = -\nabla_v \cdot (F_f f) + \Delta_v f$$

$$F_f = kP_{v^\perp} \bar{v}_f + \alpha \bar{v}_f \times v$$

$$\bar{v}_f = u_f + \varepsilon \frac{H}{|J_f|} P_{u_f^\perp} \Delta_x J_f, \quad u_f = \frac{J_f}{|J_f|}$$



The limit model is **SOH with precession**

$$\partial_t \rho + c_1 \nabla_x (\rho u) = 0$$

$$\begin{aligned} \rho \{ \partial_t u + c_2 \cos \delta (u \cdot \nabla_x) u + c_2 \sin \delta u \times ((u \cdot \nabla_x) u) \} + P_{u^\perp} \nabla_x \rho + \\ + kH \{ -(2 + c_2 \cos \delta) P_{u^\perp} \Delta_x (\rho u) + (c_2 \sin \delta - \alpha) u \times \Delta_x (\rho u) \} = 0 \end{aligned}$$

δ related to precession speed α

Special case: no self-propulsion and $\rho = 1$. Gives:

$$\partial_t u + kH \left\{ (2d + c_2 \cos \delta) (u \times (u \times \Delta_x u)) \right. \\ \left. + (c_2 \sin \delta - \alpha) (u \times \Delta_x u) \right\} = 0$$

Landau-Lifschitz-Gilbert equation

First (to our knowledge) microscopic derivation of LLG eq.

5. Conclusion

Macroscopic models of **collective dynamics**

require **new concepts** to face **new challenges**

lack of **conservation properties**, phase transitions, ...

The **Self-Organized Hydrodynamic (SOH)** model

is the paradigmatic **fluid model for collective dynamics**

Its mathematical analysis is **widely open**

It has potential to model a vast category of

self-organization phenomena