Hydrodynamics for symmetric exclusion in contact with reservoirs

Patrícia Gonçalves

TÉCNICO LISBOA

IHP, Paris

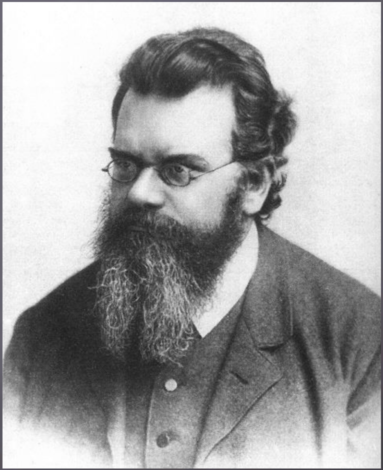
22nd and 24th May 2017

Setting

Goal: present microscopic models for the dynamics between particles to obtain the macroscopic laws for the evolution of some quantity of interest in a physical system.

Historical context

It all started with Ludwig Eduard Boltzmann (1844-1906) who was an Austrian physicist and philosopher whose greatest achievement was in the development of statistical mechanics.

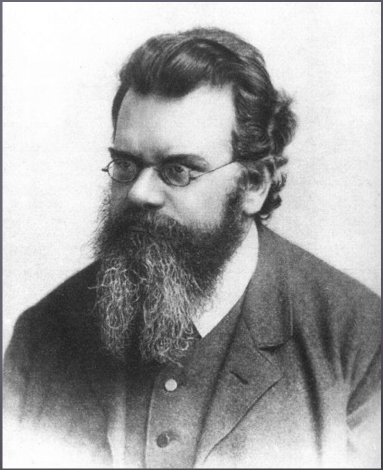


"Statistical mechanics explains and predicts how the properties of atoms determine the physical properties of matter."

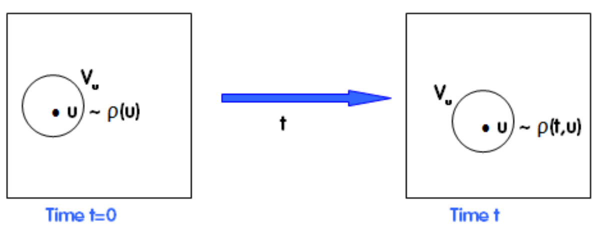
Historical context

Suppose we are interested in analyzing the evolution of a fluid or a gas. Since the number of components is huge one cannot give a precise description of the microscopic state of the system. Then we should:

- examine the equilibrium states of a system;
- characterize them through macroscopic quantities: pressure, temperature, density, etc;
- examine the system out of equilibrium.



Physical motivation



- Find the invariant states of a system.
- Characterize them by a quantity $\rho(\cdot)$.
- Fix a point $u \in V$ and a neighborhood V_u (big microscopically). Due to interaction, the system reaches an equilibrium $\rho(u)$.
- Let time evolve and now the equilibrium close to u is given by $\rho(t, u)$. How does $\rho(t, u)$ evolve?

From microscopic to macroscopic

The scenario

- Underlying scenario: we look for the evolution of a physical system, e.g. the spread of a gas confined to a finite volume.
- Two scales are considered:
  - a macroscopic one.
  - a microscopic one.
- Due to the huge number of molecules it is hard to describe precisely the microscopic state of a system.
- Goal: describe the macroscopic evolution from the microscopic interaction between particles.
- Assumption: each particle performs a *random walk* subject to some restriction.

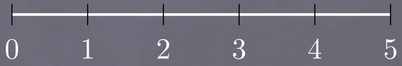
Question?

- The process for the random motion of particles is an interacting particle system.
- Two scales for space/time and a volume V.
- We discretize the volume according to a parameter N.
- In each cell we put a random number of particles.
- The dynamics conserves some quantity.
- The waiting times are given by independent Poisson processes so that the particle system becomes a Markov chain - *loss of memory*.

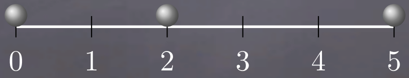
What is the law describing the evolution of the conserved quantity of the system?

Example

Let us fix our space as the set of points $\{1, 2, 3, 4\}$. This is the microscopic space. Add two end points at 0 and 5.

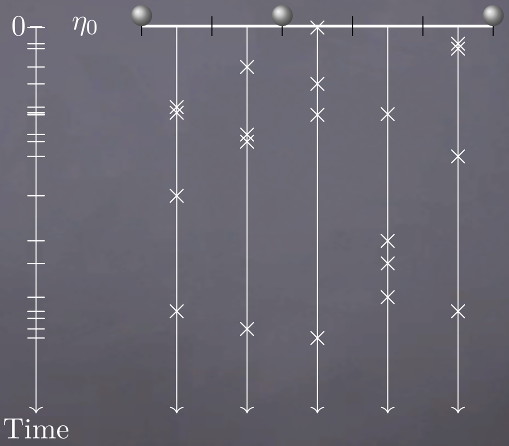


Now, for fixing the initial state we can do the following. Toss a coin, if we get head we put a particle at the site 1 and if we get a tail we leave it empty. Repeat this for each site of the discrete set. Suppose we got at the end to:

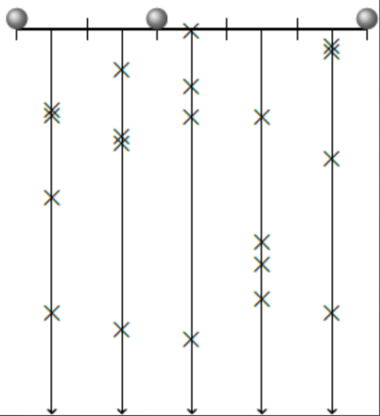


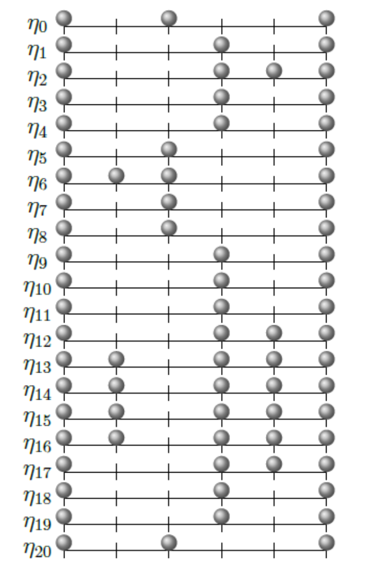
Note that 0 and 5 are occupied to mimic the role of the reservoirs.

Time between jumps



The dynamics





Particle system

1. N is a scaling parameter;
2. Microscopic space /macroscopic space;

3. Microscopic time $t\theta(N)$/macroscopic time t;
4. Each site has an exponentially distributed clock/ clocks at different sites are independent;

5. Fix a transition probability $p(x, y) = p(y - x)$.
6. $\eta_t(x)$ denotes the quantity of particles at the site x.
7. **Markov process** for which the quantity of particles $\sum_x \eta(x)$ is conserved;

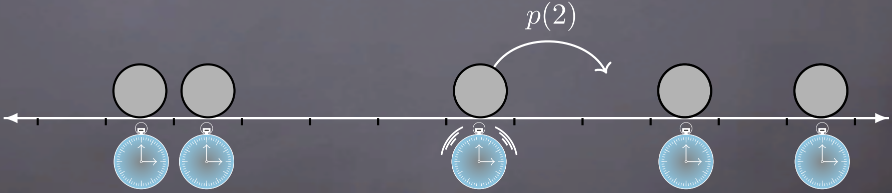
Some examples of dynamics

Exclusion processes: the initial configuration

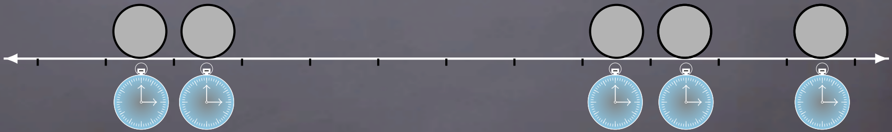
The dynamics:

After the ring of a clock the particle jumps from x to y at rate $p(y-x)$ if y is empty, otherwise the particle waits a new random time.

Exclusion processes: a ring of a clock

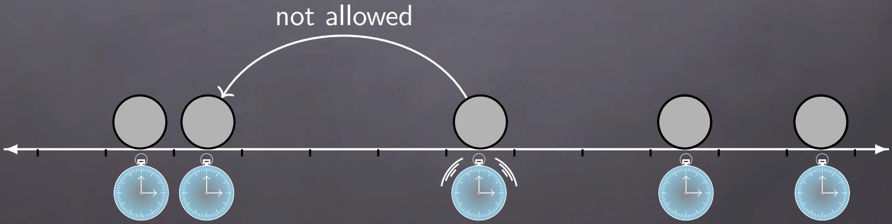


Exclusion processes: after the ring of the clock

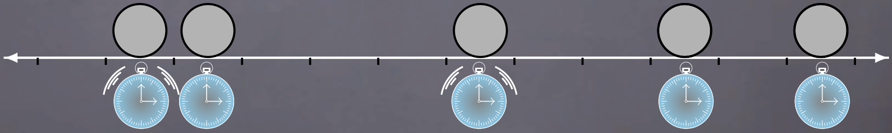


Exclusion processes: forbidden jumps

The dynamics:
Jumps to occupied sites are forbidden!



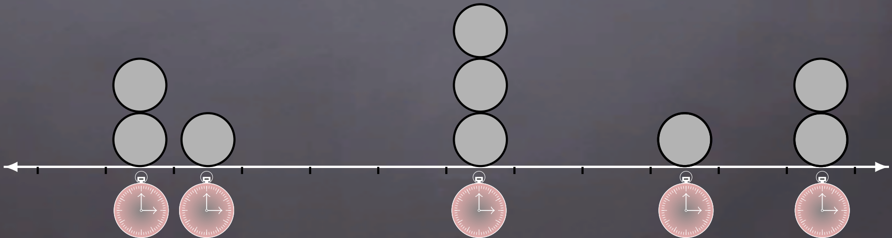
This cannot happen



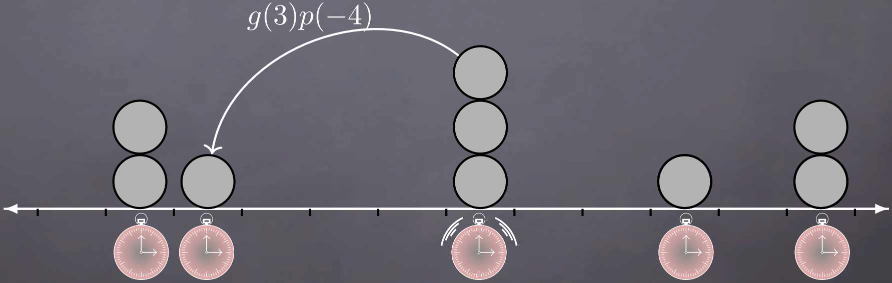
Zero-Range processes: initial configuration

The dynamics:

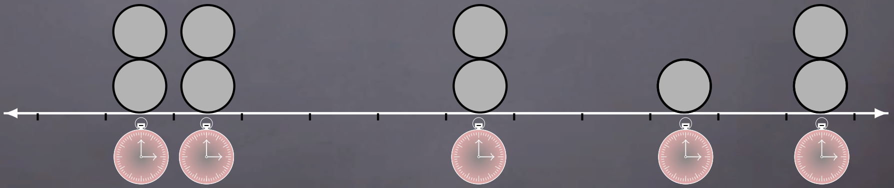
After the ring of a clock a particle jumps from x to y at rate $g(\eta(x))p(y-x)$.



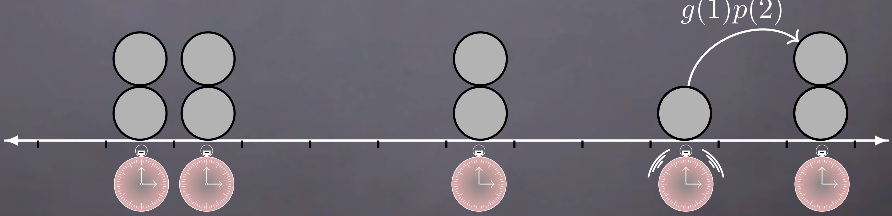
Zero-Range processes: a ring of a clock



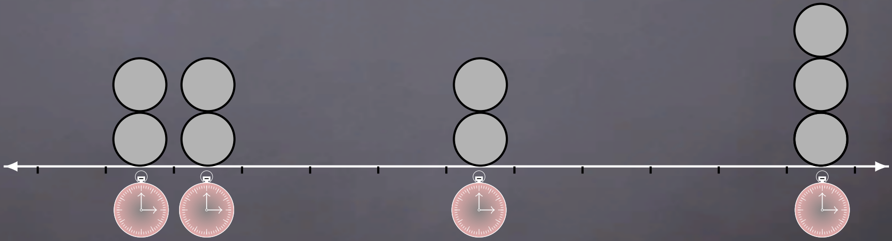
Zero-Range processes: after the ring of the clock



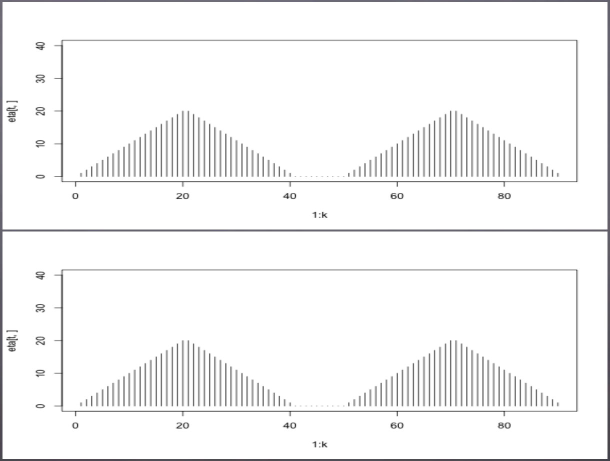
Zero-Range processes: another ring of a clock



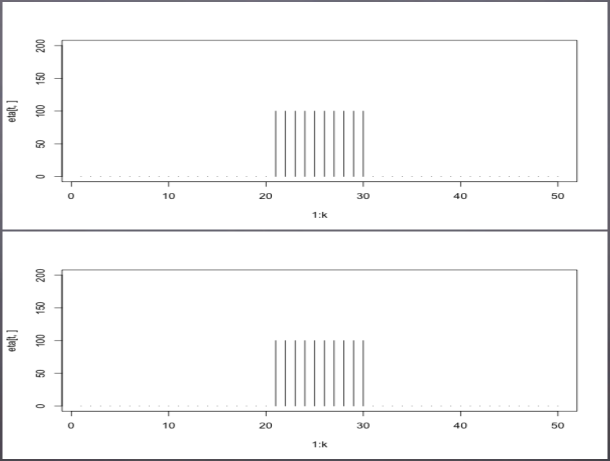
Zero-Range processes: after the ring of the clock



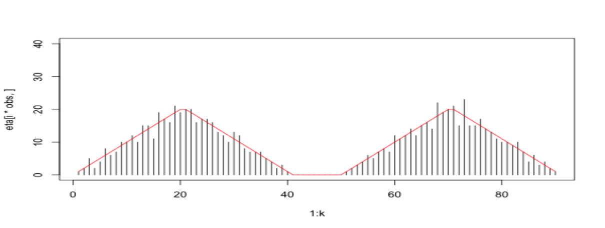
Simulation of Zero-Range: symmetric/asymmetric



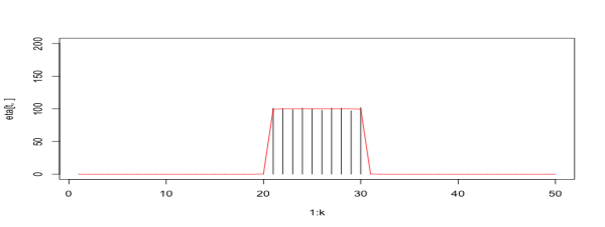
Simulation of Zero-Range: symmetric/asymmetric



The hydrodynamic equation:



The hydrodynamic equation [1]



[1] Thanks to F. Hernandez for the simulations.

Outline of the lectures:

- Lecture 1 + Lecture 2: We will analyze the hydrodynamic limit for the symmetric simple exclusion process (SSEP) in contact with stochastic (slow/fast) reservoirs.

- Lecture 3: We will analyze the hydrodynamic limit for an exclusion process in contact with stochastic reservoirs when jumps are long range given by a symmetric probability transition rate:
  - with finite variance;
  - with infinite variance.

SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

The dynamics:

- For $N \geq 1$ let $\Lambda_N = \{1, \ldots, N-1\}$.
- We denote the process by $\{\eta_t : t \geq 0\}$ which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$.
- The infinitesimal generator $\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,b}$ is given on $f : \Omega_N \to \mathbb{R}$, by

$$(\mathcal{L}_{N,0} f)(\eta) \;=\; \sum_{x=1}^{N-2} \Big(f(\eta^{x,x+1}) - f(\eta) \Big) \,,$$

$$(\mathcal{L}_{N,b} f)(\eta) \;=\; \frac{\kappa}{N^\theta} \sum_{x \in \{1, N-1\}} c_{r_x}(\eta(x)) \Big(f(\eta^x) - f(\eta) \Big) \,,$$

where for $x = 1$ and $x = N-1$,
$c_{r_x}(\eta(x)) = r_x(1 - \eta(x)) + (1 - r_x)\eta(x)$, $r_1 = \alpha$ and $r_{N-1} = \beta$.

Goal: analyze the impact of changing the strength of the reservoirs (by changing θ) on the macroscopic behavior of the system.

Invariant measures:

If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant (equilibrium measures): $\nu_\rho(\eta : \eta(x) = 1) = \rho$.
To prove this we claim that $\int_{\Omega_N} \mathcal{L}_N f(\eta)\nu_\rho(d\eta) = 0$ for any f.
Since $\eta(x) \in \{0, 1\}$ any f can be rewritten as a weighted sum of products of $\eta(x)$ or $1 - \eta(x)$. By linearity, in order to prove the claim it is enough to prove it for functions of the form $f(\eta) = \eta(x_1) \cdots \eta(x_k)$, where $x_1, ..., x_k \in \Lambda_N$.

Let us see the action of the generator! ->

Stationary measures:

If $\alpha \neq \beta$ the Bernoulli product measure is no longer invariant, but since we have a finite state irreducible Markov process we know that there exists a UNIQUE invariant measure: the stationary measure (non-equilibrium) denoted by μ_{ss}.

By the matrix ansatz method one can get information on this measure.

Hydrodynamic Limit:

- For $\eta \in \Omega_N$ let $\pi_t^N(\eta, du) = \frac{1}{N}\sum_{x=1}^{N-1} \eta_{tN^2}(x)\delta_{x/N}(du)$ be the *empirical measure. (Note the diffusive time scaling!)*

- Fix a measurable profile $g : [0,1] \to [0,1]$ and a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $\delta > 0$ and every continuous function $H : [0,1] \to \mathbb{R}$,

$$\frac{1}{N}\sum_{x=1}^{N-1} H\left(\tfrac{x}{N}\right)\eta(x) \to_{N\to\infty} \int_0^1 H(q)\, g(q) dq,$$

wrt μ_N. Then for any $t > 0$, $\pi_t^N \to \rho(t,q)dq$, as $N \to \infty$, where $\rho(t,q)$ evolves according to a PDE (the hydrodynamic equation).

Hydrodynamic Limit:

Definition: Let $g : [0,1] \to [0,1]$ be a measurable function. We say that a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ in Ω_N is associated to a profile $g(\cdot)$ if for any continuous function $H : [0,1] \to \mathbb{R}$ and every $\delta > 0$

$$\lim_{N\to\infty} \mu_N \left(\eta : \left| \frac{1}{N} \sum_{x\in\Lambda_N} H\left(\tfrac{x}{N}\right) \eta_x - \int_0^1 H(q)g(q)dq \right| > \delta \right) = 0.$$

Hydrodynamic Limit:

Theorem: Let $g : [0,1] \to [0,1]$ *be a measurable function and let* $\{\mu_N\}_{N\geq 1}$ *be a sequence of probability measures in* Ω_N *associated to* $g(\cdot)$*. Then, for any* $0 \leq t \leq T$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\eta_\cdot : \Big|\frac{1}{N}\sum_{x\in\Lambda_N} H\left(\tfrac{x}{N}\right)\eta_{tN^2}(x) - \int_0^1 H(q)\rho_t(q)dq\Big| > \delta\Big) = 0,$$

and $\rho_t(\cdot)$ *is the unique weak solution of the heat equation with different types of boundary conditions depending on the range of the parameter* θ *with initial condition* $\rho_0(\cdot) = g(\cdot)$.

Hydrodynamic equations:

- $\theta < 1$: The heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho_t(q) = \partial_q^2 \rho_t(q)\,, & \text{for } t > 0\,,\ q \in (0,1)\,, \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta\,, & \text{for } t > 0. \end{cases}$$

- $\theta = 1$: The heat equation with Robin boundary conditions

$$\begin{cases} \partial_t \rho_t(q) = \partial_q^2 \rho_t(q)\,, & \text{for } t > 0\,,\ q \in (0,1)\,, \\ \partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha)\,, & \text{for } t > 0\,, \\ \partial_q \rho_t(1) = \kappa(\beta - \rho_t(1))\,, & \text{for } t > 0. \end{cases}$$

- $\theta > 1$: The heat equation with Neumann boundary conditions

$$\begin{cases} \partial_t \rho_t(q) = \partial_q^2 \rho_t(q)\,, & \text{for } t > 0\,,\ q \in (0,1)\,, \\ \partial_q \rho_t(0) = \partial_q \rho_t(1) = 0\,, & \text{for } t > 0. \end{cases}$$

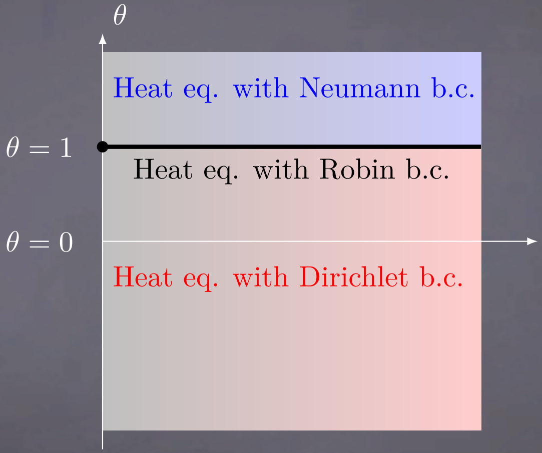


Figure: The three different hydrodynamic regimes in terms of θ.

The proof:

How do we prove the results?

Two things to do:

- Tightness;
- Characterization of limit points: limit points are concentrated on trajectories of measures that are absolutely continuous wrt the Lebesgue measure and the density is a weak solution of the corresponding PDE.

Let us focus on the second point.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HEDBC if:

1. $\rho \in L^2(0,T;\mathcal{H}^1)$;
2. ρ satisfies the weak formulation:

$$
\begin{aligned}
&\int_0^1 \rho_t(q) H_t(q)\,dq - \int_0^1 g(q) H_0(q)\,dq \\
&- \int_0^t \int_0^1 \rho_s(q) \Big(\partial_q^2 + \partial_s \Big) H_s(q)\,ds\,dq \\
&+ \int_0^t \beta \partial_q H_s(1) - \alpha \partial_q H_s(0)\,ds = 0,
\end{aligned}
$$

for all $t \in [0,T]$ and any function $H \in C_0^{1,2}([0,T] \times [0,1])$.

Other notion of solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HEDBC if:

1. $\rho \in L^2(0,T;\mathcal{H}^1)$;
2. ρ satisfies the weak formulation:

$$\begin{aligned} &\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq \\ &- \int_0^t \int_0^1 \rho_s(q) \Big(\partial_q^2 + \partial_s \Big) H_s(q)\, ds\, dq = 0, \end{aligned}$$

   for all $t \in [0,T]$ and any function $H \in C_c^{1,2}([0,T] \times [0,1])$;
3. $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$, for $t \in (0,T]$.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HERBD if:

1. $\rho \in L^2(0,T;\mathcal{H}^1)$,
2. ρ satisfies the weak formulation:

$$
\begin{aligned}
&\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq \\
&- \int_0^t \int_0^1 \rho_s(q) \Big(\partial_q^2 + \partial_s \Big) H_s(q)\, ds\, dq \\
&+ \int_0^t \{ \rho_s(1) \partial_q H_s(1) - \rho_s(0) \partial_q H_s(0) \}\, ds \\
&\qquad - \kappa \int_0^t \{ H_s(0)(\alpha - \rho_s(0)) + H_s(1)(\beta - \rho_s(1)) \}\, ds = 0,
\end{aligned}
$$

for all $t \in [0,T]$ and any function $H \in C^{1,2}([0,T] \times [0,1])$.

Characterizing limit points:

Dynkin's formula: Let $\{\eta_t\}_{t\geq 0}$ *be a Markov process with generator* $\mathcal{L}$ *and with countable state space* E*. Let* $F:\mathbb{R}^+\times E\to\mathbb{R}$ *be a bounded function such that*

- $\forall \eta\in E, F(\cdot,\eta)\in C^2(\mathbb{R}^+)$,
- *there exists a finite constant* C*, such that* $\sup_{(s,\eta)}|\partial_s^j F(s,\eta)|\leq C$*, for* $j=1,2$*.*

For $t\geq 0$*, let*

$$M_t^F = F(t,\eta_t)-F(0,\eta_0)-\int_0^t(\partial_s+\mathcal{L})F(s,\eta_s)ds.$$

Then, $\{M_t^F\}_{t\geq 0}$ *is a martingale wrt* $\mathcal{F}_s=\sigma(\eta_s; s\leq t)$*.*

Characterizing limit points:

Let us fix a test function $H : [0,1] \to \mathbb{R}$ and apply Dynkin's formula with

$$F(t, \eta_t) = \langle \pi_t^N, H \rangle = \frac{1}{N} \sum_{x=1}^{N-1} \eta_{tN^2}(x) H\left(\tfrac{x}{N}\right).$$

Note that F does not depend on time. A simple computation shows that

$$\begin{aligned} N^2 \mathcal{L}_N \langle \pi_s^N, H \rangle = {} & \langle \pi_s^N, \Delta_N H \rangle \\ & + \nabla_N^+ H(0) \eta_{sN^2}(1) - \nabla_N^- H(1) \eta_{sN^2}(N-1) \\ & + \kappa N^{1-\theta} H\left(\tfrac{1}{N}\right)(\alpha - \eta_{sN^2}(1)) \\ & + \kappa N^{1-\theta} H\left(\tfrac{N-1}{N}\right)(\beta - \eta_{sN^2}(N-1)) \end{aligned}$$

$\theta \in [0,1)$:

Take a function $H : [0,1] \to \mathbb{R}$ such that $H(0) = H(1) = 0$ and then we get

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \Delta_N H \rangle ds$$
$$- \int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1) ds + O(N^{-\theta}).$$

If we can replace $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β (this will be made rigorous ahead but only works for $\theta < 1$!) then above we have

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \Delta_N H \rangle ds$$
$$- \int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

Compare with the PDE (note that H does not depend on time).

Still $\theta \in [0,1)$:

Take the expectation above to get

$$\frac{1}{N}\sum_{x=1}^{N-1} H\Big(\tfrac{x}{N}\Big)\Big(\rho_t^N(x) - \rho_0^N(x)\Big) - \int_0^t \frac{1}{N}\sum_{x=1}^{N-1} \Delta_N H\Big(\tfrac{x}{N}\Big)\rho_s^N(x)ds$$
$$- \int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}) = 0.$$

Assume that $\rho_t^N(x) \sim \rho_t(x/N)$ and take the limit in N to get

$$\int_0^1 \rho_t(q)H(q) - \rho_0(q)H(q)dq - \int_0^t \int_0^1 \partial_q^2 H(q)\rho_s(q)dqds$$
$$- \int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0$$

Compare with the PDE (note that H does not depend on time).

$\theta < 0$:

Recall that the previous error blows up when $N \to \infty$. So now, we take a function $H : [0,1] \to \mathbb{R}$ with compact support and then we get

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \Delta_N H \rangle ds.$$

Again compare with the PDE but note that H does not depend on time.
In this case we do not see the Dirichlet boundary conditions and we need extra results to conclude.

$\theta = 1$:

Now, we take a function $H : [0,1] \to \mathbb{R}$ and we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \Delta_N H\rangle ds$$
$$- \int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1) ds$$
$$- \kappa \int_0^t H\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) + H\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) ds.$$

If we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by the average in a box around 1 (resp. $N-1$) (this works for any $\theta \geq 1$):

$$\overrightarrow{\eta}^{\epsilon N}_{sN^2}(1) := \frac{1}{\epsilon N}\sum_{x=1}^{1+\epsilon N} \eta_{sN^2}(x), \quad \overleftarrow{\eta}^{\epsilon N}_{sN^2}(N-1) := \frac{1}{\epsilon N}\sum_{x=N-1}^{N-1-\epsilon N} \eta_{sN^2}(x)$$

and noting that $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

$\theta > 1$:

Again we take a function $H : [0,1] \to \mathbb{R}$ and in this case the terms from the boundary vanish. So we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \Delta_N H\rangle ds$$
$$- \int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)ds + O(N^{1-\theta}).$$

As before, if we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by the average in a box around 1 (resp. $N-1$) and noting that $\overrightarrow{\eta}^{\epsilon N}_{sn^2}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

The replacement lemmas:

Recall that we need to prove that

For any $t > 0$, we have that:

- *for* $\theta \in [0,1)$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t (\eta_{sN^2}(1)-\alpha)\,ds\Big|\Big] = 0;$$

- *for* $\theta \geq 1$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big|\int_0^t (\eta_{sN^2}(1)-\overrightarrow{\eta}^{\epsilon N}_{sN^2}(1))\,ds\Big|\Big] = 0;$$

and the similar result for the point $N-1$.

Replacing by α:

From entropy's and Jensen's inequality, the expectation is bounded from above by

$$\frac{H(\mu_N|\nu^N_{h(\cdot)})}{BN} + \frac{1}{BN}\log \mathbb{E}_{\nu^N_{h(\cdot)}}\left[e^{BN|\int_0^t(\eta_{sN^2}(1)-\alpha)ds|}\right].$$

Above B is a positive constant and $h(\cdot)$ is a profile to choose later on. We remove the absolute value inside the exponential since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_{N\to\infty}\log(a_N+b_N) \leq \max\left\{\limsup_{N\to\infty}\log(a_N), \limsup_{N\to\infty}\log(b_N)\right\}.$$

Note that if $\alpha \leq h(\cdot) \leq \beta$, then:

$$H(\mu_N|\nu^N_{h(\cdot)}) \leq NC(\alpha,\beta).$$

Apply FK formula:

The Feynmann-Kac's formula: Assume that $\mathcal{L}$ is the generator of a Markov process $\{\eta_t\}_{t\geq 0}$ on a countable state space E. Let ν be a p.m. on E and $V : [0,\infty) \times E \to \mathbb{R}$ a bounded function. Define

$$\Gamma_t = \sup_{\{f:||f||_2=1\}} \{\langle V_t, f^2\rangle_\nu + \langle \mathcal{L}f, f\rangle_\nu\}.$$

Then $\mathbb{E}_\nu\left[e^{\int_0^t V(r,\eta_r)dr}\right] \leq e^{\{\int_0^t \Gamma_s ds\}}$.

Then we have to estimate:

$$\sup_f \left\{\langle \eta(1) - \alpha, f\rangle_{\nu^N_{h(\cdot)}} + \frac{N}{B}\langle L_N\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{h(\cdot)}}\right\},$$

where the supremum is carried over all the densities f with respect to $\nu^N_{h(\cdot)}$.

Controlling Dirichlet forms:

For a probability measure μ on Ω_N, we define

$$D_N(\sqrt{f},\mu) \quad := \quad (D_{N,0}+D_{N,b})(\sqrt{f},\mu)$$

where $D_{N,0}(\sqrt{f},\mu) := \dfrac{1}{2}\sum_{x=1}^{N-2} I_{x,x+1}(\sqrt{f},\mu)$, with
$I_{x,x+1}(\sqrt{f},\mu) = \int \left(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\right)^2 d\mu$ and

$$D_{N,b}(\sqrt{f},\mu) : = \frac{\kappa}{2N^\theta}\Big(I_1^{r_1}(\sqrt{f},\mu)+I_{N-1}^{r_{N-1}}(\sqrt{f},\mu)\Big)$$

with $I_x^{r_x}(\sqrt{f},\mu) := \int c_{r_x}(\eta(x))\left(\sqrt{f(\eta^x)}-\sqrt{f(\eta)}\right)^2 d\mu.$

We claim that for any positive constant B if $h(\cdot)$ is a Lipschitz function with $h(0)=\alpha$, $h(1)=\beta$ and locally constant at 0 and 1, then, there exists a constant $C_{\alpha,\beta,h}>0$ such that

$$\frac{N}{B}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu^N_{h(\cdot)}} \leq -\frac{N}{4B}D_N(\sqrt{f},\nu^N_{h(\cdot)})+\frac{C_{\alpha,\beta,h}}{B}.$$

Lemma: Let $T:\eta\in\Omega_N\to T(\eta)\in\Omega_N$ be a transformation and $c:\eta\to c(\eta)$ be a positive local function. Let f be a density with respect to a p.m. μ on Ω_N. Then:

$$\begin{aligned}&\Big\langle c(\eta)[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}],\sqrt{f(\eta)}\Big\rangle_\mu\\ &\leq -\frac{1}{4}\int c(\eta)\left(\left[\sqrt{f(T(\eta))}\right]-\left[\sqrt{f(\eta)}\right]\right)^2 d\mu\\ &+\frac{1}{16}\int\frac{1}{c(\eta)}\left[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2\left(\left[\sqrt{f(T(\eta))}\right]+\left[\sqrt{f(\eta)}\right]\right)^2 d\mu.\end{aligned}$$

So far we have to bound

$$\sup_f \Big\{ \langle \eta(1) - \alpha, f \rangle_{\nu^N_{h(\cdot)}} - \frac{N}{4B} D_N(\sqrt{f}, \nu^N_{h(\cdot)}) + \frac{C_{\alpha,\beta,h}}{B} \Big\},$$

where the supremum is carried over all the densities f with respect to $\nu^N_{h(\cdot)}$. To finish we use

Lemma: For any density f with respect to $\nu^N_{h(\cdot)}$ and any positive constant A, we have that

$$\Big| \langle \eta(1) - \alpha, f \rangle_{\nu^N_{h(\cdot)}} \Big| \lesssim \frac{1}{A} I_1^{r_1}(\sqrt{f}, \nu^N_{h(\cdot)}) + A + [h(\tfrac{1}{N}) - \alpha].$$

The same result holds if α is replaced by β.

Now take $A = BCN^{\theta-1}\kappa^{-1}$, which is the final error and note that it vanishes, as $N \to \infty$, if $\theta < 1$.

Hydrostatic Limit:

Theorem: Let $\{\mu_N\}_{N\geq 1}$ *be the stationary measure for the process* $\{\eta_t\}_{t\geq 0}$ *with generator* $N^2\mathcal{L}_N$*. Then,* $\{\mu_N\}_{N\geq 1}$ *is associated to the profile* $\bar{\rho}:[0,1]\to[0,1]$ *which is given for* $q\in(0,1)$ *by*

$$\bar{\rho}(q)=\begin{cases}(\beta-\alpha)q+\alpha\,;\,\theta<1,\\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\,\theta=1,\\ \frac{\beta+\alpha}{2}\,;\,\theta>1.\end{cases}$$

Note that this is a stationary solution of the hydrodynamic equation.

The empirical profile:

Fix an initial measure μ_N in Ω_N. For $x \in \Lambda_N$ and $t \geq 0$, let

$$\rho_t^N(x) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^N(0) \;=\; \alpha \text{ and } \rho_t^N(N) \;=\; \beta\,, \text{ for all } t \geq 0\,.$$

A simple computation shows that $\rho_t^N(\cdot)$ is a solution of

$$\begin{cases} \partial_t \rho_t^N(x) \;=\; (N^2 \mathcal{B}_N \rho_t^N)(x)\,, \quad x \in \Lambda_N\,, \quad t \geq 0\,, \\ \rho_t^N(0) \;= \alpha\,, \rho_t^N(N) \;=\; \beta\,, \quad t \geq 0\,, \end{cases}$$

where the operator $\mathcal{B}_N$ acts on functions $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$\begin{cases} N^2(\mathcal{B}_N f)(x) \;=\; \Delta_N f(x)\,, \quad \text{for } x \in \{2, \cdots, N-2\}\,, \\ N^2(\mathcal{B}_N f)(1) = N^2(f(2) - f(1)) + \frac{\kappa N^2}{N^\theta}(f(0) - f(1))\,, \\ N^2(\mathcal{B}_N f)(N\!-\!1) = N^2(f(N\!-\!2) - f(N\!-\!1)) + \frac{\kappa N^2}{N^\theta}(f(N) - f(N\!-\!1)). \end{cases}$$

Stationary empirical profile:

The stationary solution of the previous equation is given by

$$\rho^N_{ss}(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$$

where $a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta+\kappa(N-2)}$ and $b_N = a_N(\frac{N^\theta}{\kappa} - 1) + \alpha$.
From where we get that

$$\lim_{N\to\infty} \max_{x\in\Lambda_N} \left|\rho^N_{ss}(x) - \bar{\rho}(x/N)\right| = 0.$$

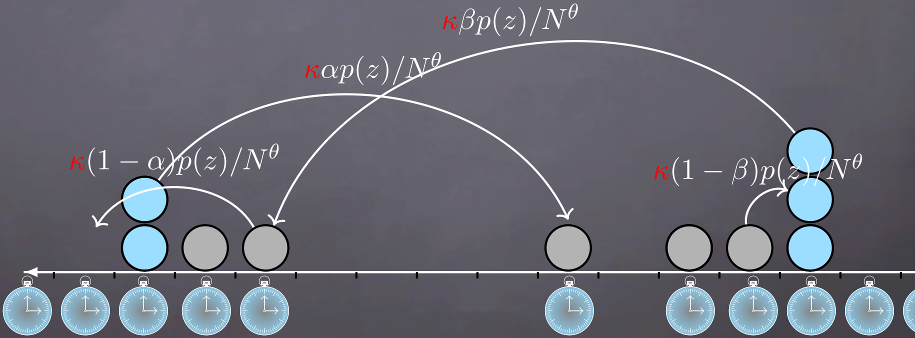
Moreover, the following estimate holds

$$\max_{1\leq x<y\leq N-1} |\mathbb{E}_{\mu_{ss}}[\eta(x);\eta(y)]| \leq \frac{C}{N^\theta + N}.$$

If we put both results together then the proof of the hydrostatic limit follows.

Long range jumps

Exclusion in contact with infinitely many reservoirs



The finite variance case

If jumps are arbitrarily big?

Let $\gamma > 2$ and $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \begin{cases} \dfrac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where c_γ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and since $\gamma > 2$ we define its variance by

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$

The infinitesimal generator:

$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,r} + \mathcal{L}_{N,\ell}$ where

$$(\mathcal{L}_{N,0} f)(\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\eta^{x,y}) - f(\eta)],$$

$$(\mathcal{L}_{N,\ell} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \le 0}} p(x-y) c_x(\eta;\alpha)[f(\eta^x) - f(\eta)],$$

$$(\mathcal{L}_{N,r} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \ge N}} p(x-y) c_x(\eta;\beta)[f(\eta^x) - f(\eta)]$$

where

$$c_x(\eta;\alpha) := (1-\eta_x)\alpha + (1-\alpha)\eta_x.$$

$$c_x(\eta;\beta) := (1-\eta_x)\beta + (1-\beta)\eta_x.$$

Hydrodynamic Limit:

Theorem: Let $g : [0,1] \to [0,1]$ *be a measurable function and let* $\{\mu_N\}_{N\geq 1}$ *be a sequence of probability measures in* Ω_N *associated to* $g(\cdot)$*. Then, for any* $0 \leq t \leq T$*,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} H\left(\tfrac{x}{N}\right)\eta_{t\Theta(N)}(x) - \int_0^1 H(q)\rho_t(q)dq\Big| > \delta\Big) = 0,$$

where the time scale is given by

$$\Theta(N) = \begin{cases} N^2, & \text{if } \theta \geq 2-\gamma, \\ N^{\gamma+\theta}, & \text{if } \theta < 2-\gamma, \end{cases}$$

and $\rho_t(\cdot)$ *is the unique weak solution of the corresponding hydrodynamic equation with initial condition* $\rho_0(\cdot) = g(\cdot)$*.*

Hydrodynamic equations:

• $\theta < 2 - \gamma$: The reaction equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho_t(q) = \kappa \Big\{ V_0(q) - V_1(q)\rho_t(q) \Big\}, & \text{for } t > 0\,,\ q \in (0,1)\,, \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & \text{for } t > 0, \end{cases}$$

where $V_1(q) = r^-(q) + r^+(q)$ and $V_0(q) = \alpha r^-(q) + \beta r^+(q)$, $r^-(q) = c_\gamma \gamma^{-1} q^{-\gamma}$ and $r^+(q) = c_\gamma \gamma^{-1}(1-q)^{-\gamma}$.

• $\theta = 2 - \gamma$: The reaction-diffusion equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q) + \kappa \Big\{ V_0(q) - V_1(q)\rho_t(q) \Big\}, & \text{for } t > 0\,,\ q \in (0,1)\,, \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & \text{for } t > 0. \end{cases}$$

Hydrodynamic equations:

- $2-\gamma<\theta<1$: The heat equation with Dirichlet boundary conditions

$$\begin{cases}\partial_t\rho_t(q)=\frac{\sigma^2}{2}\partial_q^2\rho_t(q)\,, & \text{for } t>0\,,\ q\in(0,1)\,,\\ \rho_t(0)=\alpha \quad \rho_t(1)=\beta\,, & \text{for } t>0.\end{cases}$$

- $\theta=1$: The heat equation with Robin boundary conditions

$$\begin{cases}\partial_t\rho_t(q)=\frac{\sigma^2}{2}\partial_q^2\rho_t(q)\,, & \text{for } t>0\,,\ q\in(0,1)\,,\\ \partial_q\rho_t(0)=\frac{2m\kappa}{\sigma^2}(\rho_t(0)-\alpha)\,, & \text{for } t>0\,,\\ \partial_q\rho_t(1)=\frac{2m\kappa}{\sigma^2}(\beta-\rho_t(1))\,, & \text{for } t>0.\end{cases}$$

Above $m=\sum_{y\geq1}yp(y)$.

- $\theta>1$: The heat equation with Neumann boundary conditions

$$\begin{cases}\partial_t\rho_t(q)=\frac{\sigma^2}{2}\partial_q^2\rho_t(q)\,, & \text{for } t>0\,,\ q\in(0,1)\,,\\ \partial_q\rho_t(0)=\partial_q\rho_t(1)=0\,, & \text{for } t>0.\end{cases}$$

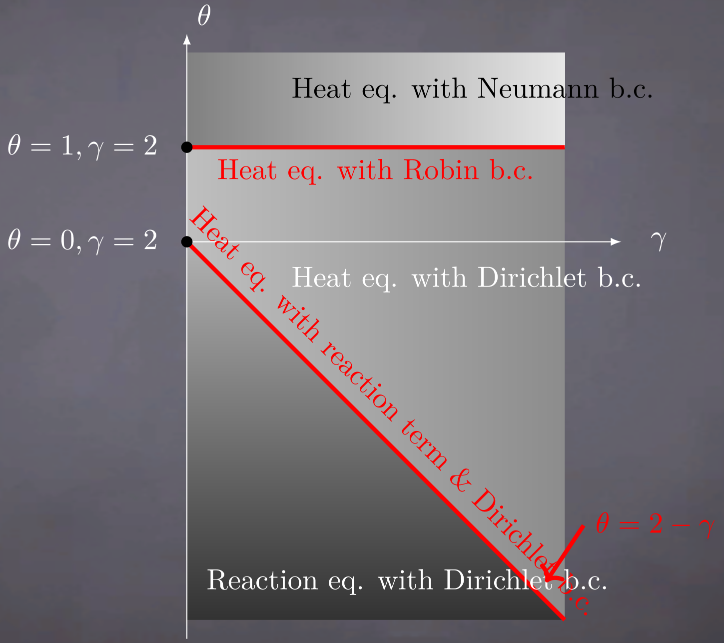


Figure: The five different hydrodynamic regimes in terms of γ and θ.

The notion of weak solution:

Let us see the notion of weak solution in the new cases, the other ones have been defined before.

Let $\hat{\sigma} \geq 0$ and $\hat{\kappa} \geq 0$ be some parameters. Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the RDEDBC

$$\begin{cases} \partial_t \rho_t(q) = \frac{\hat{\sigma}^2}{2} \Delta \, \rho_t(q) + \hat{\kappa} \Big\{ V_0(q) – V_1(q) \rho_t(q) \Big\}, \ (t,q) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in [0,T], \\ \rho_0(\cdot) = g(\cdot), \end{cases}$$

if:

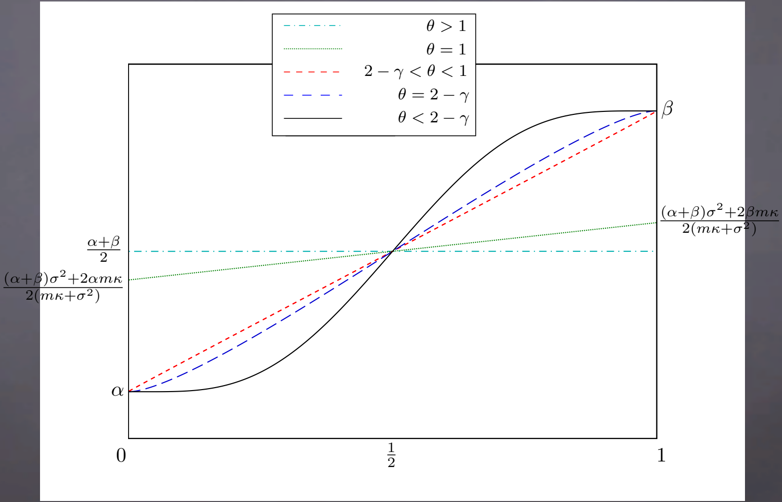
The notion of weak solution:

1. $\rho \in L^2(0,T;\mathcal{H}^1)$ if $\hat\sigma > 0$ and $\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t(q))^2}{q^\gamma} + \frac{(\beta-\rho_t(q))^2}{(1-q)^\gamma} \right\} dq\, dt < \infty$ if $\hat\kappa > 0$,
2. ρ satisfies the weak formulation:

$$\begin{aligned}
&\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq \\
&- \int_0^t \int_0^1 \rho_s(q) \Big(\frac{\hat\sigma^2}{2}\Delta + \partial_s \Big) H_s(q)\, ds\, dq \\
&- \hat\kappa \int_0^t \int_0^1 H_s(q) \left(V_0(q) - V_1(q)\rho_s(q) \right) ds\, dq = 0,
\end{aligned}$$

   for all $t \in [0,T]$ and any function $H \in C_c^{1,2}([0,T]\times[0,1])$,
3. if $\hat\sigma > 0$ then $\rho_t(0) = \alpha, \quad \rho_t(1) = \beta$ for all $t \in (0,T]$.

Stationary solutions:



Characterizing limit points:

A simple computation shows that

$$\begin{aligned}\Theta(N)\mathcal{L}_N(\langle \pi_s^N, H\rangle) =& \frac{\Theta(N)}{N}\sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)\\ &+\frac{\kappa\Theta(N)}{N^{1+\theta}}\sum_{x\in\Lambda_N}(Hr_N^-)(\tfrac{x}{N})(\alpha-\eta_s(x))\\ &+\frac{\kappa\Theta(N)}{N^{1+\theta}}\sum_{x\in\Lambda_N}(Hr_N^+)(\tfrac{x}{N})(\beta-\eta_s(x),\end{aligned}$$

where for all $x \in \Lambda_N$

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y).$$

Extend H to $\mathbb{R}$ in such a way that it remains two times continuously differentiable, and the first term at the RHS is

$$\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}(K_NH)(\tfrac{x}{N})\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\leq 0}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\geq N}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

where $(K_NH)(\frac{x}{N})=\sum_{y\in\mathbb{Z}}p(y-x)\left[H(\frac{y}{N})-H(\frac{x}{N})\right]$.

Remark: Let $H:\mathbb{R}\to\mathbb{R}$ be a two times continuously differentiable function with compact support. We have $\limsup_{N\to\infty}\sup_{x\in\Lambda_N}\left|N^2K_NH\left(\frac{x}{N}\right)-\frac{\sigma^2}{2}\Delta H(\frac{x}{N})\right|=0.$ *As a consequence when $\Theta(N)=N^{\theta+\gamma}$ and $\theta<2-\gamma$ the first term above vanishes as $N\to\infty$.*

The infinite variance case

What about $\gamma \in (1,2)$:

This is in progress. So far we know that for any $\kappa > 0$ and $\theta = 0$, we get the fractional heat equation with Dirichlet boundary conditions given by

$$\begin{cases} \partial_t \rho_t(q) = -(-\Delta)^{\gamma/2}\rho_t(q) + (1-\kappa)V_1(q)\rho_t(q) - (1-\kappa)V_0(q), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta. \end{cases}$$

See $\kappa = 1$! Above, $(-\Delta)^{\gamma/2}$ is the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1+|q|)^{1+\gamma}} du < \infty$$

by (provided the limit exists)

$$(-\Delta)^{\gamma/2} H(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$

The notion of weak solution:

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H\,(q) + V_1(q) H(q)\\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q|\geq \varepsilon} \frac{H(u)-H(q)}{|u-q|^{1+\gamma}} dy, \quad q \in (0,1).\end{aligned}$$

Observe that for any $G, H \in C_c^\infty(0,1)$ we have that

$$\langle G, -\mathbb{L}H \rangle = \langle -\mathbb{L}G, H \rangle = \langle G, H \rangle_{\gamma/2},$$

where $\langle G, H \rangle_{\gamma/2} = \dfrac{c_\gamma}{2} \iint_{[0,1]^2} \dfrac{(H(q)-H(u))(G(q)-G(u))}{|q-u|^{1+\gamma}} dq du.$

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $G : (0,1) \to \mathbb{R}$ such that $\|G\|_{\gamma/2} < \infty$ and the $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ norm is defined by

$$\|G\|^2_{\mathcal{H}^{\gamma/2}} := \|G\|^2_{L^2} + \|G\|^2_{\gamma/2}.$$

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the PDE above if:

i) $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$ and
$\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t(q))^2}{q^\gamma} + \frac{(\beta-\rho_t(q))^2}{(1-q)^\gamma} \right\} dq\, dt < \infty,$

ii) For all $t \in [0,T]$ and any function $H \in C_c^{1,\infty}([0,T] \times (0,1))$:

$$\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\partial_s + \mathbb{L} \Big) H_s(q)\, dq ds$$
$$- \kappa \int_0^t \int_0^1 V_0(q) H_s(q) dq\, ds + \kappa \int_0^t \int_0^1 V_1(q) H_s(q) \rho_s(q) dq\, ds = 0,$$

iii) $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$, for $t \in (0,T]$.

Characterizing limit points:

$$N^\gamma \mathcal{L}_N(\langle \pi_s^N, H\rangle) = \frac{N^\gamma}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)$$

$$+\frac{\kappa N^\gamma}{N} \sum_{x\in\Lambda_N} (Hr_N^-)(\tfrac{x}{N})(\alpha - \eta_s(x)) + \frac{\kappa N^\gamma}{N} \sum_{x\in\Lambda_N} (Hr_N^+)(\tfrac{x}{N})(\beta - \eta_s(x)).$$

For H with compact support in $[a, 1-a]$ for $a \in (0,1)$ we have

$$\lim_{N\to\infty} N^\gamma \sum_{y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right] = (\mathbb{L}H)(\tfrac{x}{N}),$$

$$\lim_{N\to\infty} N^\gamma (r_N^-)(\tfrac{x}{N}) = r^-(\tfrac{x}{N}),$$

$$\lim_{N\to\infty} N^\gamma (r_N^+)(\tfrac{x}{N}) = r^+(\tfrac{x}{N})$$

uniformly in $[a, 1-a]$.

Characterizing limit points:

Thus, the first term on the right hand side above can be replaced by

$$\langle \pi_t^N, \mathbb{L}H \rangle \to \int_0^1 (\mathbb{L}H)(q)\rho_t(q)dq,$$

as N goes to ∞.
The other terms can be replaced by $\kappa\langle \alpha - \pi_t^N, Hr^- \rangle + \kappa\langle \beta - \pi_t^N, Hr^+ \rangle$ which converges to

$$\kappa \int_0^1 H(q)r^-(q)(\alpha - \rho_t(q))dq + \kappa \int_0^1 H(q)r^+(q)(\beta - \rho_t(q))dq$$
$$= \kappa \int_0^1 H(q)V_0(q)dq - \kappa \int_0^1 H(q)V_1(q)\rho_t(q)dq,$$

as N goes to ∞.

Uniqueness of weak solution:

To prove it we do the following. Let $\bar{\rho} = \rho^1 - \rho^2$, where ρ^1 and ρ^2 are two weak solutions starting from g. We have $\bar{\rho}_t(0) = \bar{\rho}_t(1) = 0$. Then,

$$\langle \bar{\rho}_t, H_t \rangle - \int_0^t \langle \bar{\rho}_s, \Big(\partial_s + \mathbb{L}\Big) H_s \rangle ds + \kappa \int_0^t \langle V_1 H_s, \bar{\rho}_s \rangle ds = 0.$$

Take now $H_N(s,q) = \int_s^t G_N(r,q)\, dr$ where $(G_N)_{N\geq 0}$ is a sequence of functions in $C_c^{1,\infty}([0,T]\times(0,1))$ converging to $\bar{\rho}$. Plug H_N in the equation and take $N \to \infty$ to get

$$\int_0^t \int_0^1 \bar{\rho}_s^2(q)\, dq ds + \frac{1}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{\gamma/2}^2 + \frac{\kappa}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{V_1}^2 = 0.$$

From this we conclude the uniqueness.

Uniqueness of weak solution:

Lemma: Let $(H_N)_N$ be defined as above. We have

$$\lim_{N\to\infty} \int_0^t \int_0^1 \bar\rho_s(q)\, (\partial_s H_N)(s,q)\, dq ds = -\int_0^t \int_0^1 \bar\rho_s^2(q)\, dq ds.$$

$$\lim_{N\to\infty} \int_0^T \int_0^1 \bar\rho_s(q) \mathbb{L} H_N(s,q)\, dq ds = -\frac{1}{2} \Big\| \int_0^t \bar\rho_s ds \Big\|_{\gamma/2}^2 .$$

$$\lim_{N\to\infty} \int_0^t V_1(q) H_N(s,q) \bar\rho_s(q)\, ds = \frac{1}{2} \Big\| \int_0^t \bar\rho_s ds \Big\|_{V_1}^2 < \infty.$$

References:

1. Baldasso, R., Menezes, O., Neumann, A., Souza, R. (2017): *Exclusion process with slow boundary*, JSP, 167 no. 5, 1112–1142.
2. Franco, T., G., P., Neumann, A. (2013): *Hydrodynamical behavior of symmetric exclusion with slow bonds*, AIHP: PS, 49 n. 2, 402–427.
3. Bernardin, C., Jiménez-Oviedo, B.: *Fractional Fick's Law for the boundary driven exclusion process with long jumps*, to appear in ALEA.
4. Bernardin, C., G., P., Jiménez-Oviedo, B.: *Slow to fast infinitely extended reservoirs for the symmetric exclusion with long jumps*, online at arxiv.org.
5. Bernardin, C., G., P., Jiménez-Oviedo, B.: *A microscopic model for the regional fractional laplacian with Dirichlet boundary conditions*, in preparation.

Thank you!