Hydrodynamics with and without local equilibrium

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Single queue

- State $\eta \in \mathbb{N}$ number of clients
- Jumps $g \uparrow$, $g(0) = 0 < g(1)$, $\alpha \in (c, 1]$

$$\begin{aligned} \eta \to \eta + 1 \quad & \text{rate} \quad \lambda \\ \eta \to \eta - 1 \quad & \text{rate} \quad \alpha g(\eta) \end{aligned}$$

- Invariant measure: $\lambda < \alpha g(\infty)$,

$$\theta_\lambda^\alpha(n) = Z(\lambda/\alpha)^{-1} \frac{(\lambda/\alpha)^n}{\prod_{k=1}^n g(k)} = \theta_{\lambda/\alpha}^1(n)$$

- Outgoing flux vs. mean density

$$\begin{aligned} \lambda &= Z(\lambda/\alpha)^{-1} \sum_{n=0}^{+\infty} n\theta_\lambda^\alpha(n) =: R^\alpha(\lambda) = R^1(\lambda/\alpha) \\ \rho &= Z(\lambda/\alpha)^{-1} \sum_{n=0}^{+\infty} g(n)\theta_\lambda^\alpha(n) := (R^\alpha)^{-1}(\rho) = \alpha(R^1)^{-1}(\rho) \end{aligned}$$

Examples

1. M/M/1: $\mathcal{G}(1 - \lambda/\alpha) - 1$
   - $g(n) = \alpha n \wedge 1$
   - $R(\lambda) = \dfrac{\lambda/\alpha}{1 - \lambda/\alpha}$
   - $R^{-1}(\rho) = \alpha \dfrac{\rho}{1 + \rho}$
2. M/M/∞ (independent clients): $\mathcal{P}(\lambda/\alpha)$
   - $g(n) = \alpha n$
   - $R(\lambda) = \lambda/\alpha$
   - $R^{-1}(\rho) = \alpha\rho$

Phase transition

- Choose random server $\alpha \sim q(d\alpha)$
- Mean equilibrium density (annealed) vs. flux

$$\overline{R}(\lambda) = \int_{(c,1]} R\left(\frac{\lambda}{\alpha}\right) q(d\alpha), \quad \lambda < c$$

- Critical density $\rho_c \in (0, +\infty]$:

$$\rho_c := \overline{R}(c-) = \int_{(c,1]} R\left(\frac{c}{\alpha}\right) q(d\alpha)$$

- Finiteness depends on lower tail of q near c, e.g. MM1:

$$q(\alpha) \overset{c}{\sim} (\alpha - c)^\kappa, \ \kappa > 0 \Rightarrow \rho_c < +\infty$$

- Flux vs density only defined up to ρ_c

Zero-range process with site disorder (M. Evans '95, Ferrari & Krug '96

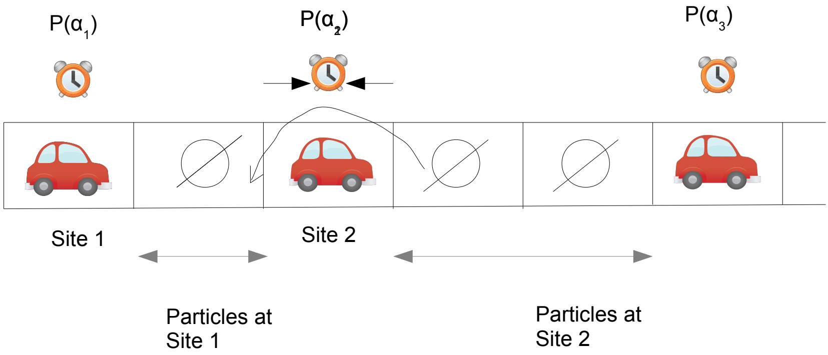
- Configuration $\eta = \{\eta(x),\, x \in \mathbb{Z}\}$
- $\eta(x) \in \mathbb{N}$ nb. particles (clients) at site (server) $x \in \mathbb{Z}$
- $p \in (1/2, 1]$ asymmetry parameter, $q = 1 - p$
- Ergodic environment α with law $Q(d\alpha)$

$$\alpha = \{ \underbrace{\alpha(x)}_{\text{server } x \text{ speed}} ,\, x \in \mathbb{Z}\} \in (c, 1]^{\mathbb{Z}}, \quad c > 0$$

- Jumps

$$\eta \to \eta - \delta_x + \delta_{x+1} \quad \text{rate} \quad p\alpha(x)\, g[\eta(x)]$$
$$\eta \to \eta - \delta_x + \delta_{x-1} \quad \text{rate} \quad q\alpha(x)\, g[\eta(x)]$$

d-ASEP with particle disorder



Invariant measures

- Product measure with marginal $\theta_{\lambda/\alpha(x)}$ at site x

  $\infty > \lambda \leq c\, g(\infty)$ mean outgoing flow

- Mean flux vs. mean density

$$\overline{R}(\lambda) := \int R\left(\frac{\lambda}{\alpha(0)}\right) dQ(\alpha)$$

$$f(\rho) := \underbrace{(p-q)}_{\text{drift}} \overline{R}^{-1}(\rho)$$

- Critical density

$$\rho_c := \lim_{\lambda \uparrow c\, g(\infty)} \overline{R}(\lambda) \in [0, +\infty]$$

The current-density function

- Current through origin starting from η

$$\Gamma_0^\alpha(t,\eta) := \textit{nb.jumps}\, 0 \to 1 - \textit{nb.jumps}\, 1 \to 0$$

- η^ρ particle configuration with density ρ:

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n}\eta(x) = \lim_{n\to+\infty} \frac{1}{n}\sum_{x=-n}^{0}\eta(x) = \rho$$

- Define current-density function $\rho \mapsto f(\rho)$

$$f(\rho) := \lim_{t\to+\infty} t^{-1}\Gamma_0^\alpha(t,\eta^\rho)$$

provided exists and depends only on ρ

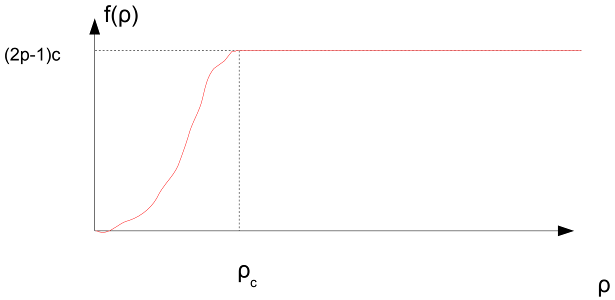
The current-density function

- Thus for $\rho < \rho_c$:

$$f(\rho) = \underbrace{(p-q)}_{\text{drift}} \overline{R}^{-1}(\rho)$$

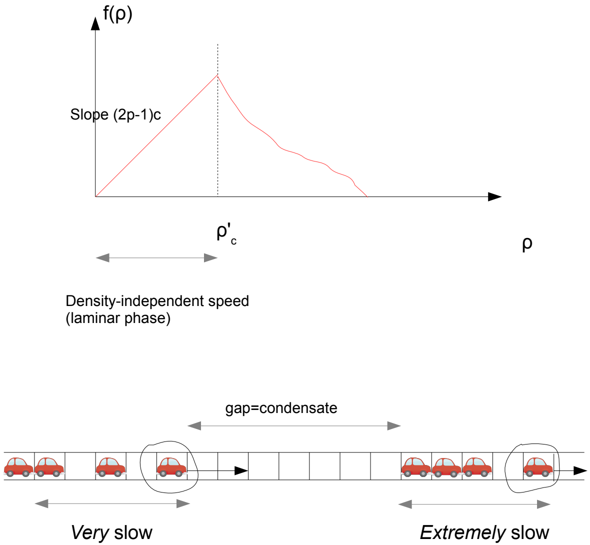
- What about $\rho \geq \rho_c$?
- From now on $g(\infty) = 1$

Flux cutoff



Can show by monotonicity arguments that After ρ_c , current cannot exceed maximum value (2p-1)c. What does it mean ?

Heuristics (d-ASEP)



Questions

1. Hydrodynamic limit including supercritical regime.
2. Quenched strong local equilibrium (subcritical and critical).
3. Loss of local equilibrium (supercritical).
4. Convergence from given initial configuration.

Hydrodynamic limit

- Empirical measure, hyperbolic scaling:

$$\pi^N(\eta_{Nt}, dx) := N^{-1} \sum_{y \in \mathbb{Z}} \eta_{Nt}(y) \delta_{y/N}(dx)$$

- Limiting equation (entropy solution)

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad \rho(0, .) = \rho_0(.) \tag{1}$$

- HDL:

$$\pi^N(\eta_0^N, dx) \to \rho_0(.)dx \Rightarrow \pi^N(\eta_{Nt}^N, dx) \xrightarrow{\mathbf{P}} \rho(t, .)dx$$

- Previous: homogeneous AZRP (Rezakhanlou '91), disordered AZRP with *unbounded g*, no hase transition (Benjamini, Ferrari, Landim '96), TAZRP with MM1 rate including phase transition (Seppäläinen '98)

Quenched strong local equilibrium *creation*

- Initial assumption Profile (no loc. eq.)

$$\pi^N(\eta_0^N, dx) \to \rho_0(.)dx$$

- Subcritical contiuity point $\rho(t,x) < \rho_c$ then $\forall h$ bounded local:

$$\lim_{N\to+\infty}\left\{\mathbb{E}\left[h(\tau_{\lfloor Nx\rfloor}\eta_{Nt})\right] - \int h(\eta)d\nu^{\tau_{\lfloor Nx\rfloor}\alpha,\rho(t,x)}(\eta)\right\} = 0$$

- Landim ('93): homogeneous, convex f, initially

$$\mu^N(d\eta) := \bigotimes_{x\in\mathbb{Z}^d} \nu_{\rho_0(x/N)}[d\eta(x)]$$

(Conservation of local equilibrium)

Dynamic mass escape

- Supercritical

$$\liminf_{(s,y)\to(t,x)} \rho(s,y) \geq \rho_c$$

- "Typical" point: x_N such that $N^{-1}x_N \to x$, $\forall$ subsequence

$$\tau_{x_N}\alpha \stackrel{\text{loc.}}{\longrightarrow} \bar{\alpha}, \quad \text{s.t.} \liminf_{x\to\pm\infty} \bar{\alpha}(x) = c$$

- Conclusion $\forall$ h bounded local:

$$\lim_{N\to+\infty} \left\{ \mathbb{E}\left[h(\tau_{\lfloor Nx\rfloor}\eta_{Nt})\right] - \int h(\eta) d\nu^{\tau_{\lfloor Nx\rfloor}\alpha,\rho_c}(\eta) \right\} = 0$$

Convergence

- Our result

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{x=-n}^{0}\eta_0(x)=\rho\Rightarrow\eta_t\stackrel{t\to+\infty}{\longrightarrow}\nu^{\alpha,\rho\wedge\rho_c}$$

- Liggett's thm ('76), homogeneous AZRP in $d\geq 1$:

$$\eta_0 \text{ spatially ergodic } \Rightarrow\eta_t\stackrel{t\to+\infty}{\longrightarrow}\int_{[0,+\infty)}\nu^\rho d\gamma(\rho)$$

- Andjel (81) $p(.)$ NN and μ spatially ergodic.

- Mountford (2000): $p(.)$ nonzero finite-mean and μ spatially ergodic.

- B. & M. (2006): ASEP, $p(.)$ nonzero finite-range, $\mu=\delta_\eta$,

$$\lim_{n\to+\infty}n^{-1}\sum_{x=0}^{n}\eta(x)=\lim_{n\to+\infty}n^{-1}\sum_{x=-n}^{0}\eta(x)=\rho$$

Escape of mass [BMRS '17]

Weak convexity assumption on g

THEOREM

1. The limit $\eta_t \to \nu_c$ holds iff.

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{x=-n}^{0} \eta(x) \geq \rho_c$$

2. Counterexample for non NN RW where $\liminf > \rho_c$ and $\eta_t \not\to \nu_c$ (but does not exceed ν_c)

Formerly Andjel et al. (2000): for MM1 with $p = 1$, limit holds if

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{x=-n}^{0} \eta(x) > \rho_c$$

Idea for loc eq.