Renormalization Group and Rough SPDE's

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White noise driven PDE's

Space time white noise $\xi(t,x)$

$$\mathbb{E}\xi(t',x')\xi(t,x) = \delta(t'-t)\delta(x'-x)$$

- Interface growth $\phi(t,x)$ interface height (KPZ)

$$\partial_t\phi = \Delta\phi + (\nabla\phi)^2 + \xi$$

- Ginzburg-Landau (GL) model $\phi(t,x)$ magnetization

$$\partial_t\phi = \Delta\phi - \lambda\phi^3 + \xi$$

- Fluctuating hydrodynamics $\phi = (\phi_1, \phi_2, \phi_3)$

$$\partial_t\vec{\phi}_\alpha = \Delta\phi_\alpha + M_\alpha^{\beta\gamma}\partial_x\phi_\beta\partial_x\phi_\gamma + \xi_\alpha$$

ξ is very rough, are these non-linear equations well-posed?

- Given a realization ξ of noise, is there a $\phi(\xi)$ solving these equations?
- How is $\phi(\xi)$ distributed? Is there a stationary state?

In general we need to **renormalize** the equations to make them well posed.

Linear case

Linear equation $x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$\partial_t \phi = \Delta \phi + \xi$$

$$\phi(0, x) = \phi_0(x)$$

solved by

$$\phi(t, x) = (e^{t\Delta}\phi_0)(x) + \eta(t, x)$$

with

$$\eta(t) = \int_0^t e^{(t-s)\Delta}\xi(s)ds$$

Free field

$\eta(t,x)$ is a random field with covariance

$$\mathbb{E}\eta(t,x)\eta(t,y) = C_t(x,y)$$

where $C_t(x,y)$ is the integral kernel of the operator

$$\int_0^t e^{2t\Delta}dt = -\tfrac{1}{2}\frac{1-e^{2t\Delta}}{\Delta}$$

$C_t(x,y)$ is **singular** in short scales:

$$\mathbb{E}\eta(t,x)\eta(t,y) \asymp \frac{1}{|x-y|^{d-2}}.$$

- $\eta(t,x)$ is **a.s.** not a function in $d \geq 2$
- $\nabla\eta(t,x)$ has same regularity as white noise for all d.

Integral equation

Consider nonlinear equation

$$\partial_t \phi = \Delta \phi + V(\phi) + \xi, \quad \phi(0, x) = 0.$$

Rewrite it as integral equation

$$\begin{aligned} \phi(t) &= \int_0^t e^{(t-s)\Delta} (V(\phi(s)) + \xi(s)) ds \\ &= \eta(t) + \int_0^t e^{(t-s)\Delta} V(\phi(s)) ds \end{aligned}$$

where $\eta(t, x)$ is the solution to the linear equation.

Fix a realization of the random field $\eta(t, x)$ and try to solve this fixed point problem in some (Banach) space of functions $\phi(t, x)$.

Perturbation theory

Study the solution iteratively:

$$\phi(t) = \eta(t) + \int_0^t e^{(t-s)\Delta} V(\eta(s)) ds + \dots.$$

This **fails**:

- For KPZ equation

  $$V(\eta(s)) = (\partial_x \eta(s,x))^2$$

  and $\partial_x \eta(s,x) =$ derivative of BM $= \infty$ almost surely.
- For GL equation

  $$V(\eta(s)) = \eta(s,x)^3 = \infty$$

  almost surely if $d \geq 2$.

Quantum Field Theory

Such divergencies are familiar from **quantum field theory**.

Formally the equation

$$\partial_t \phi = \Delta \phi - \lambda \phi^3 + \xi$$

has a **stationary law**

$$\mathbb{P}_\lambda(d\phi) \propto e^{-\frac{\lambda}{4}\int_{\mathbb{T}^d} \phi(x)^4 dx} \mathbb{P}_0(d\phi)$$

where $\mathbb{P}_0$ is the law of GFF.

Under GFF $\phi(x)^4 = \infty$ almost surely so we have a problem.

For $d < 4$ $\mathbb{P}_\lambda$ can be constructed by **renormalization**.

Renormalization

Regularize e.g. by lattice $\phi(x)$, $x \in (\epsilon\mathbb{Z})^d$, $\Delta \to \Delta_\epsilon$, $\mathbb{P}_0 \to \mathbb{P}_0^{(\epsilon)}$

Renormalize by adding a **counter term**

$$S^{(\epsilon)}(\phi) := \int_{\mathbb{T}^d} (\tfrac{\lambda}{4}\phi^4 + \tfrac{1}{2} r_\epsilon \phi^2)$$

Then

$$\mathbb{P}_\lambda \propto \lim_{\epsilon \to 0} e^{-S^{(\epsilon)}} \mathbb{P}_0^{(\epsilon)}$$

exists with

$$\begin{aligned} r_\epsilon &= m \log \epsilon & d = 2 \\ r_\epsilon &= m_1 \epsilon^{-1} + m_2 \log \epsilon & d = 3 \end{aligned}$$

Story of 60-70's.

Regularized dynamics

Consider a **regularized** equation

$$\partial_t \phi = \Delta_\epsilon \phi + V^{(\epsilon)}(\phi) + \xi_\epsilon$$

where

- $\xi_\epsilon(t, x)$ is white noise on $\mathbb{R} \times (\epsilon\mathbb{Z})^d$
- $V^{(\epsilon)}$ has ϵ-dependent terms added to V

Determine $V^{(\epsilon)}$ so that solutions converge as $\epsilon \to 0$ a.s.

Renormalized dynamics

Renormalize:

$$\begin{aligned} (\partial_x \phi)^2 &\to (\partial_x \phi)^2 + a\epsilon^{-1} \\ M_\alpha^{\beta\gamma} \partial_x \phi_\beta \partial_x \phi_\gamma &\to M_\alpha^{\beta\gamma} \partial_x \phi_\beta \partial_x \phi_\gamma + a_\alpha \epsilon^{-1} + b_\alpha \log \epsilon \\ \phi^3 &\to \phi^3 + \phi \begin{cases} m \log \epsilon & d = 2 \\ m_1 \epsilon^{-1} + m_2 \log \epsilon & d = 3 \end{cases} \end{aligned}$$

Theorem. The following holds **almost surely in ξ**:
There exists $T > 0$ s.t. the regularized equation has a unique solution $\phi_\epsilon(t, x)$ for $t \leq T$ and

$$\phi_\epsilon \to \phi \in \mathcal{D}'([0, T] \times \mathbb{T}^d)$$

where ϕ is independent of the cutoff function ρ.

Gubinelli, Imkeller, and Perkowski, Catellier and Chouk, Hairer, A.K.

Wilson RG

We prove this result using the "Wilsonian" approach to renormalization

- Proceed scale by scale to derive **effective equation** on that scale
- No new theory of distributions needed
- Standard contraction mapping theorem
- A general method to derive counterterms for subcritical nonlinearities
- A general method to study **universality**

Counter terms

Given a nonlinearity $S(\phi)$ or $V(\phi)$ how to find the counterterms?

Why is this natural ?

Both questions can be answered by considering scale dependent **effective actions** and **effective equations**.

Effective Actions

Let

$$\phi \overset{law}{=} \mathbb{P}^{(\epsilon)} = e^{-S^{(\epsilon)}(\phi)} \mathbb{P}_0^{(\epsilon)}$$

Pick a larger scale $\epsilon' > \epsilon$

Let $\phi' :=$ average of ϕ on ϵ' cubes. Call its law by $\mathbb{P}_{\epsilon'}^{(\epsilon)}$ and define $S_{\epsilon'}^{(\epsilon)}(\phi')$ by

$$\mathbb{P}_{\epsilon'}^{(\epsilon)} = e^{-S_{\epsilon'}^{(\epsilon)}(\phi')} \mathbb{P}_0^{(\epsilon')}$$

$S_{\epsilon'}^{(\epsilon)}(\phi')$ is called the **Effective Action** for scales $\geq \epsilon'$

Reformulation of $\epsilon \to 0$ limit:

- Prove: $\lim_{\epsilon \to 0} S_{\epsilon'}^{(\epsilon)}(\phi')$ exists for all scales $\epsilon' > 0$.

Good control of this limit amounts to control of $\lim_{\epsilon \to 0} \mathbb{P}^{(\epsilon)}$

Effective Equations

Regularized equation

$$\partial_t \phi = \Delta_\epsilon \phi + V^{(\epsilon)}(\phi) + \xi_\epsilon \qquad (1)$$

Let $\phi' :=$ average of ϕ on ϵ' cubes.

Derive an **Effective Equation**

$$\partial_t \phi' = \Delta_{\epsilon'} \phi' + V^{(\epsilon)}_{\epsilon'}(\phi') + \xi_{\epsilon'}$$

- Prove: **a.s.** $\lim_{\epsilon \to 0} V^{(\epsilon)}_{\epsilon'}(\phi')$ exists for all scales $\epsilon' > 0$.

Good control of this limit amounts to control of solutions of (1)

Dimensionless variables

Scale invariance of GFF: let

$$\varphi(x) = \epsilon^{\frac{d-2}{2}}\phi(\epsilon x)$$

then

$$\phi \overset{law}{=} \mathbb{P}_0^{(\epsilon)} \implies \varphi \overset{law}{=} \mathbb{P}_0^{(1)}$$

Similarly if

$$\dot{\phi} = \Delta_\epsilon \phi + \xi_\epsilon$$

then

$$\varphi(t,x) := \epsilon^{\frac{d-2}{2}}\phi(\epsilon^2 t, \epsilon x)$$

satisfies

$$\dot{\varphi} = \Delta_1 \varphi + \xi_1$$

Note: φ defined on $\epsilon^{-1}\mathbb{T}^d \cap \mathbb{Z}^d$: UV cutoff 1, IR cutoff ϵ^{-1}

Subcritical nonlinearity

$$\phi \overset{law}{=} e^{-S^{(\epsilon)}}\mathbb{P}_0^{(\epsilon)} \implies \varphi \overset{law}{=} e^{-\mathcal{S}^{(\epsilon)}}\mathbb{P}_0^{(1)}$$

with

$$\mathcal{S}^{(\epsilon)}(\varphi) = \sum_{x\in\mathbb{Z}^d\cap\epsilon^{-1}\mathbb{T}^d} (\tfrac{\lambda}{4}\epsilon^{4-d}\varphi^4 + \tfrac{1}{2}\epsilon^2 r^{(\epsilon)}\varphi)$$

and for GL equation

$$\dot{\varphi} = \Delta_1\varphi - \lambda\epsilon^{4-d}\varphi^3 - \epsilon^2 r^{(\epsilon)}\varphi + \xi_1$$

and for KPZ

$$\dot{\varphi} = \Delta_1\varphi + \epsilon^{\frac{2-d}{2}}(\nabla\varphi)^2 + \xi_1$$

In dimensionless variables nonlinearity is **small** if $d < 2$ (KPZ), $d < 4$ (GL)

Renormalization Group

In the same way define **effective action** $\mathcal{S}^{(\epsilon)}_{\epsilon'}$ and **effective equation** $\mathcal{V}^{(\epsilon)}_{\epsilon'}$ by going to dimensionless variables:

$$\mathcal{S}^{(\epsilon)}_{\epsilon'}(\varphi(\cdot)) = S^{(\epsilon)}_{\epsilon'}(\epsilon'^{\frac{d-2}{2}}\phi(\epsilon'\cdot))$$

We need to study the **renormalization group flow**

$$\epsilon' \to \mathcal{S}^{(\epsilon)}_{\epsilon'}, \mathcal{V}^{(\epsilon)}_{\epsilon'}$$

describing in dimensionless variables how the physics changes with scale.

We do this incrementally by fixing $L > 1$ and defining **RG** map $\mathcal{R}$:

$$\mathcal{R}\mathcal{S}^{(\epsilon)}_{\epsilon'} = \mathcal{S}^{(\epsilon)}_{L\epsilon'}, \;\; \mathcal{R}\mathcal{V}^{(\epsilon)}_{\epsilon'} = \mathcal{V}^{(\epsilon)}_{L\epsilon'}$$

so that if $\epsilon = L^{-N}$ and $\epsilon' = L^{-n}$

$$\lim_{\epsilon\to 0}\mathcal{S}^{(\epsilon)}_{\epsilon'} = \lim_{N\to\infty}\mathcal{R}^{N-n}\mathcal{S}^{(\epsilon)}$$

RG for QFT

$\mathcal{R}$ maps

$$e^{-\mathcal{S}(\varphi)}\mathbb{P}_0^{(1)} \to e^{-\mathcal{R}\mathcal{S}(\varphi')}\mathbb{P}_0^{(1)}$$

Decompose GFF ($[\cdot]$ denotes integer part)

$$\varphi(x) = L^{-\frac{d-2}{2}}\varphi'([\tfrac{x}{L}]) + z(x) := \varphi'_L(x) + z(x)$$

where $\varphi, \varphi' \stackrel{law}{=} \mathbb{P}_0^{(1)}$ and z fluctuates on scale L. Then

$$e^{-\mathcal{R}\mathcal{S}(\varphi')} = \mathbb{E}_z e^{-\mathcal{S}(\phi'_L+z)}$$

- The $z(x)$ variables have correlation length L
- $\mathcal{S}$ is small
- $\implies \mathcal{R}$ can be studied perturbatively

RG for SPDE

$\mathcal{R}$ maps equation

$$\dot{\varphi} = \Delta_1 \varphi + \mathcal{V}(\varphi) + \xi_1$$

to new equation

$$\dot{\varphi}' = \Delta_1 \varphi' + \mathcal{R}\mathcal{V}(\varphi') + \xi_1$$

Decompose

$$\varphi(t,x) = L^{-\frac{d-2}{2}} \varphi'(\tfrac{t}{L^2}, [\tfrac{x}{L}]) + z(t,x) := \varphi'_L(t,x) + z(t,x)$$

Solve

$$\dot{z} = \Delta_1 z + \mathcal{V}(\varphi'_L + z) + \xi_1$$

for $z = z(\varphi')$ to get

$$\mathcal{R}\mathcal{V}(\varphi') = \mathcal{V}(\phi'_L + z(\varphi'))$$

- z has scales $\in [1, L] \implies \quad \Delta_1 > L^{-2}$
- $\mathcal{V}$ is small $\implies$ $\mathcal{R}$ can be studied perturbatively

Hierarchical QFT

Hierarchical GFF

$$\varphi(x) = L^{-\frac{d-2}{2}}\varphi'([\tfrac{x}{L}]) + z(x) := \varphi'_L(x) + z(x)$$

$z(x), z(y)$ **independent** if $[\frac{x}{L}] \neq [\frac{y}{L}]$

$\mathcal{R}$ preserves **local** actions:

$$\mathcal{S}(\varphi) = \sum_x s(\varphi(x)) \implies \mathcal{R}\mathcal{S}(\varphi) = \sum_x \mathcal{R}s(\varphi(x))$$

with $s : \mathbb{R} \to \mathbb{R}$ and

$$e^{-\mathcal{R}s(\varphi)} = \mathbb{E}e^{-\sum_{y=1}^{L^d} s(L^{-\frac{d-2}{2}}\varphi' + z(y))}$$

where $\mathbb{E}$ is expectation over L^d Gaussian random variables $z(y)$

Hierarchical SPDE

Let $-\Delta_H^{-1}$:= covariance of hierarchical GFF and consider

$$\dot{\varphi} = \Delta_H \varphi + \mathcal{V}(\varphi) + \xi_1$$

Then $\mathcal{R}$ preserves **local** equations:

$$\mathcal{V}(\varphi) = \sum_x v(t, x, \varphi(\cdot, x)) \implies \mathcal{R}\mathcal{V}(\varphi) = \sum_x \mathcal{R}v(t, x, \varphi(\cdot, x))$$

with $v(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ and with $\varphi = s_L \varphi' + z$

$$\mathcal{R}v(t, x, \varphi') = L^{-\frac{d-2}{2}} \sum_{[\frac{y}{L}]=x} v(L^2 t, y, \varphi)$$

$$\dot{z}(t, x) = -L^{-2} z(t, x) + v(L^2 t, Lx, \varphi) + \xi_1$$

$z = z(t, x, \varphi')$ is solution of L^d weakly nonlinear SDE's

Linear RG flow-QFT

For small s

$$\mathcal{R}s(\varphi) = \sum_{y=1}^{L^d} \mathbb{E}s(L^{-\frac{d-2}{2}}\varphi + z(y)) + \mathcal{O}(s^2)$$

so that

$$\mathcal{R}(\lambda\varphi^4 + r\varphi^2) = L^{4-d}\lambda\varphi^4 + (L^2 r + a_1\lambda)\varphi^2 + a_2 + \mathcal{O}(\lambda^2, r^2)$$

λ and r are **relevant**: they increase under the flow.
However, λ starts very small: **let** $\epsilon = L^{-N}$ **and** $\epsilon' = L^{-n}$ and denote $s_{\epsilon'}^{(\epsilon)}$ by $s_n^{(N)}$. Then the initial condition is:

$$s^{(N)} = \lambda_N\varphi^4 + r_N\varphi^2 \quad \lambda_N = L^{-(4-d)N}\lambda, \quad r_N = L^{-2N}\mu^{(N)}$$

We try to fix $\mu^{(N)}$ so that $s_n^{(N)}$ stays small for all scales n.

Linear RG flow-QFT

We get $s_n^{(N)}(\varphi) = \lambda_n \varphi^4 + r_n \varphi^2$ with

$$\lambda_n = L^{-(4-d)n} \lambda$$
$$r_{n-1} = L^2 r_n + a_1 \lambda_n.$$

λ_n stays small but r_n **blows up**

$$r_n \sim L^{N-n}, \quad d = 3, \quad \log(N-n), \quad d = 2$$

unless fine tune initial condition: if we take

$$r_N = \lambda_N (1 - L^{d-2})^{-1} L^{d-4} a_1$$

then $r_n = \mathcal{O}(\lambda_n)$. The dimensional counterterm blows up:

$$\mu^{(\epsilon)} \propto \epsilon^{-1} \lambda, \; d = 3, \quad \mu^{(\epsilon)} \propto \lambda \log \epsilon, \; d = 3$$

However, dimensionless $s_n^{(N)} = \mathcal{O}(\lambda_n)$ is small for all n

Linear RG flow SPDE

$$\mathcal{R}v(t,x,\varphi) = L^{-\frac{d-2}{2}} \sum_{[\frac{y}{L}]=x} v(L^2 t, y, (L^{-\frac{d-2}{2}}\varphi + z(\varphi))$$

To lowest order z is solution of a linear equation

$$\dot{z}(t,x) = -L^{-2} z(t,x) + \xi(t,x)$$

for x in a L^d cube in $\mathbb{Z}^d$ so z is an O-U process. Upshot:

$$v_n(t,x;\varphi) = \lambda_n(\varphi(t,x)^3 + A_n(t,x)\varphi(t,x) + B_n(t,x)) + L^{2(N-n)} r_N \varphi(t,x).$$

$A_n(t,x)$ and $B_n(t,x))$ are **random processes**.

They are **i.i.d.** for different x, $A_n(t,x) \overset{law}{=} A_n(t)$ etc

$\mathbb{E}A_n(t)$ blows up as in QFT, cancelled by r_N and $\mathbb{E}B_n(t) = 0$

Probabilistic bounds

The variance of $\tilde{A}_n = A_n - \mathbb{E}A_n$

$$\mathbb{E}\tilde{A}_n(t)\tilde{A}_n(s) = G_n(t-s)$$

iterates as

$$G_{n-1}(t) = L^{d-4}G_n(L^2 t) + ge^{-ct}$$

Upshot: $G_n(t)$ is bounded as $N \to \infty$ and Hölder continuous

Since A_n belongs to bounded Wiener chaos of white noise

$$\mathbb{P}(\sup_{t\in[0,T]} |\tilde{A}_n(t)| > R) \leq CR^{-p}, \quad \forall p$$

Since $\lambda_n = L^{-(4-d)n}\lambda$ this leads to

Proposition Almost surely for some m the event $\mathcal{E}_m$ holds:

$$\sup_{n\geq m} \sup_{t\in[0,L^{2n}]} \sup_{x\in L^n\mathbb{T}^d} |\tilde{A}_n(t,x)| < \log \lambda_n^{-1} \qquad (\mathcal{E}_m)$$

Thus almost surely v_n has small coefficients for all $n \geq m$

Full RG for SPDE: d=2

Let $u_n(t,x,\varphi)$ be the linear RG flow. Let

$$\|f\|_n := \sup_{t\in[0,L^{2n}]} \sup_{x\in L^n\mathbb{T}^d} |f(t,x)|$$

Under $\mathcal{E}_m$, for $n \geq m$

$$\|u_n\| := \sup_{\|\varphi\|_n \leq \log \lambda_n^{-1}} \|u_n(\varphi)\|_n \leq \lambda_n \log \lambda_n^{-3}$$

Suppose $v_n = u_n + w_n$ with $\|w_n\| \leq \lambda_n^{2-\delta}$,$\delta$ small. Then z equation may be solved and we get

$$w_{n-1}(t,x,\varphi) = L^{-\frac{d-2}{2}} \sum_{[\frac{y}{L}]=x} w(L^2 t, y, (L^{-\frac{d-2}{2}}\varphi + z) + \mathcal{O}(u_n^2)$$

so that for $d = 2$ since $\lambda_n = L^{-2n}\lambda$ our bound iterates:

$$\|w_{n-1}\| \leq L^2\|w_n\| + (\lambda_n \log \lambda_n^{-3})^2 \leq \lambda_{n-1}^{2-\delta}$$

Full RG for SPDE: d=3

for $d = 3$ we get

$$\|w_{n-1}\| \leq L^{\frac{5}{2}}\|w_n\| + (\lambda_n \log \lambda_n^{-3})^2$$

Since $\lambda_n = L^{-n}\lambda$ and $\frac{5}{2} > 2$ the induction fails.

We need to solve v_n to second order in λ_n:

$$v_n = u_n + \nu_n + w_n$$

Suppose the second order term $\|\nu_n\| \leq \lambda^{2-\delta}$. Then

$$\|w_{n-1}\| \leq L^{\frac{5}{2}}\|w_n\| + \lambda_n^{3-2\delta}$$

so that $\|w_n\| \leq \lambda_n^{3-3\delta}$ iterates since $\frac{5}{2} < 3$.

Counterterms d=3

The second order term ν_n is an explicit polynomial in φ with random coefficients similar to A_n and B_n.

The coefficient of φ has expectation which blows up as $N \to \infty$: this leads to a new counterterm proportional to $\log \epsilon$

The coefficients are in bounded Wiener chaos and lead to new conditions to the event $\mathcal{E}_m$

Proof for regular (nonhierarchical) PDE almost identical: now $A_n(t, x)$ etc. are (exponentially) weakly correlated for $x \neq y$

Subcritical equations

Subcritical equation: dimensionless nonlinearity $\propto \epsilon^{\alpha}$

RG approach is the same:

One needs to compute v_n to finite order depending on the scaling dimension of the nonlinearity

Diverging expectations of random coefficients are cancelled by counterterms

Random coefficients are in bounded chaos and can be bounded

Nothing blows up: equation is weakly nonlinear throughout the iteration

Global Solutions

Above: existence time depends on the size of the noise

In QFT and SPDE the perturbative argument works for $\|\varphi\|$ not too big (above $\leq \log \lambda_n^{-1}$)

In QFT need to prove iteratively $s_n(\varphi) \geq \lambda_n \varphi^4$ for larger $\|\varphi\|$

In SPDE need to prove iteratively $\varphi v_n(\varphi) \geq \lambda_n \varphi^4 - R$ where R depends on noise.