Large deviations of surface height in the Kardar-Parisi-Zhang equation

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Outline

- ✓ The KPZ equation
- ✓ One-point height distribution
- ✓ Optimal Fluctuation Method
  - Low cumulants
  - Positive tail
  - Negative tail
  - Dynamical phase transition
  - Short times and long times
- ✓ Summary

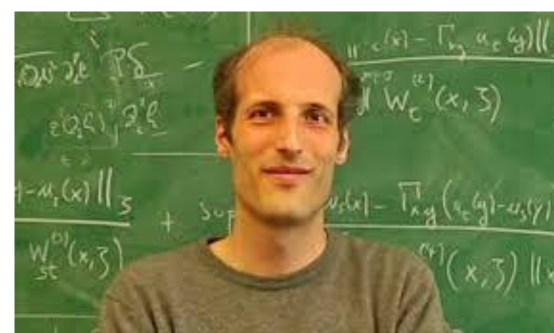
The Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \sqrt{D}\xi(x,t)$$

ξ - Gaussian noise with zero mean and covariance

$$\langle \xi(x,t)\xi(x',t') \rangle = \delta(x-x')\delta(t-t')$$

Starting from the paper of Kardar, Parisi and Zhang, PRL **56**, 889 (1986) the KPZ equation inspired many physicists and, more recently, mathematicians, culminating in a 2014 Fields Medal awarded to Martin Hairer



KPZ: a basic continuum model of surface growth

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \sqrt{D}\xi(x,t)$$

$h(x,t)$	height of a growing surface
$\nu \partial_x^2 h(x,t)$	relaxation by surface tension
$\frac{\lambda}{2}(\partial_x h(x,t))^2$	lowest-order symmetry-breaking nonlinearity
$\sqrt{D}\xi(x,t)$	deposition noise: Gaussian and delta-correlated in x and t

The stationary measure of the KPZ equation in 1d is known (and independent of λ):

$$Z^{-1} \exp\left[-\frac{\nu}{D} \int dx (\partial_x h(x))^2 \right]$$

The KPZ equation describes paper combustion, growth of bacterial colonies, rough front between stable and metastable phases of a turbulent liquid crystal, etc. It is a "standard model" of non-equilibrium statistical mechanics

1. The Hopf-Cole transformation $W(x,t) = \exp[(\lambda / 2\nu) h(x,t)]$

brings the KPZ equation to

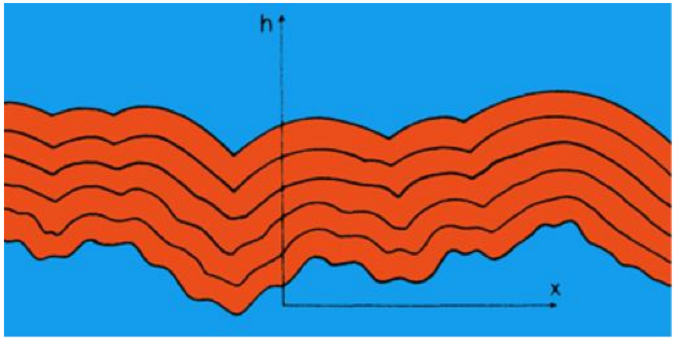
$$\partial_t W(x,t) = \nu \partial_x^2 W(x,t) + \frac{\lambda \sqrt{D}}{2\nu} \xi(x,t) W(x,t),$$

diffusion in a time-dependent random potential

2. The transformation $V(x,t) = \partial_x h(x,t)$ yields

$$\partial_t V(x,t) = \nu \partial_x^2 V(x,t) + \lambda V(x,t) \partial_x V(x,t) + \sqrt{D} \partial_x \xi(x,t) \,,$$

This is the Burgers equation with conserved noise



KPZ interface without noise (picture: APS)

The interface *roughness* is measured by the characteristic interface width

$$W(L,t)=\left\langle \frac{1}{L}\int_0^L \left(h(x,t)-\bar{h}(t)\right)^2 dx \right\rangle^{1/2}$$

Family-Viscek scaling relation $W(L,t)\approx L^{\alpha} f(t/L^{z})$ $\quad f(u)\propto \begin{cases} u^{\beta} & u \ll 1 \\ 1 & u \gg 1 \end{cases}$

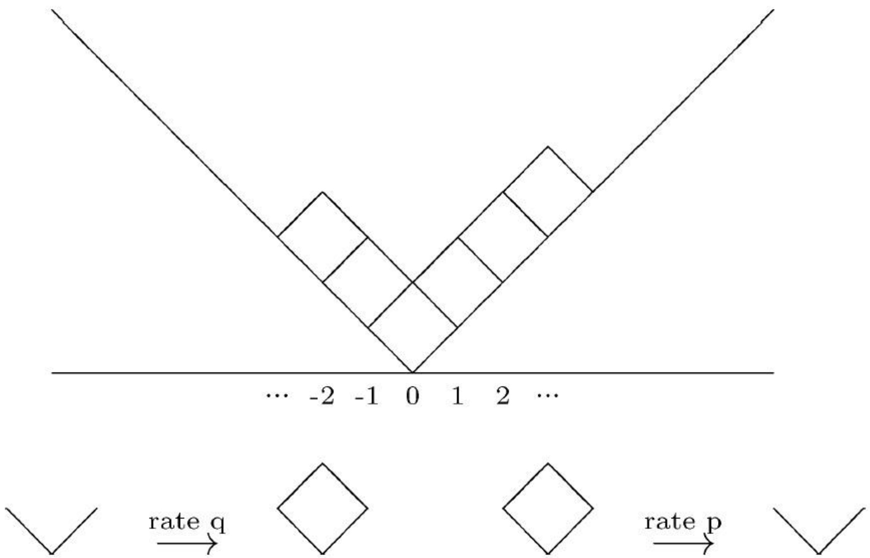
α roughness exponent β growth exponent $z=\dfrac{\alpha}{\beta}$ dynamic exponent

KPZ in 1 dimension: $\alpha=1/2$, $\beta=1/3$, $z=3/2$

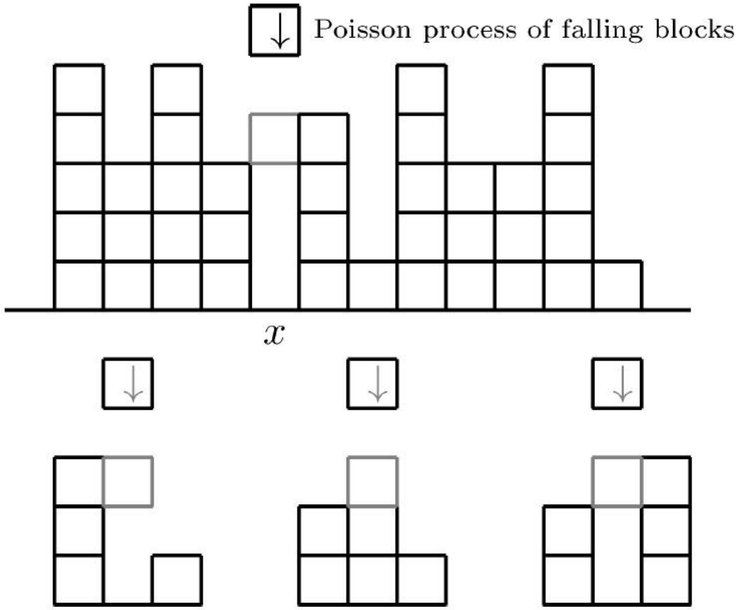
The same scaling behavior is displayed, at large scales and long times, by several "microscopic" models, including the Asymmetric Simple Exclusion Process (ASEP), the Corner Growth, and the Ballistic Deposition. These and many other models are believed to belong to the KPZ universality class, although rigorous proofs are rare

Corner Growth

Ballistic Deposition



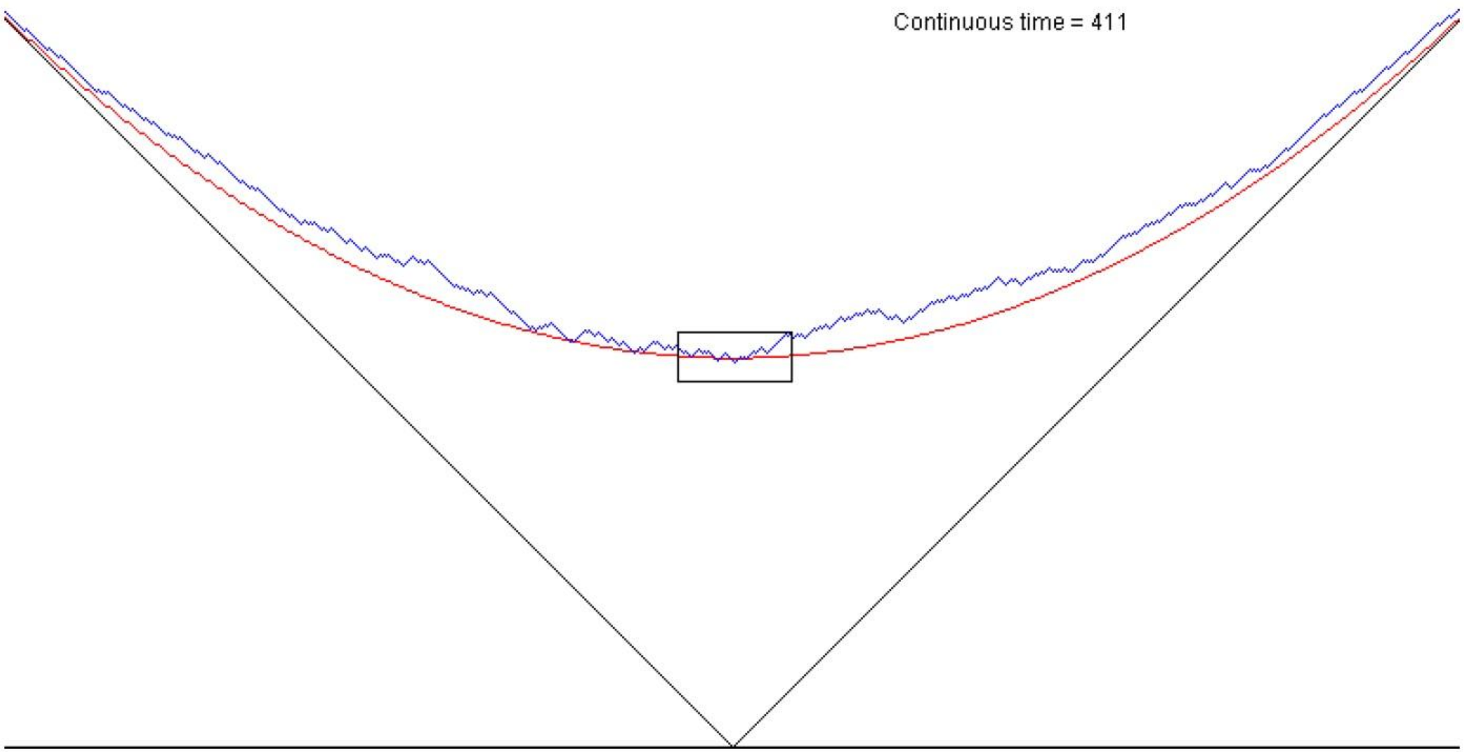
(a)



(b)

Pictures: I. Corwin

Corner Growth at long times



Picture: I. Corwin
Simulation: P. Ferrari

One-point height distribution

$W(L,t)$ is an *integral* characteristic. More recently the focus of interest is the probability density $P(H,t)$ of the surface height H at a point x in space at time t.

H. Spohn: *"KPZ is an area where full probability density functions are of considerable advantage. They seem to characterize more sharply the KPZ universality class than scaling exponents."*
ArXiv:1601.00499

Exact representations for $P(H,t)$ were recently obtained for initial conditions of three types:

- Sharp wedge $h(x,t=0)=|x|/\delta$, $\delta \ll 1$ (for $\lambda<0$)

Sasamoto and Spohn (2010), Dotsenko (2010), Calabrese, Le Doussal and Rosso (2010), Amir, Corwin and Quastel (2011)

- Flat interface $h(x,t=0)=0$

Calabrese and Le Doussal (2011,2012)

- Stationary interface: $h(x,t=0)$ is a two-sided Brownian motion pinned at $x=0$

Imamura and Sasamoto (2012,2013), Borodin, Corwin, Ferrari and Vető (2015)

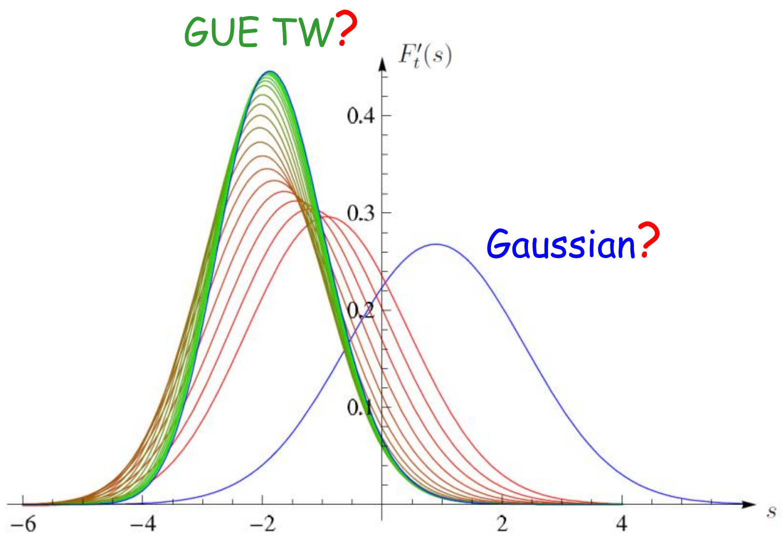
P(H,t), and even its long-time asymptotics, depend on the initial condition

Typical fluctuations at long times:

$$h(0,t) \cong ct + (\Gamma t)^{1/3}\eta$$

c and Γ depend on the initial condition

η is a non-Gaussian random quantity. Its distribution depends on the initial condition:
•Flat: Tracy-Widom dist. for Gaussian Orthogonal Ensemble (GOE)
•Sharp wedge: Tracy-Widom dist. for Gaussian Unitary Ensemble (GUE)
•Stationary: Baik-Rains dist.



Example:
Emergence of the GUE TW distribution
at long times
Prolhac and Spohn (2011)

Exact representations for $P(H,t)$ involve Fredholm determinants. Extracting asymptotics requires considerable effort. Partial results available for the distribution tails (for sharp wedge) Le Doussal, Majumdar and Schehr (2016), Le Doussal, Majumdar, Rosso and Schehr (2016)

Alternative: probe asymptotic regimes more directly. Here comes the Optimal Fluctuation Method. Additional bonus: optimal interface history $h(x,t)$

Optimal Fluctuation Method (OFM)

aka Instanton Method, Macroscopic Fluctuation Theory, Weak-Noise Theory,WKB …

Based on a leading-order saddle-point evaluation of the path integral of the KPZ equation. Leads to a variational formulation for the optimal (most likely) history of the interface height, conditioned on a given large deviation.

Previously appeared in condensed matter physics, turbulence, diffusive lattice gases, discrete-state Markov processes, …

OFM of KPZ

We first identify a formal small parameter ε, by performing rescaling

$$t/T \to t,\ x/\sqrt{\nu T} \to x,\ |\lambda| h/\nu \to h$$

$$\partial_t h(x,t) = \partial_x^2 h(x,t) - \frac{1}{2}(\partial_x h(x,t))^2 + \sqrt{\varepsilon}\xi(x,t) \qquad \varepsilon = \frac{D\lambda^2\sqrt{T}}{\nu^{5/2}} \qquad \lambda<0$$

$$\mathrm{P}[\xi(x,t)] \sim \exp\left[-\frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx\, \xi^2(x,t)\right], \qquad \xi(x,t) = \frac{\partial_t h(x,t) - \partial_x^2 h(x,t) + \frac{1}{2}(\partial_x h(x,t))^2}{\sqrt{\varepsilon}}$$

$$P[h(x,t)] \propto \exp\left(-\frac{S}{\varepsilon}\right), \qquad S = \frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx \left[\partial_t h(x,t) - \partial_x^2 h(x,t) + \frac{1}{2}(\partial_x h(x,t))^2\right]^2$$

when $S/\varepsilon >> 1$ we minimize S over all possible (constrained) histories h(x,t)

For the KPZ in 1 dimension $\varepsilon \to 0$ corresponds to short times

The minimization leads to Euler-Lagrange eqn. that can be cast into Hamiltonian form

$$\partial_t h = \frac{\delta \mathrm{H}}{\delta \rho} = \partial_x{}^2 h - \frac{1}{2}(\partial_x h)^2 + \rho,$$

$$\partial_t \rho = -\frac{\delta \mathrm{H}}{\delta h} = -\partial_x^2 \rho - \partial_x(\rho \partial_x h).$$

Fogedby (1998,1999)
Kolokolov and Korshunov (2007)

$$\mathrm{H}[h(x,t),\rho(x,t)] = \int_{-\infty}^{\infty} \rho[\partial_x{}^2 h - (1/2)(\partial_x h)^2 + \rho/2] dx$$

For a parabolic interface at *t=0*: $h(x,t=0) = x^2/L.$

L=0: sharp wedge
L=∞: flat interface

A nontrivial condition at *t=1:* $\rho(x,t=1) = \Lambda\delta(x).$

Kolokolov and Korshunov (2007)
Meerson, Katzav and Vilenkin (2016)
Kamenev, Meerson and Sasorov (2016)

Λ is ultimately set by the condition $h(0,t=1) = H.$

Once the optimal path is found:

$$-\ln P(H,T,L) \cong \frac{1}{\varepsilon} S\left(\frac{|\lambda| H}{\nu}, \frac{L}{|\lambda| T}\right) = \frac{\nu^{5/2}}{D\lambda^2\sqrt{T}} S\left(\frac{|\lambda| H}{\nu}, \frac{L}{|\lambda| T}\right)$$

$$S = \frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx\, \rho^2(x,t)$$

$$\partial_t h = \frac{\delta \mathrm{H}}{\delta \rho} = \partial_x{}^2 h - \frac{1}{2}(\partial_x h)^2 + \rho,$$

$$\partial_t \rho = -\frac{\delta \mathrm{H}}{\delta h} = -\partial_x^2 \rho - \partial_x(\rho \partial_x h).$$

$$h(x,t=0) = x^2 / L, \qquad \rho(x,t=1) = \Lambda \delta(x),$$

$$h(x,t=1) = H.$$

Fogedby (1998), Kolokolov and Korshunov (2007)
Meerson, Katzav and Vilenkin (2016)

Downhill trajectories: $\rho=0$, deterministic theory:

$$\partial_t h(x,t) = \partial_x{}^2 h - \frac{1}{2}(\partial_x h)^2.$$

Fluctuations are described by *uphill trajectories* $\rho(x,t) \neq 0$.

The "coordinate" h(x,t) describes the optimal history of the interface profile.
The "momentum density" ρ(x,t) describes the optimal history of noise.
Both are deterministic functions!

For the stationary (Brownian) interface
we also account for the "fluctuational cost" of the initial condition:

$$S_{\text{total}} = S_0 + S, \quad S_0 = \int_{-\infty}^{\infty} dx [\partial_x h(x, t=0)]^2$$

$$S = \frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx \left[\partial_t h(x,t) - \partial_x^2 h(x,t) + \frac{1}{2}(\partial_x h(x,t))^2 \right]^2$$

As a result, the initial condition involves a priori unknown h and ρ

$$\rho(x, t=0) + 2\partial_x^2 h(x, t=0) = 0,$$
$$h(x=0, t=0) = 0.$$

Janas, Kamenev and Meerson (2016)
Derrida and Gerschenfeld (2009)

$$-\ln P(H,T) \cong \frac{1}{\varepsilon} S_{\text{total}}\left(\frac{|\lambda| H}{\nu}\right) = \frac{\nu^{5/2}}{D\lambda^2 \sqrt{T}} S_{\text{total}}\left(\frac{|\lambda| H}{\nu}\right)$$

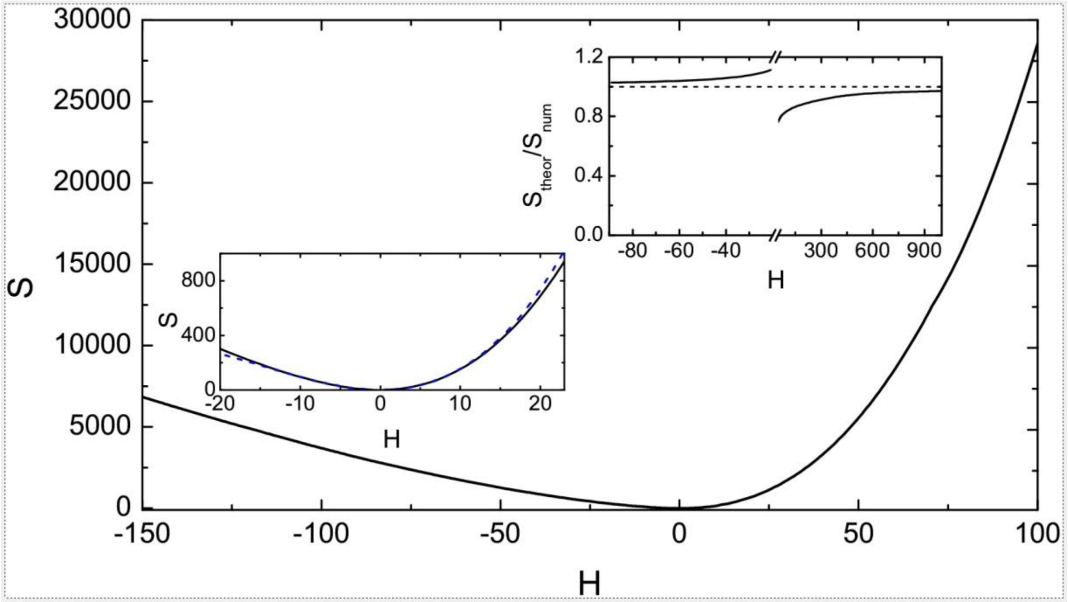
The "investment" in the initial condition pays off:
the resulting *P(H,T)* is exponentially higher

General H: no exact solutions as of today.

Numerics: back-and-forth iteration algorithm, Chernykh and Stepanov (2001)

$$\partial_t h = \partial_x{}^2 h - \frac{1}{2}(\partial_x h)^2 + \rho,$$

$$\partial_t \rho = -\partial_x^2 \rho - \partial_x(\rho \partial_x h).$$



flat interface

Meerson, Katzav and Vilenkin (2016)

$$-\ln P \cong \frac{1}{\varepsilon} S\left(\frac{|\lambda| H}{\nu}\right)$$

$$S = \frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx\, \rho^2(x,t)$$

Low cumulants: perturbation theory in H, or Λ

$$h(x,t) = h_0(x,t) + \Lambda h_1(x,t) + \Lambda^2 h_2(x,t) + \dots,$$
$$\rho(x,t) = \Lambda \rho_1(x,t) + \Lambda^2 \rho_2(x,t) + \dots,$$
$$S(\Lambda) = \Lambda^2 S_1 + \Lambda^3 S_2 + \dots$$

$$h_0(x,t) = \frac{x^2}{L+2t} + \ln\left(1 + \frac{2t}{L}\right)$$

For flat and stationary interfaces
$h_0(x,t)=0$

Flat interface $$S = \sqrt{\frac{\pi}{2}} H^2 + \sqrt{\frac{\pi}{72}} (\pi - 3) H^3 + \dots$$ Meerson, Katzav and Vilenkin (2016)

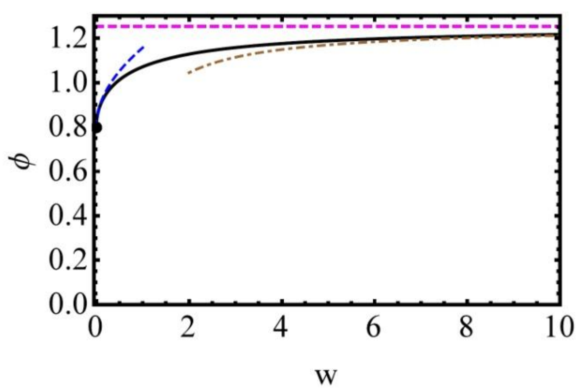
Agrees with second and third cumulant of Gueudré, Le Doussal, Rosso, Henry and Calabrese (2012) who derived it from exact representation of P(H,T)

Stationary interface $$S_{\text{total}} = \frac{\sqrt{\pi}}{2} H^2 + \dots$$ Janas, Kamenev and Meerson (2016)
Krug, Meakin and Halpin-Healy (1992)

$$\sqrt{\pi}/2 < \sqrt{\pi/2}$$

Parabolic interface

$$S = H^2 \phi\left(\frac{L}{|\lambda|\sqrt{T}}\right) + \dots$$

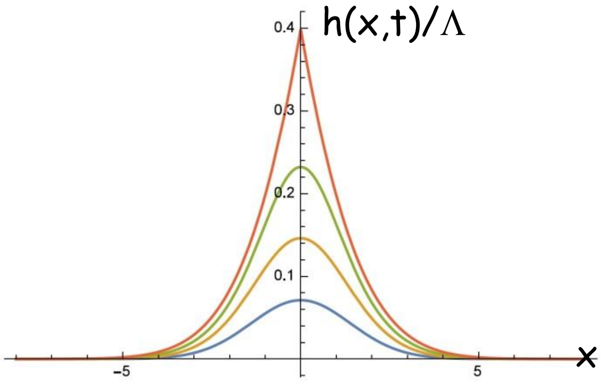


$$\phi(\infty) = \sqrt{\pi/2},$$
$$\phi(0) = \sqrt{2/\pi}$$

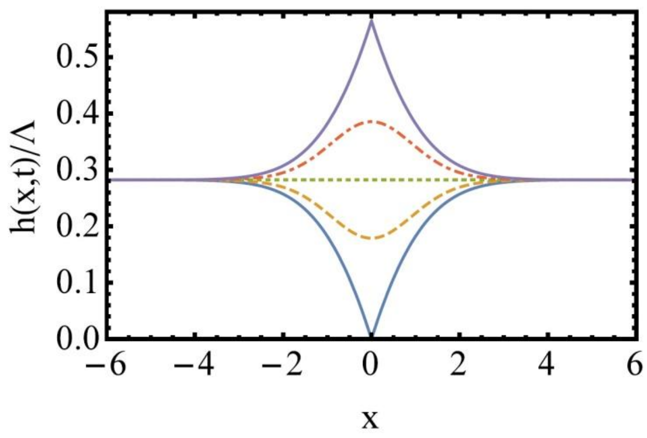
Agrees with result of Le Doussal et al (2016), extracted from exact representation of P(H,T) for sharp wedge

Kamenev, Meerson and Sasorov (2016)

Optimal path at small H, or Λ



flat interface

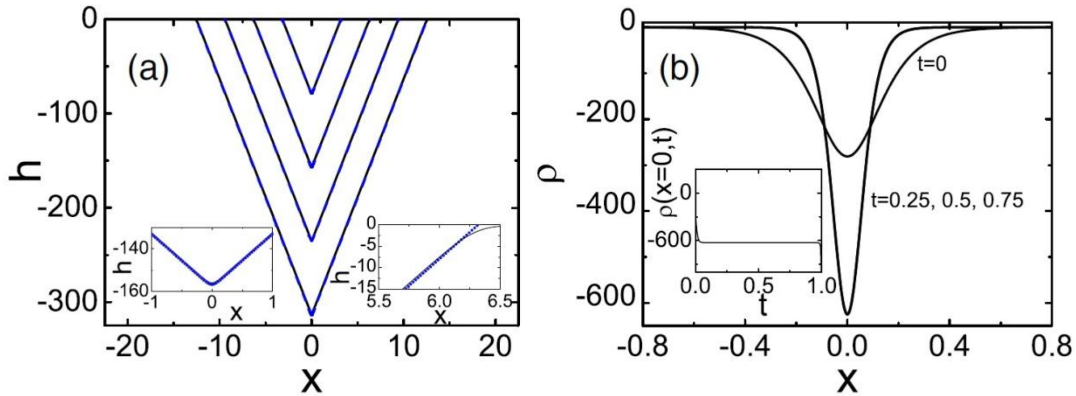


stationary interface

Optimal shape at t=0 includes
a nonzero plateau

These optimal paths are symmetric with respect to $x \to -x$
in all orders of perturbation theory

Tail H<0, |H|>>1, parabolic interface



L=∞

A standing pulse (soliton) of ρ, driving two outgoing "ramps" of h

$$\rho_{\mathrm{b1}}(x) = -2c\,\mathrm{sech}^2\left(\sqrt{c/2}\,x\right),$$

$$h_{\mathrm{b1}}(x,t) = 2\ln\cosh\left(\sqrt{c/2}\,x\right) - ct$$

This "inner" solution can be matched with the "outer" solution which solves the deterministic KPZ equation obeying the boundary conditions at |x|=∞. Diffusion can be neglected there -> shocks in $\partial_x h$ appear

The action, $S = \frac{1}{2}\int_0^1 dt \int_{-\infty}^{\infty} dx\, \rho^2(x,t)$ comes from the soliton and is independent of L. It is universal for a whole class of deterministic initial conditions

$$-\ln P(H,T) \cong \frac{8\sqrt{2}\nu\,|H|^{3/2}}{3D\,|\lambda|^{1/2}\,T^{1/2}} \sim \left(\frac{|H|}{T^{1/3}}\right)^{3/2}$$

agrees with the Tracy-Widom tail, previously observed only at long times, and only for L=0 and L=∞

Tail H<0, |H|>>1, stationary interface

The x-symmetric solution

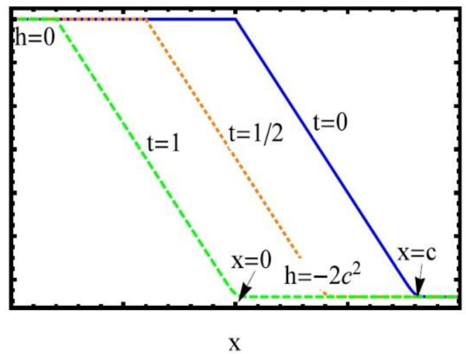
$$\rho_{\rm bl}(x) = -2c\,\mathrm{sech}^2(\sqrt{c/2}\,x),$$
$$h_{\rm bl}(x,t) = 2\ln\cosh(\sqrt{c/2}\,x) - ct$$

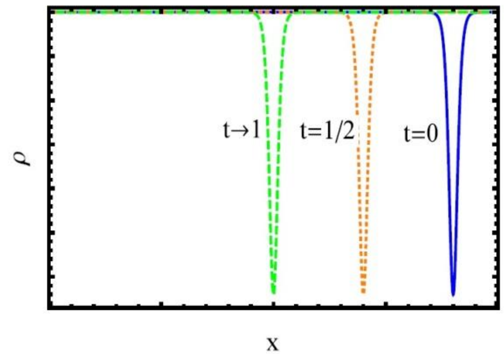
is still there

but it does not minimize the action!

A lesser action is obtained on each of two asymmetric solutions:

$$\rho(x) = -c^2\,\mathrm{sech}^2\left[\frac{c}{2}(x+ct-c)\right],$$
$$h(x,t) = 2\ln\left[e^{c(x+ct-c)}-1\right] - 2c(ct+x)$$





and its mirror reflection around x=0.

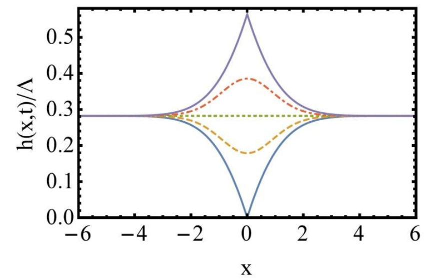
The *total* action, $S_{\rm total} = S_0 + S, \quad S_0 = \int_{-\infty}^{\infty} dx\left[\partial_x h(x,t=0)\right]^2$ is twice as small as for deterministic interface:

$$-\ln P(H,T) \cong \frac{4\sqrt{2}\nu\,|H|^{3/2}}{3D\,|\lambda|^{1/2}\,T^{1/2}} \sim \left(\frac{|H|}{T^{1/3}}\right)^{3/2}$$

agrees with the Baik-Rains tail, previously observed only at long times

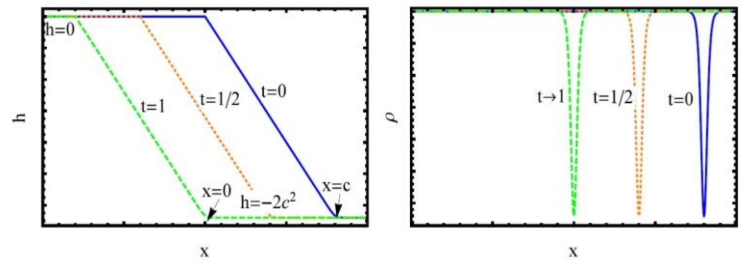
Tail H<0, stationary interface: dynamical phase transition

- At small negative H the optimal path is symmetric with respect to $x = 0$.



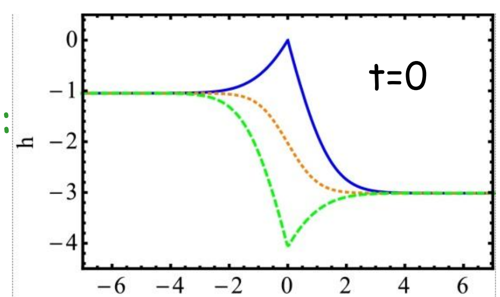
Plateau appears

- At large negative H *this symmetry is broken.*



Therefore, a symmetry-breaking transition must occur at $H = H_c < 0$, so that $|H_c| = O(1)$.

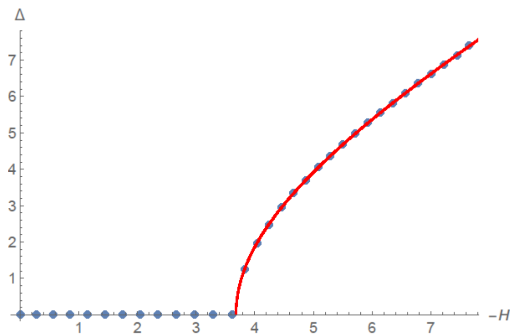
Supported by numerics:



Asymmetric plateau

Numerics shows that the transition is of second order

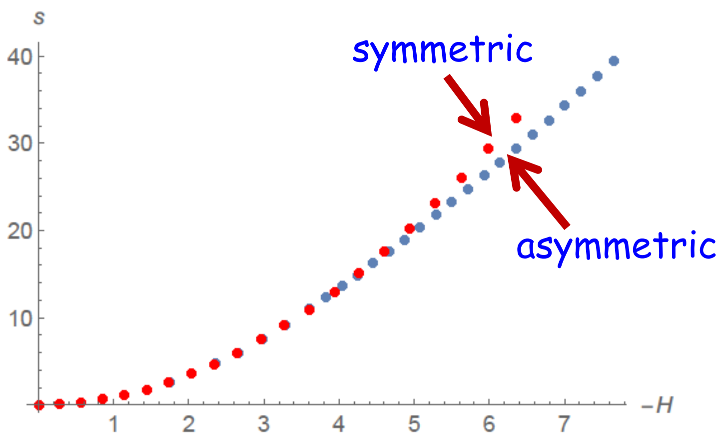
Janas, Kamenev and Meerson (2016)

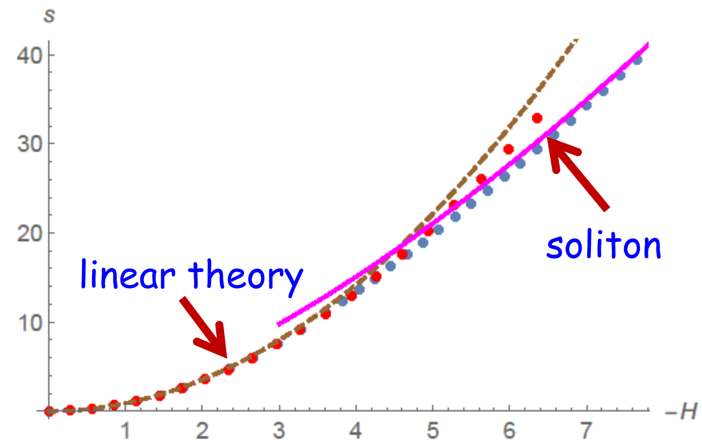


Order parameter

$$\Delta = |h(x=\infty,t=0)-h(x=-\infty,t=0)|$$

$$|H_c| \approx 3.7$$





Non-equilibrium phase transitions have been also observed in driven lattice gases: Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim (2006,2010), Bodineau and Derrida (2005), Hurtado and Garrido (2011), Bunin, Kafri and Podolsky (2013), Baek and Kafri (2015), Baek, Kafri and Lecomte (2016), …

H>>1 tail is very different: here one can drop the diffusion terms

$$\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \rho, \qquad \partial_t \rho = -\partial_x(\rho \partial_x h).$$

or

$$\partial_t V + V\partial_x V = \partial_x \rho, \qquad \partial_t \rho + \partial_x(\rho V) = 0.$$

$$V = \partial_x h$$

Inviscid flow of a 1d gas with *negative* pressure $p = -\rho^2/2$

The solution is quite simple:

$$V(x,t) = -a(t)x, \quad |x| \le l(t),$$

$$\rho(x,t) = \begin{cases} r(t)[1 - x^2/l^2(t)], & |x| \le l(t), \\ 0, & |x| > l(t). \end{cases}$$

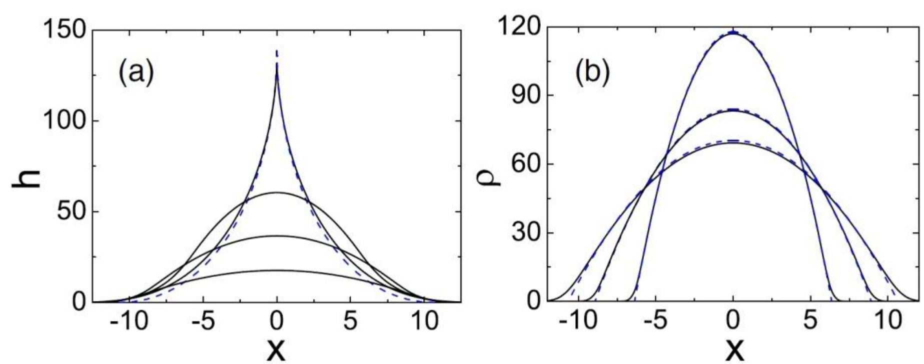
Uniform-strain flow culminating in collapse of the "gas" to the origin at t=1

$$\rho(x,t=1) = \Lambda\delta(x), \quad V(x,t=0) = 2x/L$$

for parabolic interface

$$\frac{dr}{dt} = ra, \qquad \frac{da}{dt} = a^2 + \frac{32}{9}r^3.$$

soluble

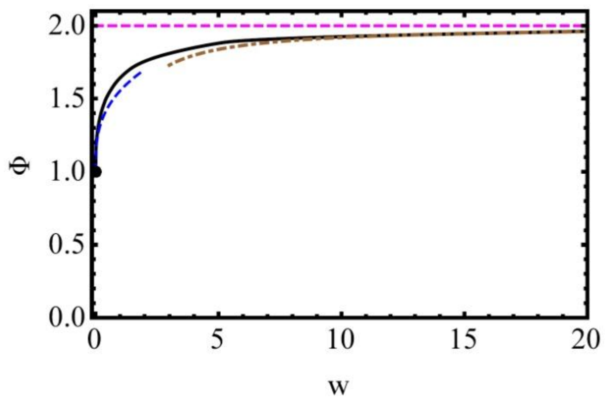


$$x/\Lambda^{1/3} \to x, \qquad h/\Lambda^{2/3} \to h, \qquad \rho/\Lambda^{2/3} \to \rho$$

H >>1 tail of P(H,T,L) for parabolic interface

Kamenev, Meerson and Sasorov (2016)

$$-\ln P(H,T,L) \cong \frac{4\sqrt{2}\,|\lambda|^{1/2}\,H^{5/2}}{15\pi D\sqrt{T}}\,\Phi\left(\frac{L}{|\lambda|T}\right),$$



Agrees with Kolokolov and Korshunov (2009) and Meerson, Katzav and Vilenkin (2016)

$$\Phi(w=\infty)=2$$

$$\Phi(w=0)=1$$

Agrees with Le Doussal et al (2016), who extracted it from exact representation of P(H,T) for sharp wedge

This tail is independent of ν; different from the Tracy-Widom tail ~ H^3/T.
It persists, at H>>T, *at all times* (proof for L=0: Sasorov, Meerson and Prolhac 2017)

H >>1 tail of P(H,T) for stationary interface

Janas, Kamenev and Meerson (2016)

$$-\ln P(H,T) \cong \frac{4\sqrt{2}\,|\lambda|^{1/2}\,H^{5/2}}{15\pi D\sqrt{T}},$$

same as for the sharp wedge, L=0;
different from the Baik-Rains tail ~ H^3/T.

Independent check: sharp wedge, long times

Extracting $H \gg T^{1/3} \gg 1$ asymptotic from exact representation for $P(H,T)$
due to Amir, Corwin and Quastel (2011)

$$-\ln P(H,T) \cong T^2 \Phi\left(\frac{H}{T}\right), \quad H \gg T^{1/3} \gg 1,$$

Sasorov, Meerson and Prolhac (2017)

$$\Phi(z) = \frac{4}{15\pi^6}(1-\pi^2 z)^{5/2} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4} z - \frac{1}{2\pi^2} z^2$$

$$-\ln P(H,T) = \begin{cases} \dfrac{2\nu^2 H^3}{3\lambda D^2 T}, & \left(\dfrac{|\lambda| D^2 T}{\nu^2}\right)^{1/3} \ll H \ll \dfrac{|\lambda|^3 D^2 T}{\nu^4}, \\ \dfrac{4\sqrt{2}\,|\lambda|^{1/2} H^{5/2}}{15\pi D\sqrt{T}}, & H \gg \dfrac{|\lambda|^3 D^2 T}{\nu^4} \end{cases}$$

Tracy-Widom tail

5/2 tail

Crossover between the tails is at H~T

Summary

• At early times P(H,t) of the KPZ equation has strongly asymmetric non-Gaussian tails

• The H<0, |H|>>1 tail $-\ln P(H,t) \sim H^{3/2}/t^{1/2}$ agrees, at any t>0, with the Tracy-Widom distribution for a whole class of deterministic initial conditions, and with the Baik-Rains distribution for the stationary interface.

• Spontaneous breaking of reflection symmetry of the optimal path, and dynamical phase transition, at H<0, $|H|=|H_c|\sim 1$ and t-> 0 for the stationary interface. Symmetry breaking is possible because of randomness of the initial condition. Other systems?

• The H>>1 tail, $$-\ln P(H,t,L) \sim \frac{H^{5/2}}{t^{1/2}} \Phi\left(\frac{L}{t}\right)$$ is controlled by λ and independent of ν.

• At long times the body of P(H,t) changes from Gaussian to Tracy-Widom. The $H^{3/2}$ and $H^{5/2}$ tails persist at all times

Future work:

Higher dimensions: no exact representations are known. OFM should be useful

Exact solution of the OF equations? Cole-Hopf canonical transformation, real NSE, symmetric structure, multi-soliton solutions, infinite number of conservation laws…

Thank you