An introduction to the inclusion process (and its scaling limits)

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Outline

- Inclusion process.
- Models related to inclusion process.
- (Self)-dualities: old and new.
- Scaling limit I: metastability.
- Scaling limit II: two particles.

1. Inclusion process

Set up

Let S finite set, $r_{x,y} \geq 0$ jump rates of an irreducible CTRW on S with reversible measure $m = (m_x)_{x \in S}$, i.e.

$$m_x r_{x,y} = m_y r_{y,x} \qquad \forall\, (x,y) \in S \times S$$

The reversible inclusion process with parameter $k \geq 0$ is the Markov jump process $\{\eta(t) \, : t \geq 0\}$ with state space $\mathbb{N}^S$ and generator

$$L f(\eta) = \sum_{x,y \in S \times S} r_{x,y}\, \eta_x \,(2k + \eta_y)\,[f(\eta^{x,y}) - f(\eta)]$$

where

$$\eta_z^{x,y} = \begin{cases} \eta_x - 1 & \text{if } z = x, \\ \eta_y + 1 & \text{if } z = y, \\ \eta_z & \text{if } z \neq \{x,y\} \end{cases}$$

Introduced in [G., Kurchan, Redig, JMP '07] for $k = 1/4$.

Reversible measure

- In the gran-canonical ensemble, a family of inhomogeneous product of Negative Binomials with parameters $2k$ and m_x, i.e.

$$\mu(\eta) = \frac{1}{Z} \prod_{x \in S} \frac{(\phi m_x)^{\eta_x}}{\eta_x!} \frac{\Gamma(\eta_x + 2k)}{\Gamma(2k)}$$

with $Z = \prod_{x \in S} (1 - \phi m_x)^{-2k}$ and $0 < \phi < (\sup_{x \in S} m_x)^{-1}$

- In the canonical ensemble with N particles, the state space is

$$E_N = \{\eta \in \mathbb{N}^S \, : \, \sum_{x \in S} \eta_x = N\}$$

and the unique reversible measure μ_N is obtained by conditioning, i.e.

$$\mu_N(\eta) = \frac{1}{Z_N} \prod_{x \in S} \frac{m_x^{\eta_x}}{\eta_x!} \frac{\Gamma(\eta_x + 2k)}{\Gamma(2k)} \mathbb{1}_{E_N}(\eta)$$

Symmetric case: SIP(k)

If the random walk is symmetric $r_{x,y} = r_{y,x}$ then:

- the random walk reversible measure m is the uniform measure

$$m_x = \frac{1}{|S|} \qquad \forall\, x \in S$$

- the process reversible measure μ is a one-parameter family of i.i.d. Neg Bin (2k,p) with $0 < p < 1$

$$\mu(\eta) = \prod_{x \in S} \frac{1}{(1-p)^{-2k}} \frac{p^{\eta_x}}{\eta_x!} \frac{\Gamma(\eta_x + 2k)}{\Gamma(2k)}$$

2. Two models related to symmetric inclusion process

Moran process

Moran model with population size N, individuals of n types and with symmetric parent-independent mutation at rate θ:

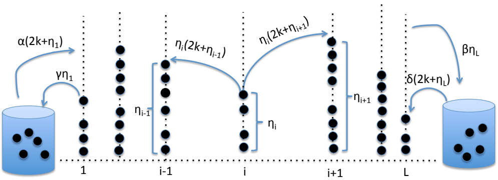
- a pair of individuals of types x and y are sampled uniformly at random, one dies with probability $1/2$ and the other reproduces
- each individual accumulates mutations at a constant rate θ and his type mutates to any of the others with the same probability.

This is the N particle symmetric inclusion process on the complete graph K_n with parameter $k = \frac{\theta}{n-1}$

$$
\begin{aligned}
L f(\eta) \;=\; \frac{1}{2} \sum_{1 \le x < y \le n} & \eta_x \left(\frac{2\theta}{n-1} + \eta_y \right) \left[f(\eta^{x,y}) - f(\eta) \right] \\
& + \eta_y \left(\frac{2\theta}{n-1} + \eta_x \right) \left[f(\eta^{y,x}) - f(\eta) \right]
\end{aligned}
$$

see [Carinci, G., Giberti, Redig, SPA '15]

Non-equilibrium statistical mechanics



- Adding reservoirs:
  - Bulk: symmetric inclusion process on one dimensional chain with nearest neighbor interaction
  - Left: birth/death process with stationary meas. Neg Bin $(2k, \frac{\alpha}{\gamma})$
  - Right: birth/death process with stationary meas. Neg Bin $(2k, \frac{\delta}{\beta})$
- If $\frac{\alpha}{\gamma} = \frac{\delta}{\beta}$ then equilibrium product measure
  If $\frac{\alpha}{\gamma} \neq \frac{\delta}{\beta}$ then non-equilibrium measure (long-range correlations)
- For $k = 1/2$ it is related to Kipnis-Marchioro-Presutti model [see Carinci, G., Giberti, Redig, JSP '13]

3. Duality: old and new

Self-duality

Let $\eta(t)$ and $\xi(t)$ be two independent copies of the SIP process. Consider the function

$$D(\eta,\xi) = \prod_x \frac{\eta_x!}{(\eta_x-\xi_x)!}\frac{\Gamma(2k)}{\Gamma(2k+\xi_x)}$$

then

$$\mathbb{E}_\eta[D(\eta(t),\xi)] = \mathbb{E}_\xi[D(\eta,\xi(t))]$$

Remark: one can compute n-point correlation functions by using only n-dual walkers. E.g.: In non-equilibrium setting, if $\gamma = 2k+\alpha$ and $\beta = 2k+\delta$ then

$$Cov(\eta_x,\eta_y) = \frac{x(L+1-y)}{(L+1)^2(2k(L+1)+1)}(\alpha-\delta)^2$$

Algebraic approach to

stochastic duality

Algebraic approach

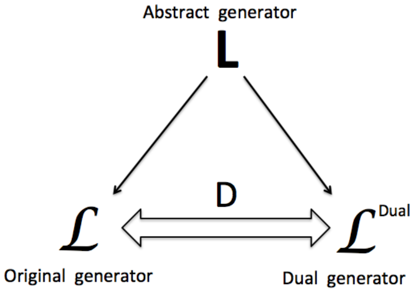
1. Write the Markov generator in abstract form, i.e. as an element of a Lie algebra, using the algebra generators (typically creation and annihilation operators).

2. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra. Duality functions are the intertwiners.

3. Self-duality is associated to symmetries, i.e. conserved quantities.

[G., Kurchan, Redig, Vafay, JSP '09]

Duality



Abstract generator
L
D
$\mathcal{L}$
$\mathcal{L}^{\text{Dual}}$
Original generator
Dual generator

Self-duality

For Markov chain with countable state space

$$LD(\cdot,\xi)(\eta) = LD(\eta,\cdot)(\xi)$$

amounts to

$$\mathbf{LD} = \mathbf{DL}^T$$

Indeed

$$\sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{D}(\eta',\xi) = LD(\cdot,\xi)(\eta) = LD(\eta,\cdot)(\xi) = \sum_{\xi'} \mathbf{L}(\xi,\xi')\mathbf{D}(\eta,\xi')$$

Trivial self-duality functions from reversible measures

From a reversible measure μ, i.e.

$$\mathbf{L}(\eta,\xi)\mu(\eta) = \mathbf{L}(\xi,\eta)\mu(\xi)$$

a trivial (i.e. diagonal) self-duality function is

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'}\mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'}\mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

Symmetries and self-duality

S: symmetry of the generator, i.e. $[\mathbf{L}, \mathbf{S}] = 0$,

$\mathbf{d}$: trivial self-duality function,

$\longrightarrow \quad \mathbf{D} = \mathbf{Sd}$ self-duality function.

Indeed

$$\mathbf{LD} = \mathbf{LSd} = \mathbf{SLd} = \mathbf{SdL}^T = \mathbf{DL}^T$$

Self-duality is related to the action of a symmetry

Construction of Markov generators with algebraic structure and symmetries

i) (*Lie Algebra*): Start from a (representation of a) Lie algebra $\mathfrak{g}$.

ii) (*Casimir*): Pick an element in the center of $\mathfrak{g}$, e.g. the Casimir C.

iii) (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making the algebra a bialgebra and conserving the commutation relations.

iv) (*Quantum Hamiltonian*): Compute the co-product $H = \Delta(C)$.

v) (*Markov generator*): Apply a ground state transform (often a similarity transformation) to turn H into a Markov generator L.

vi) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$ is a symmetry of H:

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

[Carinci, G., Redig, Sasamoto, SPA '16]

The method at work:

$\mathfrak{su}(1,1)$ Lie algebra

Algebraic structure of inclusion process

$\mathfrak{su}(1,1)$ ferromagnetic quantum spin chain

$$\mathscr{L} = \sum_{(x,y)\in E} \left(\mathcal{K}_x^+ \mathcal{K}_y^- + \mathcal{K}_x^- \mathcal{K}_y^+ - 2\mathcal{K}_x^o \mathcal{K}_y^o + 2k^2 \right)$$

with $\{\mathcal{K}_x^+, \mathcal{K}_x^-, \mathcal{K}_x^o\}_{x\in S}$ satisfying $\mathfrak{su}(1,1)$ Lie algebra

$$[\mathcal{K}_x^o, \mathcal{K}_y^\pm] = \pm\delta_{x,y}\mathcal{K}_x^\pm \qquad [\mathcal{K}_x^-, \mathcal{K}_y^+] = 2\delta_{x,y}\mathcal{K}_x^o$$

step i): representation in terms of matrices

A discrete representation of $\mathfrak{su}(1,1)$ algebra is

$$\begin{cases} K^{+} f(n) = (n+2k)\, f(n+1) \\ K^{-} f(n) = n f(n-1) \\ K^{o} f(n) = (n+k)\, f(n) \end{cases}$$

In a canonical base

$$K^{+} = \begin{pmatrix} 0 & & & \\ 2k & \ddots & & \\ & 2k+1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad K^{-} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad K^{o} = \begin{pmatrix} k & 0 & & \\ & k+1 & \ddots & \\ & & k+2 & \ddots \\ & & & \ddots \end{pmatrix}$$

step ii): Casimir element

For the $\mathfrak{su}(1,1)$ algebra the Casimir is

$$C = \frac{1}{2}(K^- K^+ + K^+ K^-) - (K^0)^2$$

C is in the center of the algebra:

$$[C, K^+] = [C, K^-] = [C, K^o] = 0$$

$$Cf(n) = k(1-k)f(n)$$

step iii): Co-product

The co-product is a morphism that turns the algebra into a bialgebra:

$$\Delta : \mathfrak{su}(1,1) \to \mathfrak{su}(1,1) \otimes \mathfrak{su}(1,1)$$

and conserves the commutations relations

$$[\Delta(K^o), \Delta(K^\pm)] = \pm\Delta(K^\pm)$$
$$[\Delta(K^-), \Delta(K^+)] = 2\Delta(K^o)$$

For classical Lie-algebras the co-product is just the symmetric tensor product with the identity

$$\Delta(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X := X_1 + X_2$$

step iv): Quantum Hamiltonian

$$\Delta(C) = \frac{1}{2}\Big(\Delta(K^-)\Delta(K^+) + \Delta(K^+)\Delta(K^-)\Big) - \Big(\Delta(K^0)\Big)^2$$

$$= K_1^- K_2^+ + K_1^+ K_2^- - 2K_1^o K_2^o + C_1 + C_2$$

$= \mathfrak{su}(1,1)$ Heisenberg ferromagnet $+$ diagonal

step v): Markov generator

There is no need of a “ground state transformation”. In the discrete representation

$$\Delta(C) = (L_{1,2}^{SIP(k)})^* + 2k(1-2k)$$

where

$$L_{1,2}^{SIP(k)} f(\eta_1, \eta_2) = \eta_1 \, (\eta_2 + 2k) \, [f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)] + \eta_2 \, (\eta_1 + 2k) \, [f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)]$$

is the generator of the Symmetric Inclusion Process SIP(k).

step vi): symmetries

As a consequence of the construction,

$\Delta(K^\alpha)$ with $\alpha \in \{+, -, o\}$ are symmetries of the process:

$$[(L_{1,2}^{SIP(k)})^*, K_1^o + K_2^o] = 0$$

$$[(L_{1,2}^{SIP(k)})^*, K_1^+ + K_2^+] = 0$$

$$[(L_{1,2}^{SIP(k)})^*, K_1^- + K_2^-] = 0$$

Proof self-duality SIP(k)

- Reversible measure is product of Negative Binomial $(p, 2k)$

$$\mu_{rev}(\eta) = \prod_x \frac{1}{(1-p)^{-2k}} \frac{p^{\eta_x}}{\eta_x!} \frac{\Gamma(2k+\eta_x)}{\Gamma(2k)}$$

- Trivial (i.e. diagonal) self-duality function

$$\mathbf{d}(\eta, \xi) = \frac{1}{\mu_{rev}(\eta)} \delta_{\eta,\xi}$$

- Symmetry

$$S = \exp \sum_x K_x^+$$

Analytical approach to duality:

orthogonal polynomials

[Franceschini, G., arXiv:1701.09115]

[Redig, Sau, arXiv:1702.07237]

Analytical approach

- Question: what is the relation between duality and stationary measure? Orthogonal polynomials?

Analytical approach

- Question: what is the relation between duality and stationary measure? Orthogonal polynomials?
- Answer: The SIP(k) is self-dual process with self-duality function

$$D(\eta,\xi) = \prod_x \frac{\Gamma(2k)}{\Gamma(2k+\xi_x)} M_{\xi_x}(\eta_x)$$

where $M_{\xi_x}(\eta_x)$ is the Meixner polynomials of degree ξ_x

$$\sum_{\eta_x=0}^{\infty} M_{\xi_x}(\eta_x) M_{\xi'_x}(\eta_x)\mu(\eta_x) = \delta_{\xi_x,\xi'_x} \frac{\xi_x!\Gamma(2k+\xi_x)}{p^{\xi_x}\Gamma(2k)}$$

with

$$\mu(\eta_x) = \frac{\Gamma(2k+\eta_x)}{\Gamma(2k)} \frac{p^{\eta_x}}{\eta_x!}(1-p)^{2k}$$

Analytical approach (cont'd)

- Hypergeometric difference equation

$$\sigma(\eta_x)\Delta\nabla M_{\xi_x}(\eta_x) + \tau(\eta_x)\Delta M_{\xi_x}(\eta_x) + \lambda_{\xi_x} M_{\xi_x}(\eta_x) = 0$$

with

$$\Delta f(n) = f(n+1) - f(n) \qquad \nabla f(n) = f(n) - f(n-1)$$

$$\sigma(n) = n \qquad \tau(n) = 2kp - n(1-p) \qquad \lambda_{\xi_x} = \xi_x(1-p)$$

- 3-point recurrence relation

$$\eta_x M_{\xi_x}(\eta_x) = \alpha_{\xi_x} M_{\xi_x+1}(\eta_x) + \beta_{\xi_x} M_{\xi_x}(\eta_x) + \gamma_{\xi_x} M_{\xi_x-1}(\eta_x)$$

with

$$\alpha_{\xi_x} = \frac{p}{p-1} \qquad \beta_{\xi_x} = \frac{\xi_x + p\xi_x + 2kp}{1-p} \qquad \gamma_{\xi_x} = \frac{\xi_x(\xi_x - 1 + 2k)}{p-1}$$

- Raising operator

$$[p(\xi_x + 2k) + \eta_x p] M_{\xi_x}(\eta_x) - \eta_x M_{\xi_x}(\eta_x - 1) = p M_{\xi_x+1}(\eta_x)$$

Analytical approach (cont'd)

Other dualities with orthogonal polynomials

- Exclusion Process $\longrightarrow$ Krawtchouk polynomials
- Independent walkers $\longrightarrow$ Charlier polynomials
- Brownian momentum process $\longrightarrow$ Hermite polynomials

$$Lf(\eta) = \sum_{(x,y)\in E} \left(\eta_x \frac{\partial}{\partial \eta_y} - \eta_y \frac{\partial}{\partial \eta_x} \right)^2 f(\eta)$$

- Brownian energy process $\longrightarrow$ Laguerre polynomials

$$Lf(\eta) = \sum_{(x,y)\in E} \left[\eta_x \eta_y \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 + 2k(\eta_x - \eta_y) \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \right] f(\eta)$$

4. Scaling limit I:

metastability

Condensation

Proposition: Consider a parameter $k = k(N)$ and define $d_N = 2k(N)$. Suppose $d_N \log N \to 0$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \mu_N(\eta^x) = \frac{1}{|S_\star|} \qquad \forall x \in S_\star$$

where

$$\eta^x_z = \begin{cases} N & \text{if } z = x, \\ 0 & \text{if } z \neq x \end{cases}$$

and

$$S_\star = argmax\{m(x) \,:\, x \in S\}$$

Proof: Consequence of Stirling's approximation, essentially proved in [Grosskinsky, Redig, Vafayi, '11].

Movement of the condensate

Theorem (Bianchi, Dommers, G., 2016). Suppose $d_N \log N \to 0$ as $N \to \infty$ and that $\eta(0) = \eta^x$ for some $x \in S_\star$. For $A \subset E_N$, let $\tau_A = \inf\{t \geq 0 \,:\, \eta(t) \in A\}$. Then

1. Average time

$$\mathbb{E}_{\eta^x}\left(\tau_{\{\bigcup_{\{y\in S_\star, y\neq x\}} \eta^y\}}\right) = \frac{1}{\sum_{y\in S_\star, y\neq x} r_{x,y}} \frac{1}{d_N}(1 + o(1))$$

2. Scaling limit

$$X_N(t) = \sum_{z\in S^*} z \mathbb{1}_{\{\eta(t)=\eta^z\}}$$

$$X_N(t/d_N) \longrightarrow X(t) \quad \text{weakly} \qquad \text{as} \quad N \to \infty$$

   where $X(t)$ is the Markov process on $S_\star$ with $X(0) = x$ and generator

$$Lf(y) = \sum_{z\in S_\star} r_{y,z}[f(z) - f(y)]$$

Comments

- In the symmetric case $S_\star = S$, item 2. recovers the result by [Grosskinsky, Redig, Vafayi 13]

- Comparison to zero-range process [Beltrán, Landim '12]:
  - Condensation if rates for a particle to move from x to y is $r_{x,y}\left(\frac{\eta_x}{\eta_x - 1}\right)^{\alpha}$ for $\alpha > 2$
  - Condensate consists of at least $N - \ell_N$ particles, $\ell_N = o(N)$; metastable states are equally probable.
  - At time scale $t \cdot N^{\alpha+1}$ the condensate moves from $x \in S_\star$ to $y \in S_\star$ at rate proportional to $cap(x, y)$, the capacity of the random walker between x and y.

Proof: key ingredients

For $F : E_N \to \mathbb{R}$ let D_N be Dirichlet form

$$D_N(F) = \frac{1}{2} \sum_{x,y \in S} \sum_{\eta \in E_N} \mu_N(\eta)\, \eta_x \left(d_N + \eta_y\right) r_{x,y} \left[F(\eta^{x,y}) - F(\eta)\right]^2$$

For two disjoint subsets $A,\ B \subset E_N$ the capacity between A and B can be computed using *Dirichlet variational principle*

$$\mathrm{Cap}_N(A,B) = \inf\{D_N(F) : F \in \mathcal{F}_N(A,B)\}$$

where

$$\mathcal{F}_N(A,B) = \{F : F(\eta) = 1 \text{ for all } \eta \in A \text{ and } F(\eta) = 0 \text{ for all } \eta \in B\}.$$

Proof: key ingredients (cont'd)

The unique minimizer of the Dirichlet principle is the *equilibrium potential*, i.e., the harmonic function $h_{A,B}$ that solves the Dirichlet problem

$$\begin{cases} L_N h(\eta) = 0, & \text{if } \eta \notin A \cup B, \\ h(\eta) = 1, & \text{if } \eta \in A, \\ h(\eta) = 0, & \text{if } \eta \in B. \end{cases}$$

It can be easily checked that

$$h_{A,B}(\eta) = \mathbb{P}_\eta(\tau_A < \tau_B)\,.$$

Capacities are related to the mean hitting time between sets [Bovier, Eckhoff, Gayrard, Klein, 01 – 04]

$$\mathbb{E}_{\nu_{A,B}}(\tau_B) = \frac{\mu_N(h_{A,B})}{\mathrm{Cap}_N(A,B)}$$

Proof: key ingredients (cont'd)

Potential theory ideas and martingale methods can be combined in order to prove the scaling limit of suitably speeded-up processes [Beltrán, Landim, 10 – 15].

Find a sequence $(\theta_N,\ N \geq 1)$ of positive numbers, such that, for any $x, y \in S_\star\,,\ x \neq y$, the following limit exists

$$p(x,y)\ :=\ \lim_{N\to\infty} \theta_N p_N\,(\eta^x,\eta^y)$$

where $p_N\,(\eta^x,\eta^y)$ are the jump rates of the original process

- (θ_N) provides the time-scale to be used in the scaling limit
- $(p(x,y))_{x,y\in S_\star}$ identifies the limiting dynamics.

Proof: key ingredients (cont'd)

Lemma

$$\begin{aligned}\mu_N(\eta^x)p_N(\eta^x,\eta^y) &= \frac{1}{2}\Bigg[\mathrm{Cap}_N\Bigg(\eta^x, \bigcup_{z\in S_\star, z\neq x}\eta^z\Bigg) \\ &+ \mathrm{Cap}_N\Bigg(\eta^y, \bigcup_{z\in S_\star, z\neq y}\eta^z\Bigg) \\ &- \mathrm{Cap}_N\Bigg(\{\eta^x,\eta^y\}, \bigcup_{z\in S_\star, z\neq y}\eta^z\Bigg)\Bigg]\end{aligned}$$

Proof: key ingredients (cont'd)

Proposition: Let $S_\star^1 \subsetneq S_\star$ and $S_\star^2 = S_\star \setminus S_\star^1$. Then, for $d_N \log N \to 0$ as $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N \left(\bigcup_{z\in S_\star^1} \eta^z, \bigcup_{z\in S_\star^2} \eta^z \right) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r_{x,y}$$

Combining Lemma and Proposition it follows

$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^x, \eta^y) = r_{x,y}$$

Proof: key ingredients (cont'd)

Lower bound by restricting the Dirichlet form to suitable subset of E_N.

Let F s.t. $F(\eta^x) = 1\ \forall x \in S_\star^1$ and $F(\eta^y) = 0\ \forall y \in S_\star^2$

$$
\begin{aligned}
D_N(F) &= \frac{1}{2}\sum_{x,y\in S}\sum_{\eta\in E_N} \mu_N(\eta)\,\eta_x\,(d_N+\eta_y)\,r_{x,y}\,[F(\eta^{x,y})-F(\eta)]^2 \\
&\geq \sum_{x\in S_\star^1}\sum_{y\in S_\star^2} r_{x,y} \sum_{\eta_x+\eta_y=N} \mu_N(\eta)\,\eta_x\,(d_N+\eta_y)\ [F(\eta^{x,y})-F(\eta)]^2 \\
&= \sum_{x\in S_\star^1}\sum_{y\in S_\star^2} r_{x,y}\sum_{i=1}^{N}\mu_N(i,N-i)\,i\,(d_N+N-i)\ [G(i-1)-G(i)]^2 \\
&\qquad \text{with}\quad G(i) = F(\eta_x = i, \eta_y = N-i) \\
&\geq \frac{d_N}{|S_\star|}\sum_{x\in S_\star^1}\sum_{y\in S_\star^2} r_{x,y}(1+o(1))
\end{aligned}
$$

Proof: key ingredients (cont'd)

Upper bound by constructing suitable test function F.

Good guess inside tubes $\eta_x + \eta_y = N$ is $F(\eta) \approx \eta_x / N$

- by construction particle moving from $x \in S_\star^1$ to $y \in S_\star^2$ give correct contribution
- unlikely to be in a configuration with particles on three sites/ sites not in $S_\star$
- unlikely for a particle to escape from a tube

Multiple timescales

On the time scale $1/d_N$ condensate jumps between site of $S_\star$.

If induced random walk on $S_\star$ is not irreducible, condensate jumps between connected components on longer time scales.

Conjecture:

- if graph distance $= 2$ then second timescale $\frac{N}{d_N^2}$
- if graph distance ≥ 3 then third timescale $\frac{N^2}{d_N^3}$

We prove this when the graph is a line with

$$S = \{1, \ldots, L\} \qquad S_\star = \{1, L\} \qquad r_{x,y} \neq 0 \quad \text{iff} \quad |x - y| = 1$$

Second time-scale

Theorem (Bianchi, Dommers, G., 2016). Suppose that $d_N \log N \to 0$ and $d_N e^{\delta N} \to \infty$ as $N \to \infty$ and $\eta_x(0) = N$ for some $x \in S_\star$. Then for one-dimensional system with $L = 3$

$$X_N(tN/d_N^2) \longrightarrow X(t) \quad \text{weakly} \qquad \text{as} \quad N \to \infty$$

where $X(t)$ is the Markov process on $S_\star = \{1, 3\}$ with $X(0) = x$ and transition rates

$$p(1,3) = p(3,1) = \left(\frac{1}{r_{1,2}} + \frac{1}{r_{3,2}}\right)^{-1} \frac{1}{1 - m_2}$$

Third time scale

Theorem (Bianchi, Dommers, G., 2016). Suppose that $d_N \log N \to 0$ as $N \to \infty$, d_N decays subexponentially and $\eta_x(0) = N$ for some $x \in S_\star$. Then for one-dimensional system with $L \geq 4$ there exists constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \leq \liminf_{N\to\infty} \frac{d_N^3}{N^2}\mathbb{E}_{\eta^1}[\tau_{\eta^L}] \leq \limsup_{N\to\infty} \frac{d_N^3}{N^2}\mathbb{E}_{\eta^1}[\tau_{\eta^L}] \leq C_2$$

Conjectured transition rates of time-rescaled process:

$$p(1, L) = p(L, 1) = 3\left(\sum_{i=2}^{L-2} \frac{(1-m_i)(1-m_{i+1})}{m_i r_{i,i+1}}\right)^{-1}$$

5. Scaling limit II:

two particles

blackboard ...

Perspectives

- Inclusion process is a novel interacting particle system with
  - several applications
  - mathematical structure of exactly solvable model (e.g. duality)
  - integrability ?

- Dynamics in the condensation regime
  - new features (i.e. multiple timescales) compared to other condensing systems, such as zero-range process
  - conjecture: three timescales as found in the one-dimensional setting
  - further problems: thermodynamic limit, coarsening, non-reversible dynamics.