Stochastic Resetting – Lectures IHP, Paris 9, 10, 12 May 2017

I Diffusion with Stochastic Resetting

IA Motivations

(i) Searching for a target – local moves (diffusion)
"intermittent search" + long range (resetting)

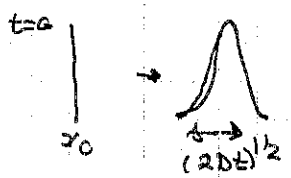
(ii) resetting a process to initial condition → NESS

IB Preliminaries – Diffusion

Diffusive Particle $x_{t+\Delta t} = x_t + \xi \qquad P(\xi) = \frac{1}{\sqrt{4\pi D \Delta t}}\, e^{-\xi^2/4D\Delta t}$

Diffusion Eq^n for pdf $\qquad \frac{\partial P(x,t|x_0)}{\partial t} = D \frac{\partial^2 P(x,t|x_0)}{\partial x^2} \qquad ①$

i.c. $\delta(x-x_0) \qquad$ sol^n $\quad P(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp -\frac{(x-x_0)^2}{4Dt} \qquad ②$



Sol^n of ① by L.T $\quad \tilde{P}(x,s|x_0) = \int_0^\infty dt\, e^{-st} P(x,t|x_0)$

$$D\frac{\partial^2 \tilde{P}}{\partial x^2} - s\tilde{P} = -\delta(x-x_0) \qquad ③$$

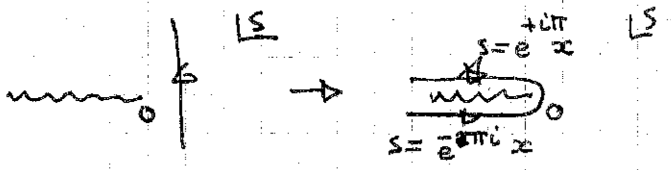
$$\to \quad \tilde{P} = \frac{1}{2(sD)^{1/2}}\, e^{-(s/D)^{1/2}|x-x_0|} \qquad ④$$

Invert ④ by 'magic integral' $\quad \int_0^\infty t^{\nu-1} e^{-\frac{\beta}{t} - st}\, dt = 2\left(\frac{\beta}{s}\right)^{\nu/2} K_\nu(2\sqrt{\beta s}) \qquad ⑤$

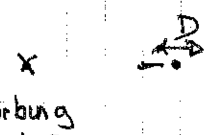
$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2}}\, e^{-z} z^{-1/2} \qquad$ then $\nu = 1/2 \quad \beta = \frac{(x-x_0)^2}{4D} \to ②$

or invert ④ by Bromwich contour

$$P(x,s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{+st}\, \tilde{P}(x,s)\, ds$$



IC Preliminaries – Survival probability



b.c. $P(x_T, t) = 0 \quad \forall t$

take $x_T = 0$

Approaches (i) solve ① + b.c. $\quad P(x,t|x_0) = \frac{1}{\sqrt{4\pi D t}}\left[\exp\left(-\frac{(x-x_0)^2}{4Dt}\right) - \exp\left(-\frac{(x+x_0)^2}{4Dt}\right)\right]$

survival prob $\quad q(t|x_0) = \int_0^\infty dx\, P(x,t|x_0) = \mathrm{erf}\left(\frac{x_0}{2\sqrt{Dt}}\right) \qquad ⑥$

(ii)

(ii) use Backward Eq^n $\quad \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x_0^2}(x,t|x_0)$ $\quad$ b.c. $P(x,0|x_0) = \delta(x-x_0)$

$P(x,t|0) = 0$

survival prob $\quad \frac{\partial q}{\partial t} = D \frac{\partial^2 q}{\partial x_0^2}$ $\quad q(0|x_0) = 1$

$q(t|0) = 0$

L.T $\quad D\frac{\partial^2 \tilde{q}}{\partial x_0^2} - s\tilde{q} = -1$

sol^n $\quad \tilde{q}(s|x_0) = \frac{1 - e^{-(s/D)^{1/2} x_0}}{s}$ $\quad$ (7)

Large t asymptotics $\sim$ small s $\quad \tilde{q}(s, x_0) \simeq \frac{1}{s^{1/2}} \frac{x_0}{D^{1/2}} + \text{const} + O(s^{1/2})$

$L^{-1}\left[\frac{1}{s^\alpha}\right] = \frac{1}{\Gamma(\alpha)} t^{\alpha - 1}$ $\quad q(t|x_0) \simeq \frac{x_0}{D^{1/2}\Gamma(1/2)} t^{-1/2} + O(t^{-3/2})$

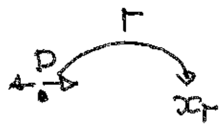
Mean time to absorption $T(x_0) = \int_0^\infty \left(-\frac{\partial q}{\partial t}\right) t\, dt = [q\, t]_0^\infty + \int_0^\infty q(t|x_0)\, dt = \tilde{q}(0|x_0) \to \infty$ $\quad$ (8)

[(iii) Renewal Eq^n $\quad P(x,t|x_0) = \int_0^t dt' \underbrace{F(x,t'|x_0)}_{\text{PDF for 1st time to reach } x} P(x, t-t'|x)$

$F(x,t|x_0) = -\frac{\partial q}{\partial t}(t|x_0)$ $\quad$ L.T. $\hat{P}(x,s|x_0) = \hat{F}(x,s|x_0)\hat{P}(x,s|x)$

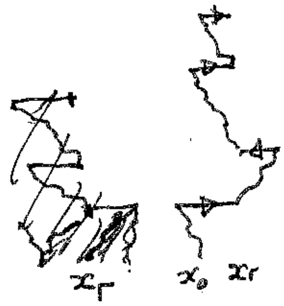
(4) $\to \hat{F}$. (5) $\to F(x,t|x_0)$] OMIT

1D Diffusion with Stochastic resetting



$x_{t+\Delta t} = x_t + \xi$ $\quad$ prob $1 - r\Delta t$

$= x_r$ $\quad$ prob $r\Delta t$

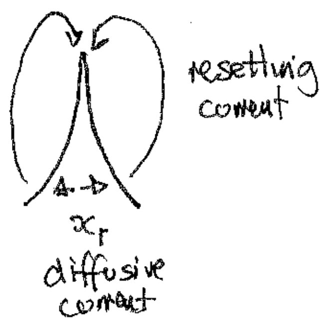


resetting is Poisson Process

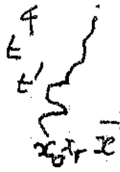
$$\frac{\partial P}{\partial t}(x,t|x_0,x_r) = D\frac{\partial^2 P}{\partial x^2} - rP + r\,\delta(x - x_r)$$ (9)

Stationary State $\quad D\frac{\partial^2 P}{\partial x^2} - rP = -r\,\delta(x - x_r)$ $\quad P = \frac{\alpha_0}{2} e^{-\alpha_0|x - x_r|}$ $\quad \alpha_0 = \left(\frac{r}{D}\right)^{1/2}$

NESS circulation of prob $\quad$ c.f. (3) $\quad$ cf (4)



Relaxation to NESS



renewal equation (suppress x_r)

$$P(x,t|x_0, x_r) = \underbrace{e^{-rt} G_D(x,t|x_0)}_{\text{no resets}} + r\int_0^t dt'\, e^{-r(t-t')} G_D(x,t-t'|x_r) \qquad (10)$$

G_D is $\frac{1}{\sqrt{4\pi D t}} \exp -\frac{(x-x_0)^2}{4Dt}$ $\qquad$ t' = time of last reset

$t \to \infty \quad P^*(x|x_r) = r\int_0^\infty dt'\, e^{-rt'} G_D(x,t'|x_r)$ $\qquad$ recovers (9)

large time: evaluate $\int_0^t dt'\, e^{-r(t-t')} G_D(x,t-t'|x_r) = \int_0^t dt'\, \frac{e^{-r(t-t') - \frac{(x-x_r)^2}{4D(t-t')}}}{\sqrt{4\pi D(t-t')}}$ by saddle point

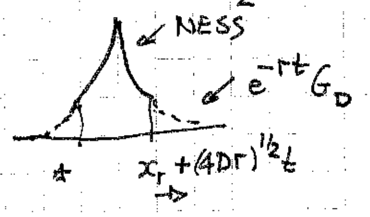
take $x - x_r = O(t)$ $\qquad$ let $t - t' = yt$ $\qquad \int_0^1 dy\, e^{-t\left[ry + \left(\frac{x-x_r}{t}\right)^2 \frac{1}{4Dy}\right]}$

$P(x,t) \simeq$

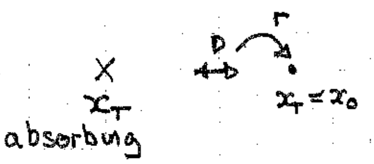
if $|x - x_r| > t(4Dr)^{1/2}$ $\qquad y = 1$ i.e. $t' = 0$ dominates $\qquad \simeq e^{-rt} G_D(x,t|x_0)$

if $\qquad < t(4Dr)^{1/2}$ $\qquad y = \frac{|x-x_r|}{t}\frac{1}{(4Dr)^{1/2}}$ dominates $\qquad \frac{1}{2}\left(\frac{r}{D}\right)^{1/2} e^{-|x-x_r|\left(\frac{r}{D}\right)^{1/2}}$

so ~~equilibration~~ "stationarity" front travels with $v = (4Dr)^{1/2}$



IE $\quad$ Diffusion + resetting + absorbing target



[here take $x_r = x_0$ to simplify] $\qquad x_T = 0$ $\qquad$ (11)

Use renewal eq^n $\quad P(x,t|x_0) = e^{-rt} G_D(x,t|x_0) + r\int_0^t dt'\, e^{-r(t-t')} G_D(x,t-t'|x_0) \underbrace{\int_0^\infty dx'\, P(x',t'|x_0)}_{\text{survival prob } q(t'|x_r)}$

G is now propagator with absorbing bdy

int wrt x $\quad q(t|x_r) = e^{-rt} q_D(t|x_0) + r\int_0^t dt'\, e^{-r(t-t')} q_D(t-t'|x_0)\, q(t'|x_0)$

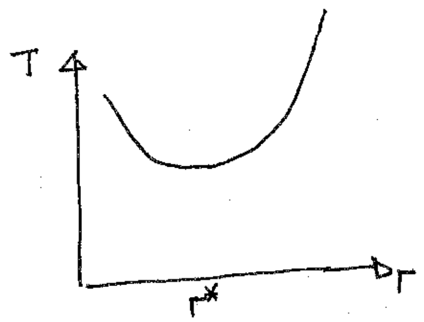
~~$\tilde{q}_D$ given by~~

L.T $\quad \tilde{q}(s|x_0) = \tilde{q}_D(s+r|x_0) + r\, \tilde{q}_D(s+r|x_0)\, \tilde{q}(s|x_0)$

$$\tilde{q}(s|x_r) = \frac{\tilde{q}_D(s+r|x_0)}{1 - r\,\tilde{q}_D(s+r|x_0)} \underset{(7)}{=} \frac{1 - e^{-\alpha x_0}}{s + r e^{-\alpha x_0}} \qquad \alpha = \left(\frac{r+s}{D}\right)^{1/2} \qquad (12)$$

$$\text{MTA} \quad \tilde{q}(s=0) = T = \frac{e^{\alpha_0 x_0} - 1}{r} \qquad (13) \qquad \alpha_0 = \left(\frac{r}{D}\right)^{1/2}$$

$$T = \frac{e^{\alpha_0 x_0} - 1}{r}$$



as fn of r

$T(0) \to \infty$

$T(\infty) \to \infty$

min at $\frac{dT}{dr} = 0$

$$\frac{dT}{dr} = 0 \Rightarrow \frac{y}{2} = 1 - e^{-y} \qquad y = \alpha_0 x_0 = \frac{x_0}{(D/r)^{1/2}} = \frac{\text{distance to target}}{\text{typical length diffused between resets}}$$

unique sol^n $y^* = 1.5936\ldots$ optimal ratio of lengths

← Finish lecture I here →

IF Diff + ~~reset + target~~ survival prob

$$\tilde{q}(x, s|x_0) = \frac{1 - e^{-\alpha x_0}}{s + r e^{-\alpha x_0}} \quad (12)$$

Inversion

Lecture II

Recap $\quad x \quad X_T \quad \xrightarrow{D} \quad x_0 \quad x_T$

Survival prob LT

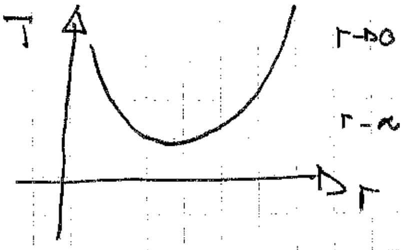
$$\tilde{q}(s|x_0) = \frac{1-e^{-\alpha x_0}}{s+re^{-\alpha x_0}} \qquad (12)$$

$$\alpha = \left(\frac{s+r}{D}\right)^{1/2}$$

take $x_T = 0 \quad x_r = x_0$

$$\to T = \tilde{q}(0|x_0) = \frac{e^{\alpha_0 x_0}-1}{r} \qquad (13) \quad (8)$$

$$\alpha_0 = \left(\frac{r}{D}\right)^{1/2}$$



$r \to 0 \quad T \simeq \frac{x_0}{r^{1/2} D^{1/2}} \to \infty$

$r \to \infty \quad T \simeq \frac{e^{\left(\frac{r}{D}\right)^{1/2} x_0}}{r} \to \infty$

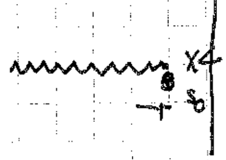
$$\frac{dT}{dr} = 0 \Rightarrow \frac{y}{2} = 1 - e^{-y} \qquad y = \alpha_0 x_0 = \frac{x_0}{(D/r)^{1/2}} = \frac{\text{distance to target}}{\text{typical length diffused between resets}}$$

unique solution $y>0$ $\qquad y^* = 1.5936$ optimal ratio of lengths

I.F Diff + reset + target survival prob

$$\text{Inversion} \quad q(t|x_0) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, e^{+st}\, \frac{1-e^{-\alpha x_0}}{s+re^{-\alpha x_0}}$$

LS



$$\text{pole at} \quad s_0 + r e^{-\left(\frac{r+s_0}{D}\right)^{1/2} x_0} = 0 \qquad (14)$$

gives dominant contribution

b.pt [check] $t^{1/2} e^{-rt}$ subdominant

$$\to q(t|x_0) \simeq e^{s_0 t}\left[1 - \exp -\left(\frac{r+s_0}{D}\right)^{1/2} x_0\right]$$

if $|s_0| \ll 1 \qquad s_0 \simeq -re^{-y} \qquad q(t|x_0) \simeq e^{-rt\, e^{-y}}$

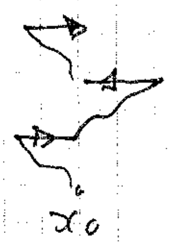
Aside on Gumbel distⁿ

N iid rv with pdf $f(x)$ $\qquad P_N(M) = \text{prob max} < M = \left[\int_{-\infty}^{M} dx\, f(x)\right]^N$

$$= \exp N \ln\left[1 - \int_M^\infty dx\, f(x)\right] \simeq \exp\left[-N\int_M^\infty dx\, f(x)\right]$$

if $f(x) \sim Ae^{-ax}$ x large $\qquad P_N(M) \simeq \exp\left[-\frac{NA}{a} e^{-aM}\right]$ $\quad$ Gumbel

Here



rt independent excursions $+ O(rt)^{1/2}$

max excursion to left $< x_0$

$$q \simeq \Big[\underbrace{\int_0^\infty dt\, e^{-rt} r\, q_D(t|x_0)}_{\text{average over duration}}\Big]^{rt}$$

$$= \left[1 - e^{-x_0 (r/D)^{1/2}}\right]^{rt} \qquad \text{using (7)}$$

$$\simeq \exp\left[-rt\, e^{-y}\right]$$

II Many Searchers

II A Many independent searchers beginning and resetting to x_i $\quad -L/2 \,|\, \cdot \; \cdot \; \cdot \; \underset{0}{\times} \; \cdot \; \dot{x}_i \; \cdot \; \cdot \,|\, L/2$

$i = 1 \ldots N$

$p(x_i)\,dx_i = \frac{dx_i}{L}$ uniform $\qquad \rho = \frac{N}{L}$ density

Survival prob. of target at origin $\quad Q(t|\{x_i\}) = \prod_{i=1}^{N} q(t|x_i)$

need to average over $\{x_i\}$

<u>Aside</u> "Quenched" and "annealed" averages

<u>Example</u> r.v. $\quad x = e^{aN}$ prob $1-p \qquad p = e^{-cN}$

$\qquad\qquad = e^{bN} \qquad p \qquad$ N large.

clearly 'typical' $x = e^{aN}$

but $\mathbb{E}[x] = (1-e^{-cN})e^{aN} + e^{(b-c)N} \simeq e^{(b-c)N}$ if $b-c > a$. 'annealed average'

but $\mathbb{E}[\ln x] = aN[1-e^{-cN}] + bN e^{-cN} \simeq aN$

so $e^{\langle \ln x \rangle}$ gives typical value "quenched average"

"annealed" ave $\quad \langle Q(t|\{x_i\})\rangle = \prod_{i=1}^{N} \langle q(t|x_i)\rangle = \exp N \ln \langle q(t|x_i)\rangle_x$

$$\langle q(t|x)\rangle_x = \frac{1}{L}\int_{-L/2}^{L/2} dx\, q(t|x) = \frac{2}{L}\int_0^{L/2} dx\,[1-(1-q(t|x))]$$

$$= 1 - \underbrace{\frac{2}{L}\int_0^{L/2} dx\,(1-q(t|x))}_{O(1)}$$

$$\therefore Q^{av} \simeq \exp N \ln\left[1 - \frac{2}{L}\int_0^{L/2} dx\,(1-q(t|x))\right] \xrightarrow[N/L=\rho]{N,L\to\infty} \exp -2\rho \int_0^{\infty} dx\,(1-q(t|x))$$

$$\equiv \exp -2\rho I_1(t)$$

$$Q^{typ} = \exp\langle \ln Q\rangle = \exp N\langle \ln q\rangle_x = \exp -2\rho I_2$$

$$I_2(t) = -\int_0^{\infty} dx\, \ln q(t|x) \quad [>0]$$

— half time break —

II B Diffusive searchers (no resetting)

$q(t|x) = \mathrm{erf}\left(\frac{x}{2(Dt)^{1/2}}\right)$ ⑥ $I_1 = 2\sqrt{Dt}^{1/2}\int_0^\infty du\,\mathrm{erfc}(u) = 2\sqrt{\frac{Dt}{\pi}}$ exercise.

$\mathrm{erfc}(u) = 1-\mathrm{erf}(u)$

$I_2 = -2\sqrt{Dt}\underbrace{\int_0^\infty du \ln \mathrm{erf}(u)}_{\text{converges}}$

$Q^{av,typ} = \exp[-\lambda^{av,typ}\rho\sqrt{Dt}]$ stretched exponential decay

IIC with resetting

$I_1 = \int_0^\infty dx\,(1-q(t|x))$ L.T. $\tilde{I}_1 = \int_0^\infty dx\left(\frac{1}{s} - \tilde{q}(s|x)\right)$

$= \int_0^\infty dx\left[\frac{(s+r)e^{-\alpha x}}{s(s+re^{-\alpha x})}\right] = \frac{r+s}{sr\alpha}\ln\left(\frac{s+r}{s}\right)$ sing at $s=0$ furthest to right

large time ~ small s $\simeq -\frac{1}{\alpha_0 s}\ln s + O(1/s)$

Now $\int_0^\infty dt\,e^{-st}\ln t = \frac{1}{s}\int_0^\infty du\,e^{-u}\ln\frac{u}{s} = -\frac{\gamma}{s} - \frac{\ln s}{s}$ $\gamma_E = -\int_0^\infty du\,e^{-u}\ln u$

$L^{-1}\left[-\frac{\ln s}{s}\right] = \ln t + \gamma_E$ $I_1 \simeq +\left(\frac{D}{r}\right)^{1/2}[\ln t + \gamma_E]$

$Q_{av} \sim$ const $\exp\left(-2\rho\left(\frac{D}{r}\right)^{1/2}\ln t\right)$ = const $t^{-2\rho(D/r)^{1/2}}$ Power law decay (slow)

exponent $(D/r)^{1/2}/(1/\rho)$ $\frac{\text{diffusive length}}{\text{spacing betwen searchers}}$

$\cancel{I_2 = -\int_0^\infty dx\,\mathrm{erf}\left(\frac{x}{2(Dt)^{1/2}}\right) = -2\sqrt{Dt}\int_0^\infty du\ln\mathrm{erf}(u)}$

$I_2 = -\int_0^\infty dx_0 \ln[q(x_0,t)]$ $t \gg 1$ $q(x_0,t) \sim e^{s_0(x)t}$

$s_0 = -r\,e^{-\left(\frac{r+s_0}{D}\right)^{1/2}x_0}$

$= -t\int_0^\infty dx_0\, s_0(x_0)$ $u = -\frac{s_0}{r}$ $u = \exp(-y(1-u)^{1/2})$ $y = \alpha_0 x_0$

$\frac{I_2}{t} = (Dr)^{1/2}\int_0^\infty dy\,u(y) = (Dr)^{1/2}\left\{\int_0^1 du\left|\frac{dy}{du}\right|u\right\}$ $y = -\frac{\ln u}{(1-u)^{1/2}}$

$= (Dr)^{1/2}\,4(1-\ln 2)$

so $Q_{av} \sim t^{-2\rho(D/r)^{1/2}}$ $Q_{typ} \sim \exp\left[-8\rho(Dr)^{1/2}[1-\ln 2]t\right]$

Explanation:
- average dominated by rare i.c. where searchers far from target
- strong memory of i.c. through resetting
- Q_{typ} decays faster than purely diffusive case

Lecture III : Generalisations of resetting and Optimization Problems

III A Resetting 2 Target distributions

- on resetting $x \to x_r$ ~~chosen~~ drawn from distribution $\mathcal{P}_r(x_r)$
  for simplicity choose distribution of i.c. (x_0) to also be $\mathcal{P}_r(x_0)$

- Having known target position is artificial, instead let
  $P_T(x_T)$ be the distribution of the target position.

Renewal equation I_e ⑪ (with absorption at fixed target x_T) generalizes to, when we integrate over $\mathcal{P}_r(x_r)$,

$$P(x,t|x_0,x_T) = e^{-rt} G_D(x,t|x_0,x_T) + r\int_0^\infty dt\, e^{-r(t-t')} \int_{-\infty}^{\infty} dx_r\, \mathcal{P}_r(x_r)\, G_D(x,t-t'|x_r,x_T) \times \int dx\, P(x,t'|x_0,x_T) \quad ①$$

Now integrate over i.c. x_0 with dist^{bn} $\mathcal{P}_r(x_0)$

$$P(x,t|x_T) = e^{-rt} G_D(x,t|x_T) + r\int_0^\infty dt\, e^{-r(t-t')} G_D(x,t-t'|x_T)\int dx\, P(x,t'|x_T)$$

where now $G_D(x,t|x_T) = \int dx_0\, \mathcal{P}_r(x_0)\, G_D(x,t|x_0,x_T)$ $\quad P(x,t|x_T) = \int dx_0\, \mathcal{P}_r(x_0)\, P(x,t|x_0,x_T)$

Finally integrate over x to get survival prob $q(t|x_T)$

$$q(t|x_T) = e^{-rt} q_D(t|x_T) + r\int_0^\infty dt\, e^{-r(t-t')} q_D(t-t'|x_T)\, q(t'|x_T) \quad ②$$

as before solve for L.T. $\tilde{q}(s|x_T)$

$$\boxed{\tilde{q}(s|x_T) = \frac{\tilde{q}_D(s+r|x_T)}{1 - r\tilde{q}_D(s+r|x_T)}} \quad ③$$

\# main result

$$q_D(t|x_T) = \int dx\, G_D(x,t|x,x_T)$$

$$T(x_T) = \frac{\tilde{q}_D(r|x_T)}{1 - r\tilde{q}_D(r|x_T)} \quad ④$$

$$= \frac{1}{r}\left[\frac{1}{\int dx_0\, \mathcal{P}_r(x_0)\, e^{-\alpha|x_0-x_T|}} - 1\right]$$

$$\tilde{q}_D(s|x_T) = \int dx_0\, \mathcal{P}_r(x_0)\, \frac{1 - e^{-\alpha_0|x_0-x_T|}}{r}$$

$$= \frac{1}{r} - \frac{1}{r}\int dx_0\, \mathcal{P}_r(x_0)\, e^{-\alpha_0|x_0-x_T|}$$

Now the stationary distribution in the absence of an absorbing target is

$$p^{*}(x) = \int dx_r \, \mathcal{P}_r(x_r) \, p^{*}(x|x_r) \qquad \text{using I.(1)}$$

$$= \frac{\alpha_0}{2}\int dx_r \, \mathcal{P}_r(x_r) \exp(-\alpha_0|x-x_r|) \qquad (5)$$

so $T(x_T) = \frac{1}{r}\left[\frac{\alpha_0}{2p^{*}(x_T)} - 1\right]$

Average over target distribution $P_T(x_T)$

$$\overline{T}(x_T) = \int dx_T \, P_T(x_T) \frac{1}{r}\left[\frac{\alpha_0}{2p^{*}(x_T)} - 1\right] \qquad (6)$$

III B Optimisation of MTA

Extremise $\overline{T}(x_T)$ for given $P_T(x_T)$ wrt $\mathcal{P}_r(x_r)$

assume $P_T(x_T)$ symmetric about 0

minimise $\int dx_T \frac{P_T(x_T)}{p^{*}(x_T)}$ subject to constraint

$$\frac{\delta}{\delta \mathcal{P}(x_r)}\left[\int dx \frac{P_T(x)}{p^{*}(x)} + \lambda \int dx \, \mathcal{P}_r(x)\right] = 0$$

$\uparrow$ Lagrange multiplier

$\rightarrow$ E-L eq^n $\quad \int dx \frac{P_T(x)}{[p^{*}_{opt}(x)]^2} e^{-\alpha_0|x_r - x|} = \frac{2\lambda}{\alpha_0}$

holds $\forall x_r \implies \frac{P_T(x)}{[p^{*}_{opt}(x)]^2} = \lambda$

$$\implies p^{*}_{opt}(x) = \frac{P_T^{1/2}(x)}{\lambda^{1/2}} = \frac{P_T^{1/2}(x)}{\int dx_T \, P_T^{1/2}(x_T)} \qquad (7)$$

so $p^{*}(x)$ should be <u>narrower</u> than target dist^n.

Solution of (5) for $\mathcal{P}_r(x_r)$

$$\mathcal{P}_r(x) = p^{*}_{opt}(x) - \frac{1}{\alpha_0^2}\frac{d^2 p^{*}_{opt}(x)}{dx^2} \qquad (8)$$

but need $\mathcal{P}_r(x_r) \geq 0$ for 'realisable p^{*}_{opt}

Example : Exponential target distbn $P_T(x) = \frac{\beta}{2} e^{-\beta|x_T|}$ small β broad

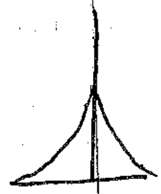
$\beta \to \infty \to \delta(x_T)$

(6)

if $p_r(x_r) = \delta(x_r)$ MFA diverges if $\alpha_0 > \beta$

small $\beta < 2\alpha_0$ (7) $\to$ $P^*_{opt}(x) = \frac{\beta}{4} e^{-\beta|x|/2}$

(8) $\to$ $p_r^{opt}(x_r) = \frac{\beta}{4} e^{-\beta|x_r|/2} \underbrace{\left[1 - \frac{\beta^2}{4\alpha_0^2}\right]}_{} + \frac{\beta^2}{4\alpha_0^2}\delta(x_r)$



$-$ve when $\beta > 2\alpha_0$

$\Rightarrow$ transition at $\beta = 2\alpha_0$ pure δ function is (locally) optimal

III C Time dependent resetting

Generalisations $r \to r(x)$ space dependent ; use Master equation

$r(t)$ time dependent

Let $r = r(\underbrace{t - t_e}_{\tau})$ t_e = time of last reset.

Prob of no resets up to t $= e^{-\int_0^t r(\tau)\, d\tau} \equiv e^{-R(t)}$

prob of next reset $t \to t+dt$ $= \left(-\frac{d}{dt} e^{-R(t)}\right) dt$ cf. ctstime r.w. "waiting time"

Noabsorption

last renewal $P(x,t|x_0) = e^{-R(t)} G_D(x,t|x_0) + \cancel{\int_0^t dt'\, r(t')\, e^{-R(t')} P(x,t-t'|x_0)}$

~~t' time~~ $+ \int_0^t dt'\, e^{-R(t-t')} \psi(t')\, G_D(x,t-t'|x_0)$

prob of last reset at t'

First renewal $P(x,t|x_0) = e^{-R(t)} G_D(x,t|x_0) + \int_0^t dt'\, \underbrace{r(t')\, e^{-R(t')}}_{\text{prob of 1st reset at } t'} P(x,t-t'|x_0)$

solve by L.T $\tilde{P}(x,s) = \dfrac{\hat{G}_D}{s\hat{A}}$ $\hat{G}_D = \int_0^\infty dt\, e^{-st-R(t)} G_D(x,t)$

$\hat{A} = \int_0^\infty dt\, e^{-st-R(t)}$

St. st $\lim_{s\to 0} [s\hat{P}]$ non-zero requires (N.&S.C) $\int_0^\infty e^{-R(t)} dt < \infty$

$\Rightarrow$ $r(\tau)$ decays more slowly than $\frac{1}{\tau}$

e.g. $r(\tau) \simeq \frac{A}{\tau}$ $R(t) \simeq A \ln t$ $e^{-R(t)} \sim t^{-A}$

$\int_0^\infty e^{-R(t)} dt$ converges if $A > 1$

~~Sur~~ Absorbing target $x_0 = x_r$

survival prob $\quad q(t|x_0) = e^{-R(t)} q_D(t|x_0) + \int_0^t dt'\, r(t') e^{-R(t')} q_D(t'|x_0)\, q(t-t'|x_c)$

$$\hat{q}(s|x_0) = \frac{\hat{q}_D(s|x_c)}{s\hat{q}_D(s|x_c) - \hat{k}(s|x_c)} \qquad \hat{q}_D = \int_0^\infty dt\, e^{-st-R(t)} q_D(t|x_c) \qquad \hat{k}_D = \int_0^\infty dt \;\; '' \;\; \frac{d}{dt} q_D(t|x_c)$$

$s \to 0$

$$T(x_0) = -\frac{\int_0^\infty dt\, e^{-R(t)} q_D(x_0, t)}{\int_0^\infty dt\, e^{-R(t)} \left[\frac{d}{dt} q_D(x_0, t)\right]} \qquad (9)$$

III D Optimal resetting rate

Extremise MTA wrt $r(t)$

E-L. $$\frac{\delta T}{\delta r(t')} = \frac{\int_{t'}^\infty dt\, e^{-R(t)} q_D(t|x_c)}{\hat{k}(0|x_c)} - \frac{\hat{q}(0|x_c)}{[\hat{k}(0|x_c)]^2} \int_{t'}^\infty dt\, e^{-R(t)} \frac{\partial q_D}{\partial t}$$

$= 0$ no solution which holds $\forall t'$

Conjecture is global optimum $\quad r(t) = \begin{cases} 0 & t < t^* \\ \infty & t > t^* \end{cases}$ (deterministic) (10)

(9) $\to$ $$T(x_0) = \frac{\int_0^{t^*} dt\, q(t|x_0)}{1 - q(t^*|x_c)} \qquad (11)$$ extremise

$t^* = \frac{x_0^2}{2D}\, 3.668$

$T = 2.671\ldots \frac{x_0^2}{2D}$

better than constant rate r_c $\qquad T = 3.0882 \frac{x_0^2}{2D}$

(10) ~~be~~ is local optimum of MTA subject to constant $r(t') > 0$

i.e. $\frac{\delta T}{\delta r(t')} > 0 \quad t < t^*$

$= 0 \quad t > t^*$