The "follow-the-leader" approximation of one-dimensional nonlinear transport equations

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Round Meanfield, crowd-opinion-cells, CNR Rome, 28/09/2022

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A 'detour' in traffic flow

- Consider *n vehicles* (particles) moving on a road, all in the same direction.
- Positions labelled by $x_1 < \ldots < x_n$.
- Positions are set in $\mathbb{R}$, $x_i \in \mathbb{R}$. Think of each position as an arc-length coordinate on a one-dimensional manifold.
- Positions depend on time, $x_i = x_i(t)$ for $i = 1, \ldots, n$.
- Let us describe their dynamics through a set of differential equations: we need to prescribe their *velocities*.

Differential equations

$$\dot{x}_i(t) = v_i(t) = \text{velocity of the } i\text{-th vehicle at time } t, \quad i = 1, \ldots, n$$

The above differential equations must be equipped with *initial conditions*

$$x_i(0) = \bar{x}_i = \text{position of the } i\text{-th vehicle at time } t = 0, \quad i = 1, \ldots, n.$$

How to model velocity?

- There are many ways to *prescribe* the velocities v_i.
- Likewise, there are many ways to *model* the movement of the n particles.

Standing assumption #1

We assume that all the velocities can be computed at any time *given the positions* of the n particles.

- In particular, we don't need to know the *initial velocities* in order to compute the solution.
- This is a *first order model*.
- There is no *inertia* in this model.

Standing assumption #2

Zero reaction time: particles adapt their velocities to their evolving positions instantaneously.

The follow-the-leader model

- For each $i = 1, \ldots, n-1$, set

$$R_i(t) = \frac{m}{d_i(t)} = \frac{m}{x_{i+1}(t) - x_i(t)}$$

where $m > 0$ is a fixed constant. m stands for a 'linear mass'.

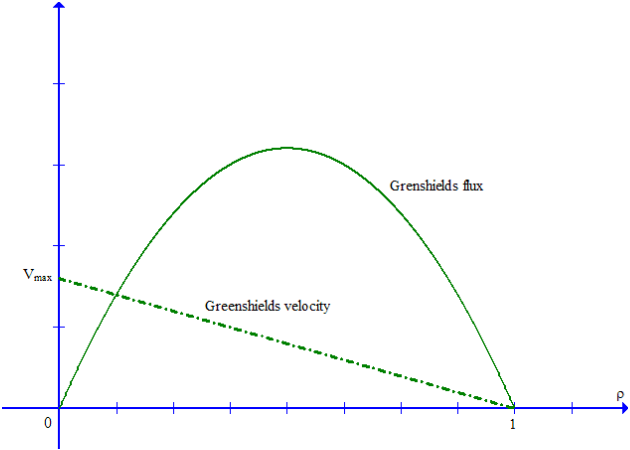
- R_i has the physical dimension of a *linear density*.
- We prescribe a *universal law* v according to which the velocity $\dot{x}_i$ depends on R_i.

$$\dot{x}_i(t) = v\left(R_i(t)\right).$$

- The velocity v is a *strictly decreasing* function of the density ρ.
- The quantity $f(\rho) = \rho v(\rho)$ is the *flux*.
- Since v decreases with ρ, $v(0) = v_{\max}$ is the *maximal velocity* (e.g. the speed limit in traffic flow).

Example of velocity law

- Greenshields (1935). $v(\rho) = v_{\max}(1 - \rho)$.



Local-in-time solution

- We prescribe the velocity law $v = v(\rho)$ once for all.
- We set the initial condition for the positions $x_i(0) = \bar{x}_i$, with $\bar{x}_1 < \ldots < \bar{x}_n$.
- The *leader* x_n doesn't have a preceding particle. Therefore, it solves the (totally decoupled) equation

$$\dot{x}_n(t) = v_{\max}\,, \qquad x_n(t) = \bar{x}_n + v_{\max} t\,.$$

- The system can be solved via *backward induction*:

$$\dot{x}_{n-1} = v\left(R_{n-1}(t)\right) = v\left(\frac{m}{x_n(t) - x_{n-1}(t)}\right), \quad x_{n-1}(0) = \bar{x}_{n-1}$$

  gives the solution for x_{n-1}. Inductively, given $x_{i+1}(t)$ we solve

$$\dot{x}_i = v\left(R_i(t)\right) = v\left(\frac{m}{x_{i+1}(t) - x_i(t)}\right), \quad x_i(0) = \bar{x}_{n-1}$$

  until we solve the whole system.
- To make sure the system has a unique local-in-time solution, we need to impose
  - All vehicles are not overlapping initially.
  - $v = v(\rho)$ is a Lipschitz function.

Global-in-time solution

The is only one (so far) possible reason for which the solution may cease to exist at a finite time: It is that *two or more particles collide*.
However, this never happens! Indeed, we can prove the following:

A discrete maximum principle

For all times $t \geq 0$ and for all $i = 1, \ldots, n-1$,

$$x_{i+1}(t) - x_i(t) \geq \min_{i=0,\ldots,n-1} (\bar{x}_{i+1} - \bar{x}_i) =: \underline{\ell}\,.$$

Alternative formulation:

$$R_i(t) \leq R_{\max} := \max_{i=1,\ldots,n-1} \frac{m}{\bar{x}_{i+1} - \bar{x}_i} = \max_{i=1,\ldots,n-1} R_i(0)\,.$$

This estimate provides a uniform-in-time bound for all densities $R_i(t)$, only depending on the maximal initial density.

Sketch of the proof

Case $i = n-1$:

$$x_n(t) - x_{n-1}(t) = \bar{x}_n - \bar{x}_{n-1} + \int_0^t (v_{\max} - v(R_{n-1}(s)))\, ds \geq \bar{x}_n - \bar{x}_{n-1}\,.$$

Assume for all $t \geq 0$

$$x_{i+2}(t) - x_{i+1}(t) \geq \underline{\ell}$$

and assume by contradiction that $x_{i+1}(t) - x_i(t)$ is

- $= \underline{\ell}$ at some time t_1,
- $< \underline{\ell}$ on $t \in (t_1, t_2)$.

For $t \in (t_1, t_2)$ we therefore have

$$v(R_{i+1}(t)) \geq v\left(\frac{m}{\underline{\ell}}\right)$$

$$v(R_i(t)) < v\left(\frac{m}{\underline{\ell}}\right)$$

$$\begin{aligned} x_{i+1}(t) - x_i(t) &= x_{i+1}(t_1) - x_i(t_1) + \int_{t_1}^t [v(R_{i+1}(s)) - v(R_i(s))]\, ds \\ &> x_{i+1}(t_1) - x_i(t_1) = \underline{\ell}\,, \end{aligned}$$

a contradiction.

Conclusion on FtL

- Particles are proven to preserve their order and to never collide.
- The discrete particle density is uniformly bounded in terms of the initial density.
- The maximum principle relies on the monotonicity of v and on the fact that particles 'look ahead'. A scheme in which vehicles 'look behind' would not have the same property.
- The model is a set of ODEs which can be implemented very easily numerically.

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The continuum approach

- Approach through *partial differential equations of transport type*.
- The ensemble of particles is treated as a fluid moving in a one-dimensional domain.
- Use of *Eulerian approach*. Observable variables are computed as functions of a space position x and time t.
- Relevant variables are
  - the density of vehicles $\rho(x,t)$,
  - the velocity $u(x,t)$.
- Assuming that the total *mass* of vehicles is constant, we use a *continuity equation*

$$\partial_t \rho(x,t) + \partial_x(\rho(x,t)u(x,t)) = 0\,.$$

- Similarly to the FtL, we prescribe a law for the velocity as a function of the density

$$u = v(\rho)\,.$$

Scalar conservation law

$$\rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho v(\rho), \qquad x \in \mathbb{R}, \quad t \geq 0.$$

Relevant parameters.

- Maximum density $\rho_{\max} > 0$. We normalize $\rho_{\max} = 1$.
- Maximum possible speed $v_{\max} > 0$.
- Total mass (constant in time) $\int_{\mathbb{R}} \rho(t,x)dx = L > 0$.

Main assumptions on the velocity map v

- $v \in C^1([0,1];[0,v_{\max}])$,
- v strictly decreasing on $[0,1]$,
- $v(0) = v_{\max}$, $v(1) = 0$.

Initial condition

- $\rho(t=0) = \bar{\rho} \in L^\infty(\mathbb{R})$, $\bar{\rho} \geq 0$, $\bar{\rho}$ with compact support.

A quick review of the mathematical theory.

- For smooth initial data, solution via *characteristics*

$$\rho_t + f'(\rho)\rho_x = 0$$
$$\dot{x}(t) = f'(\rho(x(t),t))$$
$$\frac{d}{dt}\rho(x(t),t) = 0$$

- Simple examples show that discontinuities (*shocks*) may form in finite time.
- Need a concept of *weak solution* (distributional): for all $\varphi \in C^1_c([0,+\infty) \times \mathbb{R})$,

$$\int_{\mathbb{R}} \int_0^{+\infty} \Big[\rho(t,x)\,\varphi_t(t,x) + f\left(\rho(t,x)\right)\varphi_x(t,x)\Big]\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \bar{\rho}(x)\,\varphi(0,x)\,\mathrm{d}x = 0$$

A quick review of the mathematical theory.

- Existence of weak solutions can be proven in many ways, for example via *vanishing viscosity*.
- Simple examples show *non uniqueness of weak solutions*.
- Concept of *entropy solution*, due to Oleinik 1963 and Kružkov 1970: for all test functions $\varphi \geq 0$ and for all $k \in \mathbb{R}$,

$$\int_{\mathbb{R}} \int_0^{+\infty} \Big[|\rho(t,x) - k| \, \varphi_t(t,x) + \mathrm{sgn}\,(\rho(t,x) - k) \, [f\,(\rho(t,x)) - f(k)] \, \varphi_x(t,x) \Big] \, \mathrm{d}t \, \mathrm{d}x + \int_{\mathbb{R}} \varphi(0,x) |\bar{\rho}(x) - k| \, \mathrm{d}x \geq 0 \tag{1}$$

- Entropy solutions are unique.
- *Oleinik condition*. Entropy solutions are characterised by

$$\rho \text{ is a weak solution} \quad \text{and} \quad f'(\rho)_x \leq \frac{1}{t}, \qquad \text{in } \mathcal{D}'((0,+\infty) \times \mathbb{R}).$$

Construct solutions in the entropy sense

Regularization strategy.

- Vanishing viscosity, see Dafermos (2000) and the references therein.

Numerical methods.

- Finite differences. Glimm (1965).
- Wave front tracking method (Dafermos 1972).

Mesoscopic approximation.

- Kinetic approximation. Lions, Perthame, Tadmor 1994.

Microscopic probabilistic approach.

- Exclusion processes (list incomplete!). Rost (1982), Ferrari and Fouque (1987), Ferrari (1986).

Heuristics.

- In the LWR model, the velocity in the continuity equation is $u = v(\rho)$.
- A *Lagrangian* version of LWR is

$$\dot{x}(t) = v(\rho(x(t), t))$$

  where $x(t)$ are the 'particles trajectories'.
- This suggests the FtL as a 'many particle' approximation for the LWR model.

A new method: the FtL approximation[1]

- $\bar{\rho} \in L^\infty_c$ initial condition with unit mass.
- Fix $N \in \mathbb{N}$. Set $m = 1/N$ (mass of each particle).
- Atomization of $\bar{\rho}$:

$$\bar{x}_N = \sup(\mathrm{spt}(\bar{\rho}))$$

$$\bar{x}_i \quad \text{defined such that} \quad \int_{\bar{x}_i}^{\bar{x}_{i+1}} \bar{\rho}(y)dy = m, \qquad i = 0, \dots, N-1.$$

- Consider $N+1$ *ordered particles* $x_0, x_1, \dots, x_N$ with mass m each, with initial positions $\bar{x}_i$, $i = 0, \dots, N$.
- Let $x_0, \dots, x_N$ evolve by

$$\dot{x}_i(t) = v\left(\frac{m}{x_{i+1}(t) - x_i(t)}\right), \qquad i = 0, \dots, N-1,$$

$$\dot{x}_N(t) = v_{\max},$$

[1] DF-Rosini, ARMA 2016

Atomization of the initial condition

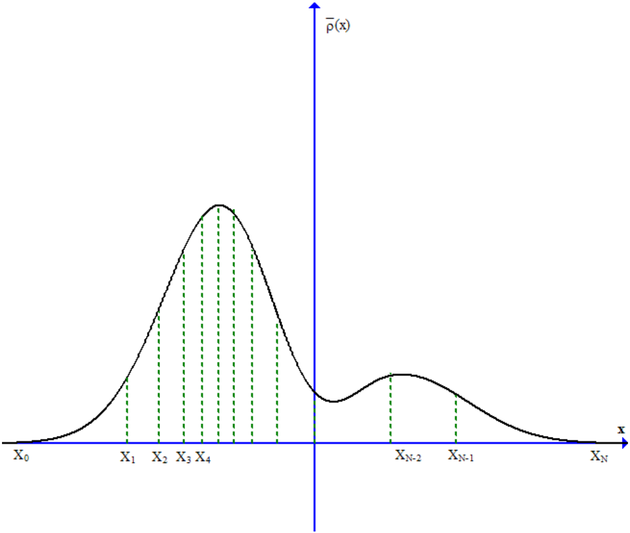

Figure: The initial condition is split into N parts with equal integral.

Many-particle limit

Empirical measure

$$\tilde{\rho}^N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}$$

Piecewise constant density

$$\hat{\rho}^N(t,x) = \sum_{i=0}^{N-1} R_i^N(t)\chi_{[x_i(t),x_{i+1}(t))}(x)$$

$$R_i^N(t) = \frac{1}{N(x_{i+1}(t) - x_i(t))}$$

Problem:
Prove that $\tilde{\rho}^N(t)$ and $\hat{\rho}^N(t,\cdot)$ converge to the unique entropy solution ρ to the LWR equation with $\bar{\rho}$ as initial condition.

Convergence Theorem

Theorem

Assume either

- $\bar{\rho} \in BV(\mathbb{R})$,

or

- $[0,1] \ni \rho \mapsto \rho v'(\rho)$ *non-increasing (Lagrangian convexity assumption)*

Then,

- *the discrete density* $\hat{\rho}^N$ *converges to* ρ *the unique entropy solution to the LWR equation with initial condition* $\bar{\rho}$ *almost everywhere and in* $L^1_{loc}([0,+\infty) \times \mathbb{R})$.
- *the empirical measure* $\tilde{\rho}^N$ *converges to the same* ρ *in the topology* $L^1_{loc}([0,+\infty); d_{1,L})$, *where* $d_{1,L}$ *is the* 1*-Wasserstein distance.*

Lagrangian convexity assumption

$$\underbrace{\rho_t + v(\rho)\rho_x}_{\frac{D\rho}{Dt}\ \text{material derivative}} + \overbrace{\rho v'(\rho)}^{\text{derivative of the Lagrangian flux}} \rho_x = 0$$

Strategy of the proof

(i) Discrete maximum principle: the discrete density $\hat{\rho}^N$ is uniformly bounded w.r.t. N.

(ii) Strong compactness of $\hat{\rho}^N$:

- Case $\bar{\rho} \in BV$: prove direct uniform BV estimate of $\hat{\rho}^N$.
- 'Lagrangian' convex case: prove a *discrete Oleinik condition* for $\hat{\rho}^N$.

(iii) Consistency: Prove that ρ satisfies the entropy inequality up to a negligible error.

Discrete equations for the density (very useful):

$$\dot{R}_i^N(t) = -NR_i^N(t)^2 \left(v(R_{i+1}^N(t)) - v(R_i^N(t)) \right), \qquad i = 0, \dots, N-2,$$
$$\dot{R}_{N-1}^n(t) = -NR_{N-1}^N(t)^2 \left(v_{\max} - v(R_{N-1}^N(t)) \right).$$

BV contraction for BV initial data

Proposition (BV contractivity for BV initial data)

Assume $\bar{\rho} \in BV$. Then for any $n \in \mathbb{N}$

$$TV\left[\hat{\rho}^N(t)\right] \leq TV\left[\bar{\rho}\right] \qquad \text{for all } t \geq 0.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t} TV\left[\hat{\rho}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[R_0 + R_{N-1} + \sum_{i=0}^{N-2} |R_i - R_{i+1}|\right]$$

$$= \dot{R}_0 + \dot{R}_{N-1} + \sum_{i=0}^{N-2} \operatorname{sgn}\left[R_i - R_{i+1}\right]\left[\dot{R}_i - \dot{R}_{i+1}\right]$$

$$= \left[1 + \operatorname{sgn}\left[R_0 - R_1\right]\right]\dot{R}_0 + \left[1 - \operatorname{sgn}\left[R_{N-2} - R_{N-1}\right]\right]\dot{R}_{N-1}$$

$$+ \sum_{i=1}^{N-2}\left[\operatorname{sgn}\left[R_i - R_{i+1}\right] - \operatorname{sgn}\left[R_{i-1} - R_i\right]\right]\dot{R}_i .$$

BV contraction for BV initial data

Proof (continued): We use that v is *non-increasing*:

$$\left[1+\operatorname{sgn}\left[R_0-R_1\right]\right]\dot{R}_0=-\left[1+\operatorname{sgn}\left[R_0-R_1\right]\right]\frac{R_0^2}{\ell}\left[v(R_1)-v(R_0)\right]\leq 0,$$

$$\left[1-\operatorname{sgn}\left[R_{N-2}-R_{N-1}\right]\right]\dot{R}_{N-1}=-\left[1-\operatorname{sgn}\left[R_{N-2}-R_{N-1}\right]\right]\frac{R_{N-1}^2}{\ell}\left[v_{\max}-v(R_{N-1})\right]\leq 0,$$

$$\left[\operatorname{sgn}\left[R_i-R_{i+1}\right]-\operatorname{sgn}\left[R_{i-1}-R_i\right]\right]\dot{R}_i$$

$$=-\left[\operatorname{sgn}\left[R_i-R_{i+1}\right]-\operatorname{sgn}\left[R_{i-1}-R_i\right]\right]\frac{R_i^2}{\ell}\left[v(R_{i+1})-v(R_i)\right]\leq 0.$$

Therefore, $TV\left[\hat{\rho}(t)\right]\leq TV\left[\bar{\rho}\right]$ for all $t\geq 0$.

Discrete Oleinik condition

Lemma (Discrete Oleinik-type condition)

Assume v satisfies $\rho \mapsto \rho v'(\rho)$ non-increasing. Then, for any $i = 0, \ldots, N_n - 2$ we have

$$t\, R_i^N(t) \left[v\left(R_{i+1}^N(t)\right) - v\left(R_i^N(t)\right)\right] \leq \ell \qquad \text{for all } t \geq 0. \tag{2}$$

Condition (2) in terms of $x_i(t)$

$$\frac{v\left(R_{i+1}^N(t)\right) - v\left(R_i^N(t)\right)}{x_{i+1}(t) - x_i(t)} \leq \frac{1}{t} \qquad \text{for all } t \geq 0. \tag{3}$$

(3) is a discrete counterpart of

$$v(\rho)_x \leq \frac{1}{t}.$$

Recall the sharp Oleinik condition for the scalar conservation law (cf. Hoff 1983)

$$f'(\rho)_x = (v(\rho) + \rho v'(\rho))_x \leq \frac{1}{t}.$$

Entropy solutions in the limit

Entropy condition

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big[\Big|\hat{\rho}^N(t,x)-k\Big|\,\varphi_t(t,x)$$
$$+\,\mathrm{sgn}(\hat{\rho}^N(t,x)-k)\,[f(\hat{\rho}^n(t,x))-f(k)]\,\varphi_x(t,x)\Big]\,\mathrm{d}x\,\mathrm{d}t\geq o(1)$$

- What makes us obtain this inequality is the *upwind direction* of the FtL scheme.
- Here the upwind direction is determined by the sign of v', hence by the sign of the Lagrangian characteristic velocity $\rho v'(\rho)$.
- Unlike classical numerical schemes, here we are *not tracking shock waves*.
- Oleinik condition makes this result catch the L^∞-BV smoothing effect of a nonlinear conservation law with a strictly concave flux.

Remarks

- Alternative approach: *specific volume* as key Lagrangian variable instead of the density. This leads to the formulation of a Lagrangian conservation law. However, an estimate of the specific volume $\tau(t,x)$ requires an initial estimate for $\|\tau(0,\cdot)\|_{L^\infty}$, which is only possible if the initial density is detached from zero.
- Having the specific volume as key variable and assuming BV initial data implies one can rely on a well known result by Wagner, which allows to switch from Lagrangian to Eulerian variables after passing to the limit as $N \to +\infty$. With our approach, the use of Wagner's result is far from being trivial, that's why we prefer to prove the entropy condition via a direct evaluation of the entropy condition.
- Holden and Risebro published two papers on this method in 2018, one on the time-step discretisation of the scheme away from the vacuum, another one with a partly alternative proof of our result.

Numerical simulations - Rarefaction wave

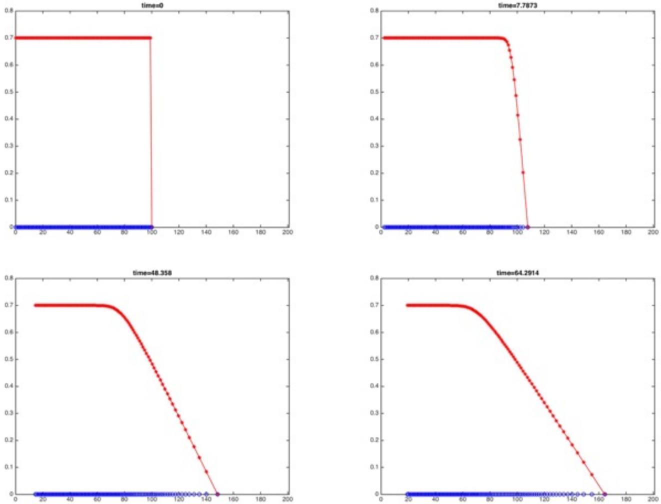

Figure: Initial density with a decreasing jump

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Extensions and applications.

- DF-Fagioli-Rosini-Russo 2018: extension of the result to bounded interval with Dirichlet BC.
- DF-Fagioli-Rosini 2017: extension of the method to the Aw-Rascle-Zhang model for traffic flow (second order).
- DF-Stivaletta 2020: case of x-dependent velocities.
- DF-Stivaletta 2022: we caught the sharp version of the Oleinik condition for power-law fluxes.

In the next we will briefly review two extentions:

(i) to the Hughes model for pedestrian movements,

(ii) to a nonlocal transport equation with applications in biology.

Hughes' model for pedestrian movements

$$\rho_t - \mathrm{div}\left(\rho v(\rho)\frac{\nabla\varphi}{\|\nabla\varphi\|}\right) = 0\,, \quad \|\nabla\varphi\| = c(\rho)\,.$$

- $x \in \Omega$, $t \geq 0$
- $\rho(t,x)$ density of pedestrians
- potential $\varphi(t,x)$, estimated exit time
- $v(\rho)$ decreasing velocity map
- $c(\rho)$ increasing (and convex) running cost function
- suitable boundary conditions in order to model exits and walls

Model formulated by Roger L. Hughes (2002). No existence theory. Big problem: discontinuity in the velocity field.

The one-dimensional theory

- One-dimensional interval $x \in [-1, 1]$
- Dirichlet boundary conditions $\rho = 0$ at ± 1 (in the sense of [Bardos et al. 1979])
- Two exits: $\varphi = 0$ at $x = \pm 1$
- The eikonal equation can be explicitly solved in the class of semi-concave functions. One obtains

$$\int_{-1}^{\xi(t)} c(\rho(t,x))dx = \int_{\xi(t)}^{1} c(\rho(t,x))dx\,,$$

  assuming $\varphi(t,\cdot)$ continuous at $x = \xi(t)$.
- Continuity equation becomes

$$\rho_t + (\mathrm{sign}(x - \xi(t))\rho v(\rho))_x = 0\,.$$

- $\xi(t)$ is the *turning point*. It depends non-locally on ρ.

A Follow-The-Leader approach

Hinted by the previous results on conservation laws, we consider

$$\begin{cases} \dot{x}_0 = -v(0) \\ \dot{x}_i = -v\left(\frac{\ell}{x_i - x_{i-1}}\right) & i = 1, \dots, I_0 \\ \dot{x}_i = v\left(\frac{\ell}{x_{i+1} - x_i}\right) & i = I_0 + 1, \dots, N-1 \\ \dot{x}_N = v(0) \end{cases}$$

$$[x_0 + 1] + \sum_{i=1}^{I_0+1} [x_{i+1} - x_i] c\left(\frac{\ell}{x_{i+1} - x_i}\right) + [\xi(t) - x_{I_0}]$$

$$= [x_{I_0+1} - \xi(t)] + \sum_{i=I_0+1}^{N-1} [x_{i+1} - x_i] c\left(\frac{\ell}{x_{i+1} - x_i}\right) + [1 - x_N]$$

The latter is a discretization of the integral equation defining $\xi(t)$. It is used as long all particles belong to the interval $[-1, 1]$. The exiting particles are removed by the formula. The index I_0 is uniquely determined at $t = 0$ as the largest integer $I \in \{0, \dots, N\}$ such that

$$[x_0 + 1] + \sum_{i=1}^{I_0+1} [x_{i+1} - x_i] c\left(\frac{\ell}{x_{i+1} - x_i}\right) < \sum_{i=I_0+1}^{N-1} [x_{i+1} - x_i] c\left(\frac{\ell}{x_{i+1} - x_i}\right) + [1 - x_N]$$

Results

- If $c = 1$ then $\varphi = \mathrm{dist}(x, \{-1, 1\})$. Two independent one-sided LWR problems are generated.
- Relevant example: $c(\rho) = 1/v(\rho)$. Studied (together with other nontrivial cases) in
   (i) [DF, Markowich, Pietschmann, and Wolfram - JDE 2011]. Viscous regularization
   (ii) [Amadori, DF - Acta M. Sci. 2012], [El-Khatib, Goatin, and Rosini - Z. Angew. Math. Phys. 2013]. Riemann type solutions
   (iii) [Goatin, and Mimault - SIAM J. Sci. Comput. 2013], [Twarogowska, Goatin, and Duvigneau - Appl. Math. Model. 2014]. Numuerical simulations
   (iv) [Amadori, Goatin, and Rosini - J. Math. Anal. Appl. 2014]. Existence in a *small data* setting.
   (v) [DF-Fagioli-Rosini-Russo 2018]. Convergence of the FtL scheme to solutions for which existence is known.

(vi) [Andreianov-Rosini-Stivaletta, preprint]. Existence of entropy solutions in the case of linear cost via FtL approximation.

A nonlocal transport model

We study[2]

$$\begin{aligned} \partial_t \rho - \partial_x(\rho v(\rho) K' * \rho) = 0 \\ \rho(t=0) = \bar{\rho} \in L^\infty_c(\mathbb{R}\,;\,[0,\rho_{\max}]) \cap BV(\mathbb{R}) \end{aligned}$$

Assumptions on K

$K \in C^2(\mathbb{R})$, $K(-x) = K(x)$, $K' > 0$ on $(0,+\infty)$, $K'' \in \mathrm{Lip}_{loc}(\mathbb{R})$.

Assumptions on v

$v \in C^1([0,+\infty))$, v decreasing on $[0,\rho_{\max}]$, $v \equiv 0$ on $[\rho_{\max},+\infty)$.

Same atomization algorithm as before $\Rightarrow$ Initial positions of $N+1$ particles $\bar{x}_0 < \ldots < \bar{x}_N$.

[2] DF, Fagioli, Radici - JDE 2019

The discrete model

$$\dot{x}_i(t) = -\frac{1}{N} \underbrace{v(R_i(t))}_{\text{forward density}} \sum_{j>i} K'(x_i(t) - x_j(t)) - \frac{1}{N} \underbrace{v(R_{i-1}(t))}_{\text{backward density}} \sum_{j<i} K'(x_i(t) - x_j(t))$$

Discrete density: $$R_i(t) = \frac{1}{N(x_{i+1}(t) - x_i(t))}$$

Properties:

- Discrete maximum principle $x_{i+1}(t) - x_i(t) \geq \frac{M}{\rho_{\max} N}$, i.e. $R_i(t) \leq \rho_{\max}$
- Global existence
- $x_0(t) \geq \bar{x}_0$, $x_N(t) \leq \bar{x}_N$ (confined support)

Estimates

$$\rho^N(t,x) = \sum_{i=0}^{N-1} R_i(t)\chi_{[x_i(t),x_{i+1}(t))}$$

$$TV[\rho^N(\cdot,t)] = R_0(t) + \sum_{i=0}^{N-2} |R_{i+1}(t) - R_i(t)| + R_{N-1}(t)$$

Uniform BV estimate

There exists a constant $C > 0$ depending only on K, v, and $\mathrm{meas}(\mathrm{supp}[\bar{\rho}])$, such that

$$TV[\rho^N(\cdot,t)] \leq TV[\bar{\rho}]e^{Ct}, \quad \text{for all } t \geq 0.$$

- The proof crucially uses the monotonicity of v and the splitting of the use of 'upwind' densities in the velocity field.
- The overcrowding prevention effect is also crucial: without it, particles would collapse into a single point mass.

Convergence of the scheme

- The previous BV-estimate allows to control space-oscillations.
- As for time-oscillations, we prove Lipschitz equi-continuity in the 1-Wasserstein distance
$$W_1(\rho^N(t,\cdot),\rho^N(s,\cdot)) \leq C|t-s|,$$
with $C>0$ independent of N.
- Rossi-Savaré 2003 (Aubin/Lions-type compactness theorem) implies strong compactness of ρ^N in $L^1([0,+\infty)\times\mathbb{R})$.

Entropy solutions

Similarly to scalar conservation laws, we define

Definition

$\rho : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is an entropy solution with initial condition $\bar{\rho}$ if $\rho \in L^\infty([0, +\infty); L^1 \cap L^\infty(\mathbb{R}))$ and, for all constants $c \geq 0$ and for all $\varphi \in C_c([0, +\infty) \times \mathbb{R})$ with $\varphi \geq 0$ one has

$$\int_{\mathbb{R}} |\bar{\rho}(x) - c|\varphi(0, x)dx + \int_0^{+\infty} \int_{\mathbb{R}} (|\rho - c|\varphi_t \\ - \mathrm{sign}(\rho - c) \left[(f(\rho) - f(c))K' * \rho\varphi_x - f(c)K'' * \rho\varphi\right]) \, dxdt \geq 0,$$

where $f(z) = zv(z)$.

Notice that entropy solutions are weak solutions.

Theorem

- There exists no more than one entropy solution with initial condition $\bar{\rho}$
- $\rho^N \to \rho$ as $N \to +\infty$ and ρ is an entropy solution.

Non uniqueness of weak solutions

Consider $v(\rho) = (1-\rho)_+$ and the initial condition

$$\bar{\rho}(x) = \chi_{[-1,-1/2]} + \chi_{[1/2,1]}.$$

Let $\rho_s(t,x) = \bar{\rho}(x)$ for all $t \geq 0$.

- ρ_s is a (stationary) weak solution.
- ρ_s is *not* an entropy solution. Proof: use test functions that concentrate around $-1/2$ and $1/2$ to violate the entropy condition. Extra assumption needed: $K'' > 0$ on the support of $\bar{\rho}$.
- We know the scheme converges to an entropy solution, therefore there are *at least two weak solutions* with this initial condition.
- Why is ρ_s not satisfying the entropy condition: the discontinuities at $\pm 1/2$ are *not admissible*.
- The scheme catches this behavior because particles at $\pm 1/2$ are forced to *move*.

End of the talk

Thanks for your attention!