Some results related to consensus formation in graphon dynamics

(in collaboration with N. Pouradier Duteil and M. Sigalotti)

Benoît Bonnet-Weill
CNRS Reseacher – LAAS-CNRS

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Ypatia Laboratory of Mathematical Sciences

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Outline of the talk

Multi-agent systems – From microscopic to macroscopic models

Review of consensus methods for microscopic cooperative systems

Consensus analysis in the context of graphon dynamics

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Multi-agent systems – *Finite dimensional setting*

Multi-agents dynamics can be described by **systems of ODEs**

$$\dot{x}_i(t) = \boldsymbol{v}_i(t, \boldsymbol{x}(t), x_i(t)), \qquad i \in \{1, \dots, N\},$$

where

- $\boldsymbol{x} = (x_1, ..., x_N) \in (\mathbb{R}^d)^N$ encodes the **states** of the agents,
- $\boldsymbol{v}_i : [0, T] \times (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}^d$ are **non-local velocity fields**.

Breadcrumb trail example (Time-dependent cooperative dynamics)

$$\boldsymbol{v}_i(t, \boldsymbol{x}, x_i) = \frac{1}{N} \sum_{j=1}^{N} a_{ij}(t) \phi(|x_i - x_j|)(x_j - x_i).$$

Central observation (pattern formation)

Simple **microscopic interactions** $\rightsquigarrow$ rich **macroscopic structures**.

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- **Flocking** (everybody goes in the same direction)
- **Synchronisation** (periodic motions arise in the system)

Macroscopic approximations (Main motivations)

- Interest for **global** patterns, i.e. that involve **many agents**,
- N is usually **very large** $\rightsquigarrow$ **numerical issues**

Today: Consensus for **micro** and **macro** cooperative systems.

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Multi-agent systems – *General cooperative dynamics*

We consider the **cooperative** dynamics

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{ij}(t)\phi(|x_i(t) - x_j(t)|)(x_j(t) - x_i(t)),$$

where

- $\phi \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ encodes **distance-based interactions**,
- $a_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0,1])$ represent **communication links**.

Definition (Graph-Laplacian operators)

The **graph-Laplacian** $\boldsymbol{L}_N(t, \boldsymbol{x}) : (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N$ is defined by

$$\boldsymbol{L}_N(t,\boldsymbol{x})\boldsymbol{y} = \left(\frac{1}{N}\sum_{i=1}^{N} a_{ij}(t)\phi(|x_i - x_j|)(y_j - y_i)\right)_{1\leq i\leq N}.$$

$\hookrightarrow$ Semilinear reformulation of the dynamics

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}_N(t, \boldsymbol{x}(t))\boldsymbol{x}(t). \qquad \text{(CS)}$$

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Multi-agent systems – *Reformulation as graphons (1)*

Idea: Study consensus for the **mean-field approximation**

$$\partial_t \mu_N(t) + \mathrm{div}_x\big(\Phi(t) \star \mu_N(t)\mu_N(t)\big) = 0$$

[Ha&Liu'09, Carrillo,Fornasier,Rosado&Toscani'10, Piccoli,Rossi&Trélat'15].

Problem: Mean-field **needs indistinguishability**, i.e. $a_{ij}(t) = 1$.

Definition (Graph limit)[LS'07,M'14]

Given a solution $x(\cdot)$ of (CS), define the **piecewise constant** maps

$$i \in I \mapsto x_N(t,i) := \sum_{k=1}^{N} x_k(t)\, \mathbb{1}_{\left[\frac{k-1}{N},\frac{k}{N}\right)}(i)$$

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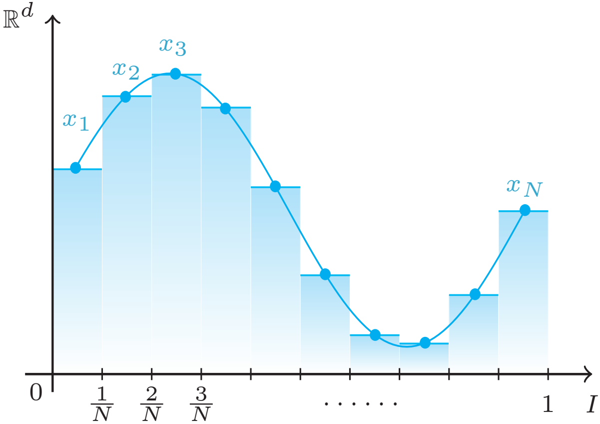

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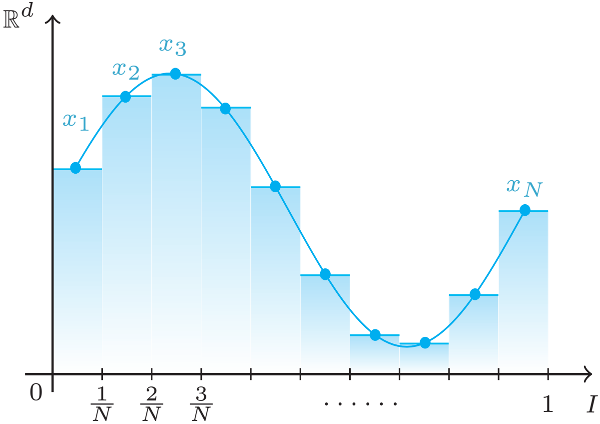

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Consensus – *Definition and main ideas*

Definition (Asymptotic consensus formation)

A solution $\boldsymbol{x}(\cdot)$ of (CS) converges to **consensus** if

$$\lim_{t \to +\infty} |x_i(t) - x^\infty| = 0,$$

for all $i \in \{1, \dots, N\}$ and some $x^\infty \in \mathbb{R}^d$.

Idea: **Quantitative** convergence results $\rightsquigarrow$ **Lyapunov** methods!

Definition (Candidate energy functionals)

We define the **variance functional**

$$\mathcal{V}(\boldsymbol{x}) := \frac{1}{2N} \sum_{i=1}^{N} |x_i - \bar{\boldsymbol{x}}|^2 \qquad (\ell_2\text{-convergence}),$$

and the **diameter**

$$\mathcal{D}(\boldsymbol{x}) := \max_{i,j \in \{1,\dots,N\}} |x_i - x_j| \qquad (\ell_\infty\text{-convergence}),$$

$\hookrightarrow$ Decay prescribed by two **intrinsic scalar quantities**.

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Definition (Scrambling coefficient)[Seneta'79]

The **scrambling** of a graph $\boldsymbol{A}_N = (a_{ij})_{i,j=1}^N$ satisfying $a_{ii} = 1$ is

$$\eta(\boldsymbol{A}_N) := \min_{1 \leq i,j \leq N} \frac{1}{N} \left(\sum_{k=1, k \neq i,j}^{N} \min \left\{ a_{ik}, a_{jk} \right\} + a_{ij} + a_{ji} \right)$$

$\hookrightarrow$ Positive if each (i,j) either **interact** or **follow** the same k.

Theorem (Quantitative diameter decay)[Motsch&Tadmor'14]

For each $\boldsymbol{x}^0 \in (\mathbb{R}^d)^N$, there exists $\phi_0 > 0$ such that

$$\mathcal{D}(\boldsymbol{x}(t)) \leq \mathcal{D}(\boldsymbol{x}^0) \exp \left(- \phi_0 \int_0^t \eta(\boldsymbol{A}_N(s)) \mathrm{d}s \right).$$

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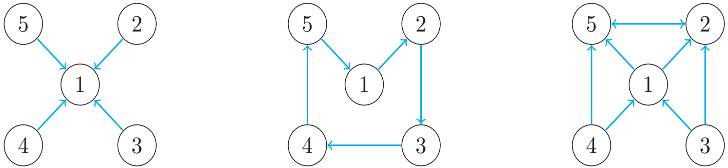

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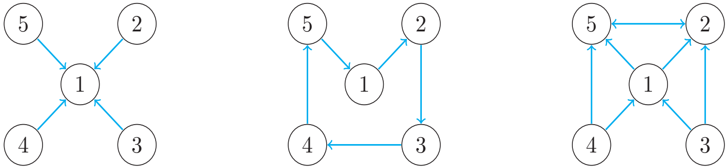

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Consensus – *Fiedler number and variance estimates*

Definition (Algebraic connectivity of a graph)[Fiedler'73, Mohar'91]

The **Fiedler number** of a symmetric graph $\boldsymbol{A}_N = (a_{ij})_{i=1}^N$ is

$$\lambda_2(\boldsymbol{A}_N) = \inf_{\boldsymbol{x} \in \mathscr{C}_N^\perp} \frac{\langle \boldsymbol{L}_N \boldsymbol{x}, \boldsymbol{x} \rangle_N}{|\boldsymbol{x}|_N^2},$$

where $(\boldsymbol{L}_N \boldsymbol{x})_i = \frac{1}{N} \sum_{j=1}^N a_{ij}(x_i - x_j)$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_N := \frac{1}{N} \sum_{i=1}^N \langle x_i, y_i \rangle$ and

$$\mathscr{C}_N := \left\{ \boldsymbol{x} \in (\mathbb{R}^d)^N \text{ s.t. } x_1 = \cdots = x_N \right\}$$

is the **consensus manifold**.

Theorem (Quantitative variance decay)[Motsch&Tadmor'14]

Suppose that $\boldsymbol{A}_N(t) = (a_{ij}(t))_{i,j=1}^N$ is symmetric for a.e. $t \in \mathbb{R}_+$. Then for each $\boldsymbol{x}^0 \in (\mathbb{R}^d)^N$, there exists $\phi_0 >$ such that

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Consensus – *Idea of the proof in the symmetric case*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(\boldsymbol{x}(t)) = \frac{1}{N}\sum_{i=1}^{N}\left\langle \dot{x}_i(t), x_i(t) - \bar{\boldsymbol{x}}\right\rangle$$

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Theorem (Characterisation of graph connectivity)[Mohar'91]

A symmetric graph $\boldsymbol{A}_N = (a_{ij})_{i,j=1}^N$ is **strongly connected**, i.e.

for all i, j there exists $i = k_1, \dots, k_m = j$ s.t. $a_{k_l k_{l+1}} > 0$

if and only if $\lambda_2(\boldsymbol{A}_N) > 0$.

Question: What happens when $\boldsymbol{A}_N$ is not symmetric ?

Theorem (Characterisation of graph connectivity)[Wu'05]

A graph $\boldsymbol{A}_N$ is a **disjoint union** of **str. connected components** ("DUSCC") **if and only if** there exists $(v_1, \dots, v_N) \in (\mathbb{R}_+^*)^N$ s.t.

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Remark: $\boldsymbol{L}_N^*(1, \dots, 1) = 0$ if $\boldsymbol{A}_N$ is symmetric $\rightsquigarrow$ always DUSCC!

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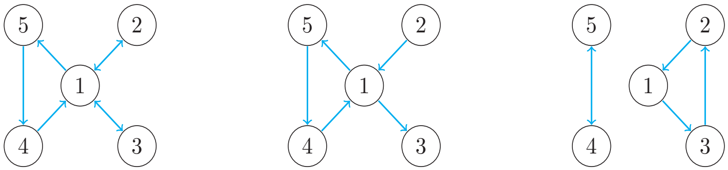

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The **algebraic connectivity** of a DUSCC graph $\boldsymbol{A}_N$ is

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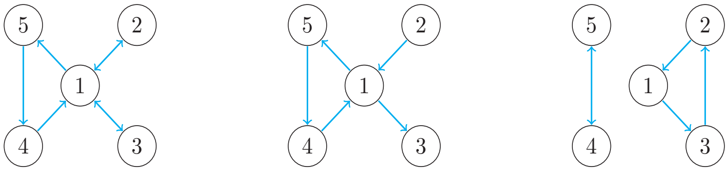

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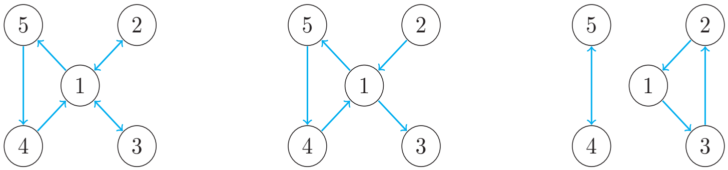

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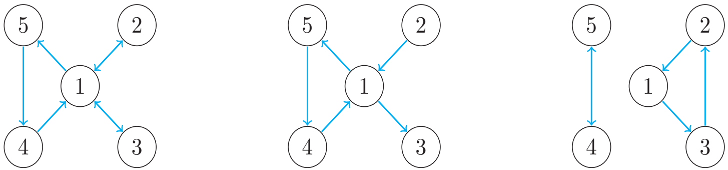

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Outline of the talk

Multi-agent systems – From microscopic to macroscopic models

Review of consensus methods for microscopic cooperative systems

Consensus analysis in the context of graphon dynamics

Graphon dynamics – *Adjacency and graph-Laplacian*

We consider the graphon dynamics

$$\partial_t x(t,i) = \int_I a(t,i,j)\phi(|x(t,i)-x(t,j)|)(x(t,j)-x(t,i))\mathrm{d}j$$

where $a(t) \in L^\infty(I\times I,[0,1])$ represents the **communications**.

Definition (Adjacency, degree and graph-Laplacian operators)
We define the **adjacency** operator by

$$\mathcal{A}(t,x)\,y : i \in I \mapsto \int_I a(t,i,j)\phi(|x(i)-x(j)|)y(j)\mathrm{d}j,$$

as well as the **graph-Laplacian** $\mathbb{L}(t,x) : L^2(I,\mathbb{R}^d) \to L^2(I,\mathbb{R}^d)$

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$$\dot{x}(t) = -\mathbb{L}(t,x(t))x(t). \qquad \text{(GD)}$$

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We define the **scrambling coefficient** of a graphon $\mathcal{A}$ by

$$\eta(\mathcal{A}) := \inf_{i,j \in I} \int_I \min\{a(i,k), a(j,k)\} \mathrm{d}k,$$

as well as the **diameter** of a map $x \in L^\infty(I, \mathbb{R}^d)$

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Theorem (Quantitative diameter decay)[BPDS'22]

For each $x^0 \in L^\infty(I, \mathbb{R}^d)$, there exists $\phi_0 > 0$ such that

$$\mathcal{D}(x(t)) \leq \mathcal{D}(x^0) \exp\left(-\phi_0 \int_0^t \eta(\mathcal{A}(s)) \mathrm{d}s \right).$$

Two technical novelties

- ◇ No **stochastic normalisation** trick ⇝ **Geometric** argument.
- ◇ $t \mapsto \mathcal{D}(x(t))$ not diff. ⇝ approx. with **Scorza-Dragoni**.

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$$\mathcal{D}(x(t)) \leq \mathcal{D}(x^0) \exp\left(- \phi_0 \int_0^t \eta(\mathcal{A}(s)) \mathrm{d}s \right).$$

Two technical novelties

- No **stochastic normalisation** trick $\rightsquigarrow$ **Geometric** argument.
- $t \mapsto \mathcal{D}(x(t))$ not diff. $\rightsquigarrow$ approx. with **Scorza-Dragoni**.

Graphon dynamics – *Strong connectivity for graphons*

Definition (Graphon connectivity)[Boudin,Salvarini&Trélat'21]

A graphon $\mathcal{A}$ is **strongly connected** if the following holds.

(i) (**Connectivity**) For $\mathscr{L}^1$-almost every $i, j \in I$, there exists $i = k_1, \ldots, k_m = j$ such that $k_{l+1} \in \operatorname{supp}(a(k_l, \cdot))$.

(ii) (**Degree lower-bound**) $\inf_{i \in I} \int_I a(i,j)\mathrm{d}j \geq \delta > 0$.

Theorem (Canonical kernel of $\mathbb{L}^*$)[Boudin,Salvarini&Trélat'21]

If $\mathcal{A}$ is strongly connected, there exists a unique $v \in L^2(I, \mathbb{R}_+^*)$ s.t.

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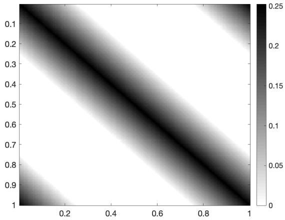

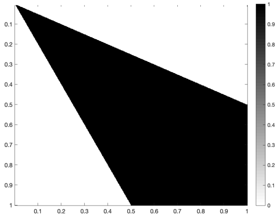

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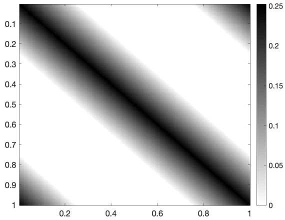

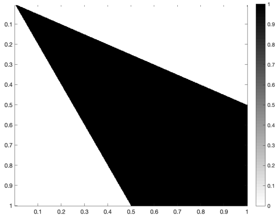

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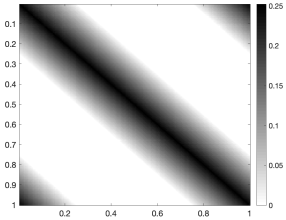

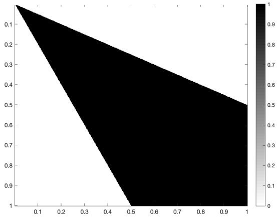

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Graphon dynamics – *Algebraic and graphon connectivity*

Definition (Generalised algebraic connectivity)

We define the **algebraic connectivity** of a DCUSCC graphon $\mathcal{A}$ by

$$\lambda_2(\mathcal{A}) := \inf_{x \in \mathscr{C}^\perp} \frac{\langle \mathbb{L}_v\, x, x \rangle_{L^2(I)}}{\|x\|^2_{L^2(I)}}$$

where

- $\mathscr{C} := \{ x \in L^2(I, \mathbb{R}^d) \text{ constant} \}$ is the **consensus manifold**,
- $\mathbb{L}_v := \mathcal{M}_v \mathbb{L}$ the **renormalised graph-Laplacian**.

Theorem (On algebraic and graphon connectivity)[BPDS'22]

For a graphon $\mathcal{A}$, the following connectivity characterisations hold.

- If $\mathcal{A}$ is **symmetric**, strong connectedness $\Longleftrightarrow \lambda_2(\mathcal{A}) > 0$.
- If $\mathcal{A}$ is **DCUSCC**, strong connectedness $\Longleftrightarrow \lambda_2(\mathcal{A}) > 0$.

Open problem: Is $\mathcal{A}$ DCUSCC whenever $v \in \mathrm{Ker}(\mathbb{L}^*)$ exists ?

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Theorem (Variance decay for symmetric graphons) [BPDS'22,BF'21]

Suppose that $\mathcal{A}(t)$ is symmetric for a.e. $t \in \mathbb{R}_+$. Then for each $x^0 \in L^\infty(I, \mathbb{R}^d)$ and every $\tau > 0$, there exist $\phi_0, \alpha_\tau, \gamma_\tau > 0$ s.t.

$$\mathcal{V}(x(t)) \leq \alpha_\tau \mathcal{V}(x^0) \exp\left(- \phi_0 \gamma_\tau \int_0^t \lambda_2\Big(\tfrac{1}{\tau}\textstyle\int_s^{s+\tau} \mathcal{A}(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}s \right).$$

$$\mathcal{V}(x) := \frac{1}{2}\int_I \int_I |x(i) - x(j)|^2 \mathrm{d}j\mathrm{d}i$$

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Idea: Exponential consensus with mere **average connectivity**.

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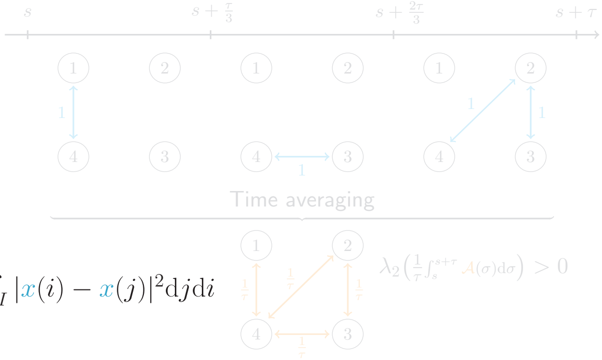

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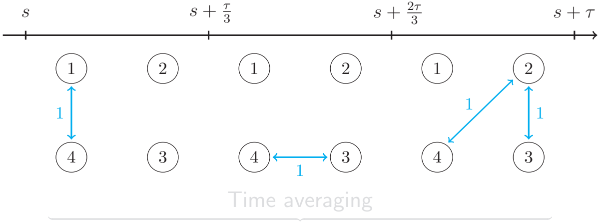

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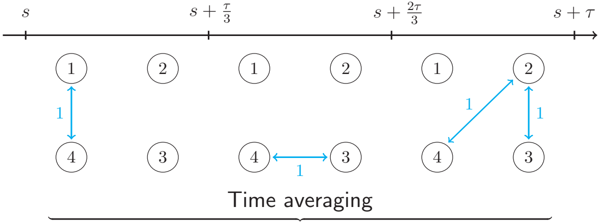

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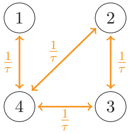

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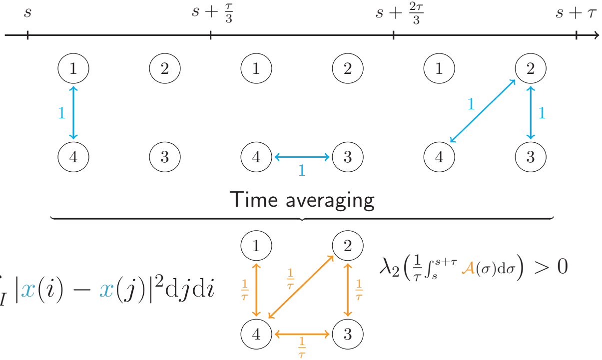

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Graphon dynamics – *Variance decay and connectivity (2)*

Definition (Balanced interaction topology)

A graphon $\mathcal{A}$ is said to be **balanced** if $\mathbb{L}^* 1 = 0$, namely

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$\hookrightarrow$ Equality between the **in-degree** and **out-degree** at each node.

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Suppose that $\mathcal{A}(t)$ is balanced for $\mathscr{L}^1$-almost every $t \in [0,T]$ and that $\phi(\cdot) \equiv 1$. Then for each $x^0 \in L^\infty(I, \mathbb{R}^d)$, it holds that

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Issue: If $\mathcal{A}(t)$ is DCUSCC $\rightsquigarrow$ $\mathcal{V}(\cdot)$ **not** **Lyapunov** anymore!

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Graphon dynamics – *Link between L^2- and L^∞-consensus*

Observation: Under the **sufficient condition** for L^2-consensus

$$\lambda_2\left(\frac{1}{\tau}\int_t^{t+\tau}\mathcal{A}(s)\mathrm{d}s\right)\geq\mu \quad \text{or} \quad \frac{1}{\tau}\int_t^{t+\tau}\lambda_2(\mathcal{A}(s))\mathrm{d}s\geq\mu$$

we **numerically** observed L^∞-consensus $\rightsquigarrow$ true in general ?

Theorem (Equivalence between L^2- and L^∞-consensus) [BPDS'22]

Suppose that there exist constants $(\tau,\mu)\in\mathbb{R}_+^*\times(0,1]$ s.t.

$$\frac{1}{\tau}\int_t^{t+\tau}\int_I a(s,i,j)\mathrm{d}j\mathrm{d}s\geq\mu$$

for $\mathscr{L}^1$-almost every $i\in I$. Then

$$\|x(t)-x^\infty\|_{L^2(I)}\underset{t\to+\infty}{\longrightarrow}0$$

for some $x^\infty\in\mathbb{R}^d$ **if and only if**

$$\|x(t)-x^\infty\|_{L^\infty(I)}\underset{t\to+\infty}{\longrightarrow}0.$$

Graphon dynamics – *Link between L^2- and L^∞-consensus*

Observation: Under the **sufficient condition** for L^2-consensus

$$\lambda_2\left(\frac{1}{\tau}\int_t^{t+\tau} \mathcal{A}(s)\mathrm{d}s\right) \geq \mu \quad \text{or} \quad \frac{1}{\tau}\int_t^{t+\tau} \lambda_2(\mathcal{A}(s))\mathrm{d}s \geq \mu$$

we **numerically** observed L^∞-consensus $\rightsquigarrow$ true in general ?

Theorem (Equivalence between L^2- and L^∞-consensus)[BPDS'22]

Suppose that there exist constants $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$ s.t.

$$\frac{1}{\tau}\int_t^{t+\tau}\int_I a(s,i,j)\mathrm{d}j\mathrm{d}s \geq \mu$$

for $\mathscr{L}^1$-almost every $i \in I$. Then

$$\|x(t) - x^\infty\|_{L^2(I)} \underset{t\to+\infty}{\longrightarrow} 0$$

for some $x^\infty \in \mathbb{R}^d$ **if and only if**

$$\|x(t) - x^\infty\|_{L^\infty(I)} \underset{t\to+\infty}{\longrightarrow} 0.$$

Conclusion – *That's all friends!*

Wrap-up (Summary of presentation)

1. Convergence to **consensus** in **micro** and **macro** dynamics.
2. Generalisation of the **scrambling** and **Fiedler** numbers.
3. Some interesting **open problems** to investigate!

Thank you for your attention !

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