YPATIA WORKSHOP

2022

Round Mean-field: crowd-opinion-cell

Rome, 27-29 September 2022

Mean-Field limit for hybrid models with chemotaxis: theory and numerics

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*Mean-Field limit for hybrid models with chemotaxis: theory and **numerics***

Outline of the talk

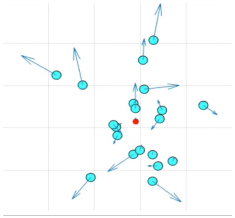
Numerical simulations comparing the different scales

- Basic Hybrid model: discrete particles + chemotaxis
- Mean-field kinetic model: Vlasov equation + chemotaxis
- Hydrodynamic Euler limit: the non monokinetic case

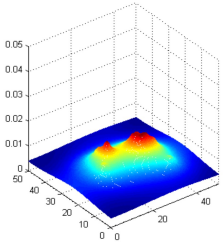
M. Menci , R. Natalini and T. Paul, *Hybrid and mean-field models of collective motions with chemotaxis: a numerical study*. In preparation.

Basic Discrete model with chemotaxis

Mechanical interactions + Chemotaxis



Evolution in time of each agent's state



Evolution in time of the concentration of the chemoattractant

Basic Discrete model with chemotaxis

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) \\ \partial_t \varphi = D\Delta\varphi - \kappa\varphi + f(x, X(t)) \end{cases}$$

Microscopic

Macroscopic

- x_i Position at time t of the i-th cell
- v_i Velocity at time t of the i-th cell
- φ Concentration of a chemoattractant produced by the cells
- $\gamma(v, x)$ Interaction function among the cells
- $f(x, X)$ Production of chemoattractant

Alignment Models: Cucker Smale (2007)

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) \end{cases}$$

$$\gamma(v_i - v_j, x_i - x_j) = \frac{1}{(1+||x_i - x_j||^2)^{\beta}}\ (v_i - v_j)$$

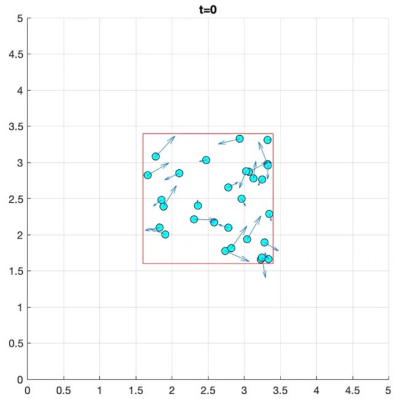
Cucker-Smale influence function

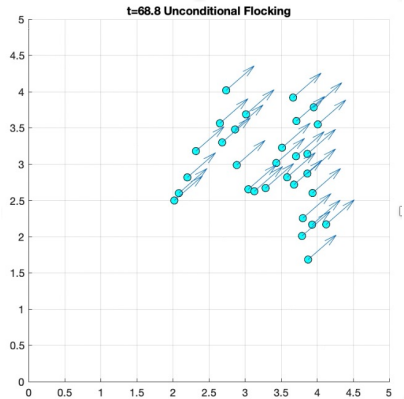
$\beta \leq \frac{1}{2}$ Unconditional flocking behavior

$\beta > \frac{1}{2}$ Conditional flocking behavior

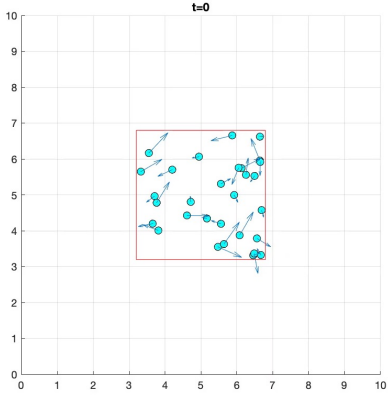
F. Cucker, S. Smale *Emergent Behavior in Flocks*, IEEE Transactions On Automatic Control 52 (5), 852–862 (2007).

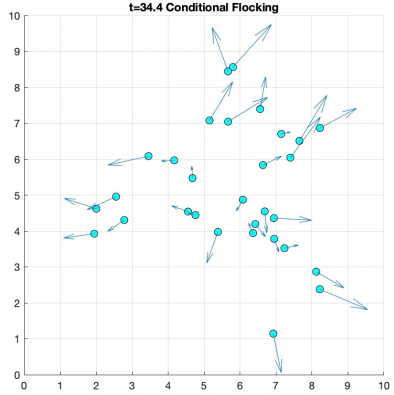
Unconditional Flocking





Conditional Flocking





The effect of chemicals

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) \\ \partial_t \varphi = D\Delta\varphi - \kappa\varphi + f(x, X(t)) \end{cases}$$

$$\gamma(v_i - v_j, x_i - x_j) = \frac{1}{(1+||x_i - x_j||^2)^{\beta}}(v_i - v_j)$$

Cucker-Smale influence function

$$f(x, X(t)) = \sum_{j=1}^{N} \chi_{\mathbf{B}(\mathbf{X}_j, R)}$$

sum of characteristic functions
on balls of radius R, centered in each particle

Cells = regions from which signal arises

The effect of chemicals

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) \\ \partial_t \varphi = D\Delta\varphi - \kappa\varphi + f(x, X(t)) \end{cases}$$

Analytical Results

- Global existence and uniqueness of the solution in $\mathbb{R}^2$
- Asymptotic properties

Di Costanzo E., **Menci M.**, Messina E., Natalini R. and Vecchio A. (2020). *A hybrid model of collective motion of discrete particles under alignment and continuum chemotaxis*. Discrete Cont. Dyn-B, 25(1), 443–472. (doi: 10.3934/dcdsb.2019189)

Asymptotic properties on the linearised system

$$\mathbf{X}_{CM}(t) := \frac{1}{N}\sum_{i=1}^{N}\mathbf{X}_i(t),$$

$$\mathbf{V}_{CM}(t) := \frac{1}{N}\sum_{i=1}^{N}\mathbf{V}_i(t),$$

1) Velocity and position of all particles converge to the same values (their centre of mass)

2) The velocity of the centre of mass decays to zero

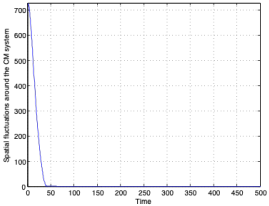
✓ Time-asymptotic flocking condition

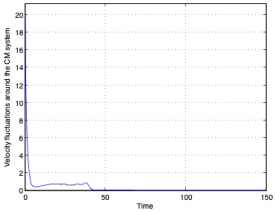
✓ Stronger condition: Particles converge to their centre of mass

Velocity decays to zero

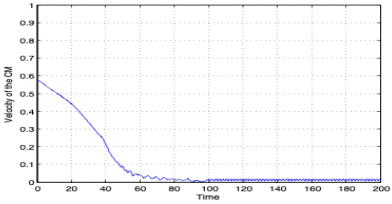
Di Costanzo E., **Menci M.,** Messina E., Natalini R. and Vecchio A. (2020). *A hybrid model of collective motion of discrete particles under alignment and continuum chemotaxis*. Discrete Cont. Dyn-B, 25(1), 443–472. (doi: 10.3934/dcdsb.2019189)

Agreement between analytical and numerical results on the full system

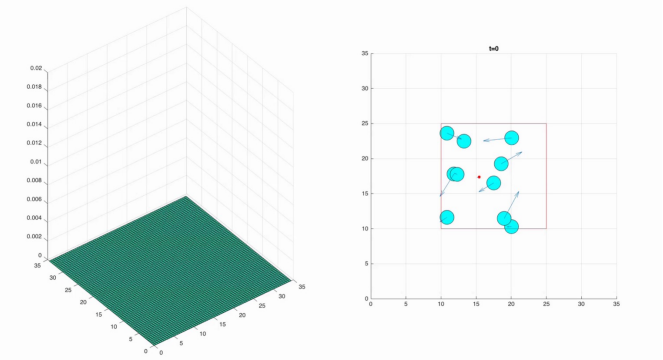




- $Fl_x(t) := \sum_{i=1}^{n} \|\mathbf{x}_i(t) - \mathbf{x}_{CM}(t)\|^2 \to 0$
- $Fl_v(t) := \sum_{i=1}^{n} \|\mathbf{v}_i(t) - \mathbf{v}_{CM}(t)\|^2 \to 0$
- $||\mathbf{v}_{CM}(t)||^2 \to 0$



Chemotaxis plays a crucial role: time-asymptotic flocking behavior is ensured **for any value of $\beta > 0$**



Di Costanzo E., **Menci M.,** Messina E., Natalini R. and Vecchio A. (2020). *A hybrid model of collective motion of discrete particles under alignment and continuum chemotaxis.* Discrete Cont. Dyn-B, 25(1), 443–472. (doi: 10.3934/dcdsb.2019189)

Mean-field kinetic equation

$$\partial_t \rho^t(x,v) + v \cdot \nabla_x \rho^t(x,v) + \nabla_v \cdot \left(\rho^t(x,v) \int_{2d} \psi(x-y)(v-w)\rho^t(y,w)dydw \right) = 0$$

Numerical aspects

J.A. Carrillo, M. Fornasier, G. Toscani, and F, Vecil. *Particle, kinetic, and hydrodynamic models of swarming*. Mathematical modeling of collective behavior in socio-economic and life sciences (2010)

G. Albi, and L. Pareschi. *Binary interaction algorithms for the simulation of flocking and swarming dynamics*. Multiscale Modeling & Simulation (2013)

- Different binary interaction methods for solving the kinetic equation
- Comparison of Cucker-Smale model in absence/ presence of visual limitations (perception cone)

1D Numerical Simulations

$$\partial_t \rho^t(x,v) + v \cdot \nabla_x \rho^t(x,v) + \nabla_v \cdot \left(\rho^t(x,v) \int_{2d} \psi(x-y)(v-w)\rho^t(y,w)dydw \right) = 0$$

$$\rho^0(x,v) = \frac{1}{2\pi\sigma_x\sigma_v} e^{\frac{-x^2}{2\sigma_x^2}} \left(e^{\frac{-(v+v_0)^2}{2\sigma_v^2}} + e^{\frac{-(v-v_0)^2}{2\sigma_v^2}} \right)$$

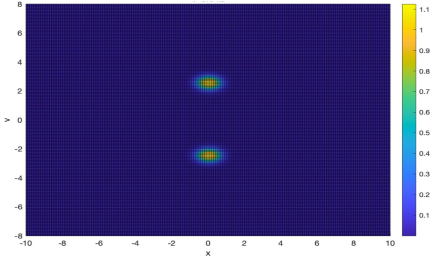
Finite Difference scheme

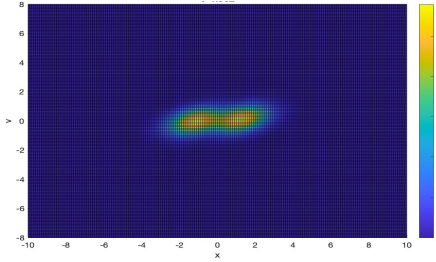
No perception cone

Simulations for $\beta = 0.05$ and $\beta = 0.95$

$$\partial_t \rho^t(x,v) + v \cdot \nabla_x \rho^t(x,v) + \nabla_v \cdot \left(\rho^t(x,v) \int_{2d} \psi(x-y)(v-w)\rho^t(y,w)dydw \right) = 0$$

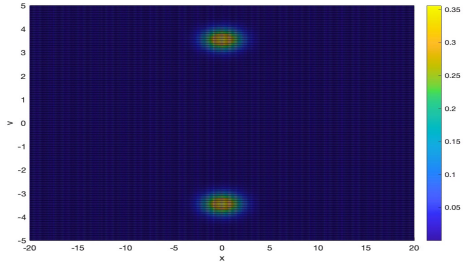
1D Cucker-Smale flocking in the phase space ($\beta = 0.05$)

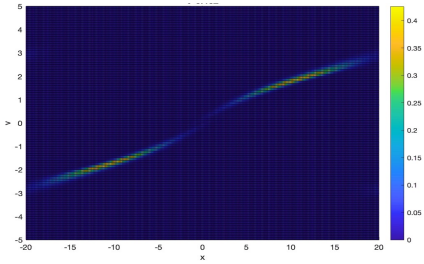




$$\partial_t \rho^t(x,v) + v \cdot \nabla_x \rho^t(x,v) + \nabla_v \cdot \left(\rho^t(x,v) \int_{2d} \psi(x-y)(v-w)\rho^t(y,w)dydw \right) = 0$$

1D Cucker-Smale flocking in the phase space ($\beta = 0.95$)





The effect of chemotaxis

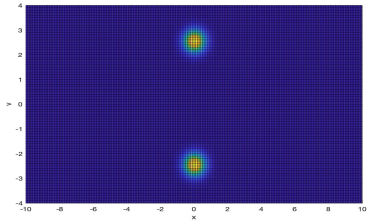
$$\begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v (\nu(t,x,v)\rho^t), \ \ \rho^0 = \rho^{in} \\ \nu(t,x,v) = \gamma(x,v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa\psi + g(z,\rho^s), \ \ \psi^0 = \varphi^{in} \end{cases}$$

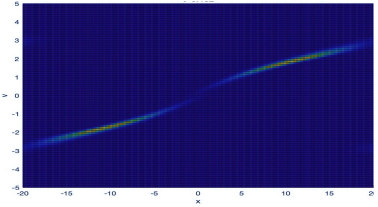
Neglet external forces

Alignment interaction

As in the microscopic scale, chemotactic effects leads to consensus

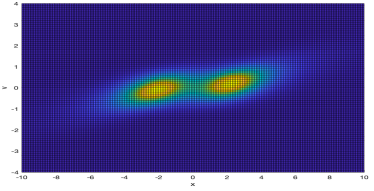
R. Natalini and T. Paul. *The Mean-Field limit for hybrid models of collective motions with chemotaxis*. arXiv preprint arXiv:2107.14645 (2021).





$\eta = 0$

No flocking



$\eta > 0$

Flocking

Non local hybrid Euler system

$$\begin{cases} \partial_t \mu^t + \nabla_x (u^t \mu^t) = 0 \\ \partial_t (\mu^t u^t) + \nabla (\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy + \mu^t \nabla \psi^t \\ \partial_s \psi^s(z) = D \Delta_z \psi - \kappa \psi + \chi * \mu^s \end{cases}$$

Pressureless Euler-type system + Chemotaxis

R. Natalini and T. Paul. *The Mean-Field limit for hybrid models of collective motions with chemotaxis*. arXiv preprint arXiv:2107.14645 (2021).

Pressureless Euler-type system

Negletting chemotaxis

$$\begin{cases} \partial_t \mu^t + \nabla_x (u^t \mu^t) = 0 \\ \partial_t (\mu^t u^t) + \nabla (\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy \end{cases}$$

Condition on initial data for global consensus.

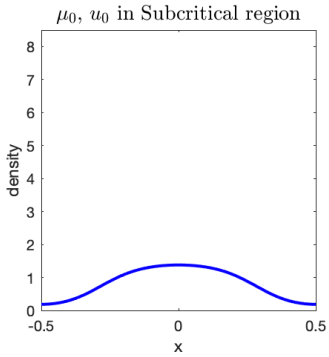
Finite-time blow up of the solutions.

Relaxation Schemes for conservation laws (Aregba-Driollet & Natalini 2000)

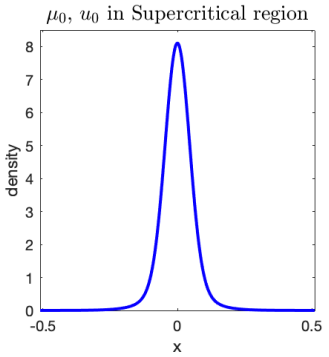
J. A. Carrillo, Y.-P. Choi, E. Tadmor, and C. Tan, *Critical thresholds in 1D Euler equations with nonlocal forces*, Math. Mod. Meth. Appl. Sci., 26:185–206 (2016)

Pressureless Euler-type system

Consensus



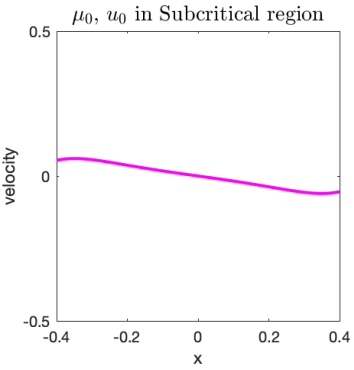
Finite-time blow up



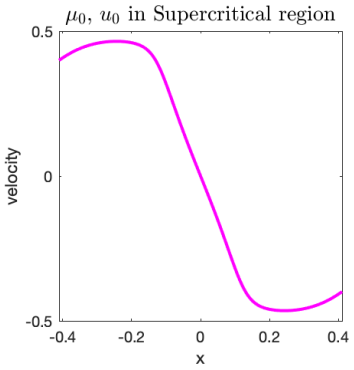
density value at the origin gets larger and larger…

Pressureless Euler-type system

Consensus



Finite-time blow up



the derivative of the velocity becomes sharper and sharper at the origin

The effect of chemotaxis

$$\begin{cases} \partial_t \mu^t + \nabla_x (u^t \mu^t) = 0 \\ \partial_t (\mu^t u^t) + \nabla (\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy + \mu^t \nabla \psi - \nu \mu^t u^t \\ \partial_s \psi^s(z) = D \Delta_z \psi - \kappa \psi + \chi * \mu^s \end{cases}$$

Does the damping term and/or chemotaxis prevent blow up phenomena?

Damping term	Chemotactic gradient	Finite-time blow up
		✓
	✓	
✓		
✓	✓	

Analytical result: the monokinetic case

Theorem 7.1. *Let μ^t, u^t, ψ^t solves the following system*

$$\begin{cases} \partial_t \mu^t + \nabla_x (u^t \mu^t) = 0 \\ \partial_t (\mu^t u^t) + \nabla (\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy + \nabla \psi^t + F \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + \chi * \mu^s, \ s \in [0,t], \\ (\mu^0, u^0, \psi^0) = (\mu^{in}, u^{in}, \psi^{in}) \in H^s, \ s > \frac{d}{2} + 1. \end{cases}$$

where $\mu^t, u^t \in C([0,t]; H^s) \cap C^1([0,T]; H^{s-1})$, $\psi^t \in C([0,t]; H^s) \cap C^1([0,T]; H^{s-2}) \cap L^2(0,T; H^{s+1})$ [3].

Then $\rho^t(x,v) := \mu^t(x)\delta(v - u^t(x))$ *solves the following system*

$$\begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v (\nu(t,x,v) \rho^t), \\ \nu(t,x,v) = \gamma(x,v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho^s), \ \psi^0 = \psi^{in} \\ \rho^0(x,v) = \mu^{in}(x) \delta(v - u^{in}(x)). \end{cases}$$

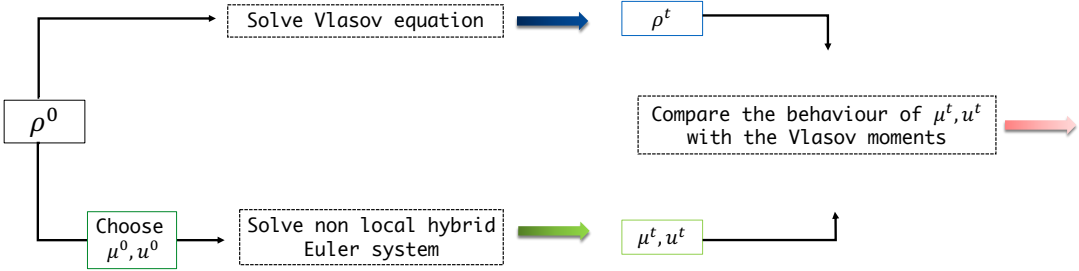
R. Natalini and T. Paul. *The Mean-Field limit for hybrid models of collective motions with chemotaxis*. arXiv preprint arXiv:2107.14645 (2021).

Numerical results: the general case

NEW!

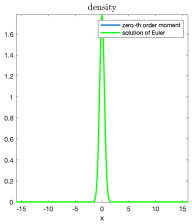
In the monokinetic case, Vlasov moments and Euler solutions coincide.
What about the non monokinetic case?

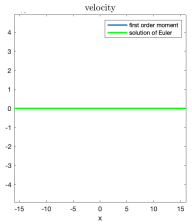
Numerical comparison between moments of Vlasov equation and solutions of the non local hybrid Euler system

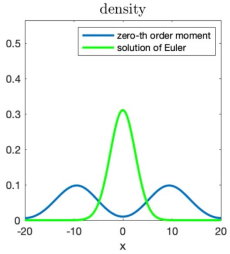


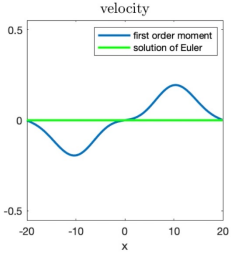
Numerical results: the general case

Negletting chemotaxis







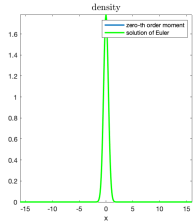


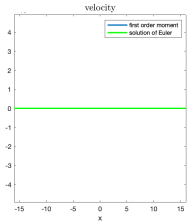
Numerical evidence:

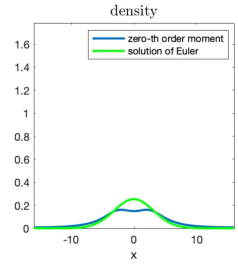
No good agreement in the non monokinetic case

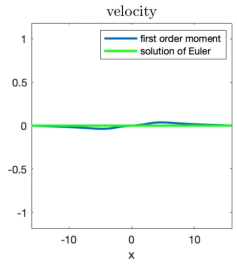
Numerical results: the general case

With chemotaxis









Numerical evidence:

good agreement even in the non monokinetic case

*Mean-Field limit for hybrid models with chemotaxis: theory and **numerics***

Conclusions and beyond

- The different features of the microscopic dynamics are still present in numerical simulations of Vlasov and non local hybrid Euler system
- Numerical investigations on the non monokinetic case
- The role of chemotaxis at different scale
- Different kind of interactions
- Compare the non local hybrid Euler system and Euler models with pressure for chemotaxis

YPATIA WORKSHOP

2022

Round Mean-field: crowd-opinion-cell

Rome, 27-29 September 2022

Thanks for your attention

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