Mean-field limit for particle systems with topological interactions

jointly with D. Benedetto and E. Caglioti

Stefano Rossi
stef.rossi@uniroma1.it

Department of Mathematics, Sapienza University of Rome

Round meanfield - CNR Rome
29th September 2022

SAPIENZA
UNIVERSITÀ DI ROMA

Overview

"Topological" interaction

We consider a Cucker-Smale type model for the motion of N agents $((X_i, V_i) \in \mathbb{R}^d \times \mathbb{R}^d)$, in the mean-field scaling,

$$\begin{cases} \dot{X}_i(t) = V_i(t) \\ \dot{V}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} p_{ij}(V_j(t) - V_i(t)). \end{cases}$$

Here

$$p_{ij} = K\Big(M(X_i, |X_i - X_j|)\Big),$$

$$M(X_i, r) = \frac{1}{N} \sum_{k=1}^{N} \chi\{|X_k - X_i| \le r\}$$

Here $K : [0,1] \to \mathbb{R}^+$ is decreasing and smooth.

"Topological" interaction

We consider a Cucker-Smale type model for the motion of N agents $((X_i, V_i) \in \mathbb{R}^d \times \mathbb{R}^d)$, in the mean-field scaling,

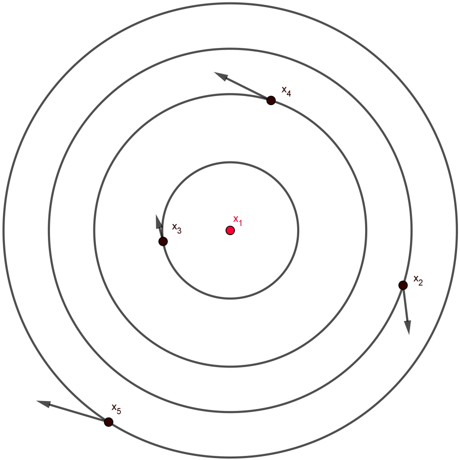
$$\begin{cases} \dot{X}_i(t) = V_i(t) \\ \dot{V}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} p_{ij}(V_j(t) - V_i(t)). \end{cases}$$

Here

$$p_{ij} = K\Big(M(X_i, |X_i - X_j|)\Big),$$

$$M(X_i, r) = \frac{1}{N} \sum_{k=1}^{N} \chi\{|X_k - X_i| \le r\}$$

Here $K\colon [0,1] \to \mathbb{R}^+$ is decreasing and smooth.



$$p_{12} = K(4/5), p_{13} = K(2/5), p_{14} = K(3/5), p_{15} = K(1).$$

Motivations

- The Cucker-Smale model with metric weights was introduced in [Cucker, Smale: IEEE Trans. Automat. Contr.,2007], in this model the interaction is such that neighboring birds tend to align their velocities.
- [Ballerini et al., PNAS, 2008], using extensive observations of starling flocks, observed that starling interactions are scale-free and their strength depends on the so-called topological distance between individuals, measured in units of average bird separation.
- In other words, the relevant quantity is how many intermediate individuals separate two birds, not how far apart they are in the metric sense.

Motivations

- The Cucker-Smale model with metric weights was introduced in [Cucker, Smale: IEEE Trans. Automat. Contr.,2007], in this model the interaction is such that neighboring birds tend to align their velocities.
- [Ballerini et al., PNAS, 2008], using extensive observations of starling flocks, observed that starling interactions are scale-free and their strength depends on the so-called topological distance between individuals, measured in units of average bird separation.
- In other words, the relevant quantity is how many intermediate individuals separate two birds, not how far apart they are in the metric sense.

Motivations

- The Cucker-Smale model with metric weights was introduced in [Cucker, Smale: IEEE Trans. Automat. Contr.,2007], in this model the interaction is such that neighboring birds tend to align their velocities.
- [Ballerini et al., PNAS, 2008], using extensive observations of starling flocks, observed that starling interactions are scale-free and their strength depends on the so-called topological distance between individuals, measured in units of average bird separation.
- In other words, the relevant quantity is how many intermediate individuals separate two birds, not how far apart they are in the metric sense.

Mathematical literature

- [Haskovec, 2016]: introduced the topological Cucker-Smale model. Velocity consensus and mean-field limit for a regularized interaction are proved.
- [Blanchet, Degond:2018]: consider jump processes with topological interactions (*smooth rank-based* dynamics and *nearest-neighbor* dynamics). The kinetic limit equations are derived formally.
- [Degond, Pulvirenti: 2019]: propagation of chaos is proved for the smooth rank-based model.

Mathematical literature

- [Haskovec, 2016]: introduced the topological Cucker-Smale model. Velocity consensus and mean-field limit for a regularized interaction are proved.
- [Blanchet, Degond:2018]: consider jump processes with topological interactions (*smooth rank-based* dynamics and *nearest-neighbor* dynamics). The kinetic limit equations are derived formally.
- [Degond, Pulvirenti: 2019]: propagation of chaos is proved for the smooth rank-based model.

Existence of the particle dynamics

One of the difficulties is that the dynamics is not always well defined.

Example

Consider three agents $\{X_i\}_{i=1}^3$ on a line interacting via a communication weight that consider only the nearest neighbour, with initial data

$$X_1(0) = -1, \quad X_2(0) = \epsilon, \quad X_3(0) = 1$$
$$V_1(0) = -1, \quad V_2(0) = 0, \quad V_3(0) = 1$$

$\epsilon \in (-1, 1)$.
If $\varepsilon = 0$ there is not a single way to define the dynamics.
Moreover if $\varepsilon < 0$ [$\varepsilon > 0$] the trajectories asymptotically evolve in the negative [positive] semi-axis. So the dynamics is not continuous w.r.t the initial datum.

Existence of the particle dynamics

One of the difficulties is that the dynamics is not always well defined.

Example

Consider three agents $\{X_i\}_{i=1}^3$ on a line interacting via a communication weight that consider only the nearest neighbour, with initial data

$$X_1(0) = -1, \quad X_2(0) = \epsilon, \quad X_3(0) = 1$$
$$V_1(0) = -1, \quad V_2(0) = 0, \quad V_3(0) = 1$$

$\epsilon \in (-1, 1)$.
If $\varepsilon = 0$ there is not a single way to define the dynamics. Moreover if $\varepsilon < 0$ [$\varepsilon > 0$] the trajectories asymptotically evolve in the negative [positive] semi-axis. So the dynamics is not continuous w.r.t the initial datum.

Existence of the particle dynamics

Nevertheless, the N-particle dynamics is almost everywhere well defined.

Theorem

Except for a set of Lebesgue measure zero, given $(X, V) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, there exists a unique global solution

$$\Big(X^N(t, X, V), V^N(t, X, V)\Big) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

with initial datum (X, V).
Moreover, we have that

$$|X_i(t)| \leq R_x + tR_v, \ |V_i(t)| \leq R_v$$

for any i, if $|x_i| \leq R_x$ and $|v_i| \leq R_v$.

Idea of the proof

In the iso-rank manifold

$$\mathcal{S} = \{(X, V) : \text{exist } i \neq j \neq k \ : |x_i - x_k| = |x_j - x_k|\}$$

we have to take care of the points $(X, V) \in \mathcal{S}$ such that

$$x_i \neq x_j \neq x_k \quad \text{and} \quad (v_i - v_k) \cdot \hat{n}_{ik} = (v_j - v_k) \cdot \hat{n}_{jk}$$

where $\hat{n}_{ab} = (x_a - x_b)/|x_a - x_b|$.

The subset of initial data (X, V) such that the trajectory, at a first time in the future or in the past, intersects this patological set, has Lebesgue measure zero (codimension 1).

Idea of the proof

In the iso-rank manifold

$$\mathcal{S} = \{(X, V) : \text{exist } i \neq j \neq k \; : |x_i - x_k| = |x_j - x_k|\}$$

we have to take care of the points $(X, V) \in \mathcal{S}$ such that

$$x_i \neq x_j \neq x_k \quad \text{and} \quad (v_i - v_k) \cdot \hat{n}_{ik} = (v_j - v_k) \cdot \hat{n}_{jk}$$

where $\hat{n}_{ab} = (x_a - x_b)/|x_a - x_b|$.

The subset of initial data (X, V) such that the trajectory, at a first time in the future or in the past, intersects this patological set, has Lebesgue measure zero (codimension 1).

Empirical Measure

$\{(X_i^N(t), V_i^N(t)\}_{i=1}^N \in \mathbb{R}^{2dN}$ is a solution of the particle equations, if and only if

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}\delta_{V_i^N(t)}$$

solves (in the weak sense) the following nonlinear PDE

$$\partial_t f_t + v\cdot\nabla_x f_t + \nabla_v\cdot\left(W[Sf_t, f_t](x,v)f_t\right) = 0,$$

where $Sf_t(x) = \int f_t(x,v)\,\mathrm{d}v$ and where

$$W[Sf,f](x,v) = \int K\left(M[Sf](x,|x-y|)\right)(w-v)f(y,w)\,\mathrm{d}y\,\mathrm{d}w,$$

with

$$M[Sf](x,r) = \int_{|x'-x|\leq r}(Sf)(x')\,\mathrm{d}x'.$$

Mean-field limit

We want to justify the description given by $f(t,x,v)$ via the mean-field Vlasov equation by a rigorous derivation from the N-particle dynamics as $N \to \infty$.

This question can be translated in the continuity w.r.t the initial datum of the solutions of the Vlasov equation: if

$$\mu_0^N \rightharpoonup f_0$$

then

$$\mu_t^N \rightharpoonup f_t?$$

Mean-field limit

We want to justify the description given by $f(t,x,v)$ via the mean-field Vlasov equation by a rigorous derivation from the N-particle dynamics as $N \to \infty$.

This question can be translated in the continuity w.r.t the initial datum of the solutions of the Vlasov equation: if

$$\mu_0^N \rightharpoonup f_0$$

then

$$\mu_t^N \rightharpoonup f_t?$$

Excursus: mean-field limit in the regular case

Reviewing the regular case:

$$\begin{cases} \dot{X}_i(t) = V_i(t) \\ \dot{V}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} F_{j\to i}. \end{cases}$$

Suppose that the system interacts by a regular two-body potential so that $F_{j\to i} = -\nabla\phi(X_i - X_j)$ ($\nabla\phi$ Lipschitz).

Wasserstein distance

To quantify the convergence rate as $N \to \infty$ we introduce a distance on $\mathcal{P}_1(\mathbb{R}^d)$:

$$\mathcal{W}(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\,\pi(x,y)$$

where $\Pi(\mu,\nu)$ is the set of all possible *couplings* between μ and ν.
(From now on, we consider measures with support on a compact subset.)
It holds the following duality result

Theorem (Kantorovich duality)

$$\mathcal{W}(\mu,\nu) = \sup_{\psi : Lip(\psi) \leq 1} \int \psi d\,(\mu - \nu)$$

From which follows easily that

$$\mu_n \rightharpoonup \mu \iff \mathcal{W}(\mu_n,\mu) \to 0$$

Wasserstein distance

To quantify the convergence rate as $N \to \infty$ we introduce a distance on $\mathcal{P}_1(\mathbb{R}^d)$:

$$\mathcal{W}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\,\pi(x, y)$$

where $\Pi(\mu, \nu)$ is the set of all possible *couplings* between μ and ν.
(From now on, we consider measures with support on a compact subset.)
It holds the following duality result

Theorem (Kantorovich duality)

$$\mathcal{W}(\mu, \nu) = \sup_{\psi : Lip(\psi) \leq 1} \int \psi d\,(\mu - \nu)$$

From which follows easily that

$$\mu_n \rightharpoonup \mu \iff \mathcal{W}(\mu_n, \mu) \to 0$$

Dobrushin's estimate

Theorem (Dobrushin 1979)

Given an initial datum $f_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and a smooth potential ϕ, there exists a unique global solution of the Vlasov mean-field equation.
Moreover the solutions are weakly-continuous w.r.t the initial datum.

Corollary (Mean-field limit)

Fixed $T > 0$, let f_t be a solution of the Vlasov equation with initial datum f_0 and let μ_t^N be a solution of the particle system with initial datum μ_0^N. Then, for $0 \leq t \leq T$,

$$\mathcal{W}(f_t, \mu_t^N) \leq \mathrm{e}^{cT} \mathcal{W}(f_0, \mu_0^N)$$

Dobrushin's estimate

Theorem (Dobrushin 1979)

Given an initial datum $f_0 \in \mathcal{P}(\mathbb{R}^{2d})$ *and a smooth potential* ϕ*, there exists a unique global solution of the Vlasov mean-field equation.*
Moreover the solutions are weakly-continuous w.r.t the initial datum.

Corollary (Mean-field limit)

Fixed $T > 0$*, let* f_t *be a solution of the Vlasov equation with initial datum* f_0 *and let* μ_t^N *be a solution of the particle system with initial datum* μ_0^N*. Then, for* $0 \leq t \leq T$*,*

$$\mathcal{W}(f_t, \mu_t^N) \leq \mathrm{e}^{cT} \mathcal{W}(f_0, \mu_0^N)$$

Continuity w.r.t the initial datum

Let $T_t[g]$ be the flow evolving via the force field defined by $g_t \in C_w([0,T];\mathcal{P}(\mathbb{R}^{2d}))$:

$$\dot{X}(t) = V(t), \quad \dot{V}(t) = F[g_t](X(t)) = -\nabla\phi * g_t(X(t)).$$

We denote by $\mu \circ g^{-1}$ the push-forward of the measure μ along the map g.

We estimate $\mathcal{W}(f_t, \mu_t^N)$ using the triangular inequality

$$\mathcal{W}(f_t, \mu_t^N) = \mathcal{W}(f_0 \circ T_t^{-1}[f_t], \mu_0^N \circ T_t^{-1}[\mu_t^N])$$

$$\leq \mathcal{W}(f_0 \circ T_t^{-1}[f_t], f_0 \circ T_t^{-1}[\mu_t^N]) + \mathcal{W}(f_0 \circ T_t^{-1}[\mu_t^N], \mu_0^N \circ T_t^{-1}[\mu_t^N])$$

Using the Lipschitzianity of the flow in the second term

$$\mathcal{W}(f_t, \mu_t^N) \leq \int \int_0^t |T_s[f_s] - T_s[\mu_s^N]| d\, f_0 + e^{Ct} \mathcal{W}(f_0, \mu_0^N)$$

We estimate $\mathcal{W}(f_t, \mu_t^N)$ using the triangular inequality

$$\mathcal{W}(f_t, \mu_t^N) = \mathcal{W}(f_0 \circ T_t^{-1}[f_t], \mu_0^N \circ T_t^{-1}[\mu_t^N])$$

$$\leq \mathcal{W}(f_0 \circ T_t^{-1}[f_t], f_0 \circ T_t^{-1}[\mu_t^N]) + \mathcal{W}(f_0 \circ T_t^{-1}[\mu_t^N], \mu_0^N \circ T_t^{-1}[\mu_t^N])$$

Using the Lipschitzianity of the flow in the second term

$$\mathcal{W}(f_t, \mu_t^N) \leq \int \int_0^t |T_s[f_s] - T_s[\mu_s^N]| d\, f_0 + e^{Ct} \mathcal{W}(f_0, \mu_0^N)$$

Let

$$\mathcal{K}[g](x,v) = (v, F[g](x))$$

The key estimates are:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|T_t[g_t^1] - T_t[g_t^2]| &\leq |\mathcal{K}[g_t^1]\circ T_t[g_t^1] - \mathcal{K}[g_t^2]\circ T_t[g_t^2]| \\
&\leq |\mathcal{K}[g_t^1]\circ T_t[g_t^1] - \mathcal{K}[g_t^1]\circ T_t[g_t^2]| \\
&+ |\mathcal{K}[g_t^1]\circ T_t[g_t^2] - \mathcal{K}[g_t^2]\circ T_t[g_t^2]| \\
&\leq C|T[g_t^1] - T[g_t^2]| + \mathrm{Lip}(\nabla\phi)\mathcal{W}(g_t^1, g_t^2)
\end{aligned}$$

and using the Kantorovich duality in the second term.

By the Gronwall's lemma we get

$$|T_t[g_t^1] - T_t[g_t^2]| \leq C \int_0^t \mathrm{e}^{C(t-s)} \mathcal{W}(g_s^1, g_s^2)\, ds$$

We arrive at

$$\mathcal{W}(f_t, \mu_t^N) \leq \mathrm{e}^{Ct} \mathcal{W}(f_0, \mu_0^N) + C \int_0^t \mathrm{e}^{C(t-s)} \mathcal{W}(f_s, \mu_s^N)\, ds$$

Using the Gronwall's lemma, we obtain the thesis

$$\mathcal{W}(f_t, \mu_t^N) \leq \mathrm{e}^{2Ct} \mathcal{W}(f_0, \mu_0^N).$$

Beyond MFL for regular potentials

This approach depends on the regularity of the potential.

- Difficult to extend to the case of Coulomb/Newton interaction. Only partial results for $d \geq 2$.
- Validity of the mean-field limit for the 1d Vlasov-Poisson equation on S^1: [Trocheris 1986], [Hauray 2012].
- Here $F(x) = -x + \text{sign}(x)/2$.

Beyond MFL for regular potentials

This approach depends on the regularity of the potential.

- Difficult to extend to the case of Coulomb/Newton interaction. Only partial results for $d \geq 2$.
- Validity of the mean-field limit for the 1d Vlasov-Poisson equation on S^1: [Trocheris 1986], [Hauray 2012].
- Here $F(x) = -x + \text{sign}(x)/2$.

Beyond MFL for regular potentials

This approach depends on the regularity of the potential.

- Difficult to extend to the case of Coulomb/Newton interaction. Only partial results for $d \geq 2$.
- Validity of the mean-field limit for the 1d Vlasov-Poisson equation on S^1: [Trocheris 1986], [Hauray 2012].
- Here $F(x) = -x + \text{sign}(x)/2$.

Discrepancy distance

$$M[\rho](x,r) = \int_{|x'-x|\leq r} \rho(x')\,\mathrm{d}x'.$$

is not continuous w.r.t. $\mathcal{W}$.

It is natural to work with another distance, the discrepancy

$$\mathcal{D}(\rho_1,\rho_2) = \sup_{x,r>0} \left| \int_{B_r(x)} \mathrm{d}\rho_1 - \int_{B_r(x)} \mathrm{d}\rho_2 \right|.$$

Indeed, by definition, it holds that

$$|M[\rho_1](x,r) - M[\rho_2](x,r)| \leq \mathcal{D}(\rho_1,\rho_2).$$

Duality formulation

Let X be the subset of $C_b^1([0,+\infty);\mathbb{R})$, and define

$$\|\phi\|_X = \int_0^{+\infty} |\phi'(r)| \,\mathrm{d}r.$$

Then

$$\mathcal{D}(\rho_1, \rho_2) = \sup_{\phi \in X \colon \|\phi\|_X \leq 1} \sup_x \int \phi(|x-y|)\,(\mathrm{d}\rho_1(y) - \mathrm{d}\rho_2(y)).$$

The discrepancy distance is not equivalent to the Wasserstein: consider two deltas δ_{x_1}, δ_{x_2} while $x_1 \to x_2$.
However the equivalence holds when one of the two measures is in L^∞.

Lemma (Weak-Strong continuity)

Let ρ and ν be two probability measures on $\mathbb{R}^d$ with support in a ball B_R and such that $\rho \in L^\infty(\mathbb{R}^d)$. Then

$$\mathcal{D}(\nu, \rho) \le C(\|\rho\|_\infty, R)\sqrt{\mathcal{W}(\nu, \rho)}.$$

MFL for topological interactions

We can use this fact to prove the MFL.

Theorem (Benedetto, Caglioti, R.)

Fixed $T > 0$, let f_t be a solution of the mean-field topological equation with initial datum $f_0 \in L^\infty$ and let μ_t^N be a solution of the N-particle dynamics with initial datum μ_0^N.
Then, for $0 \leq t \leq T$,

$$\mathcal{W}(f_t, \mu_t^N) \leq C(T) max \left\{ \mathcal{W}(f_0, \mu_0^N), \sqrt{\mathcal{W}(f_0, \mu_0^N)} \right\}.$$

Sketch of the proof

A first step is to estimate the discrepancy distance between two empirical measures when they are supported on points which are close.

Lemma (Proximity Lemma)

Let $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ *and* $\nu^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ *be two empirical measures on* $\mathbb{R}^d$ *and take* $\delta > 0$ *such that* $|x_i - y_i| \leq \delta$ *for all* $i = 1, \ldots, N$.
Then, for any probability measure $\rho \in L^\infty(\mathbb{R}^d)$ *supported on a ball* B_R,

$$\mathcal{D}(\mu^N, \nu^N) \leq cR^{d-1}\delta\|\rho\|_\infty + c\mathcal{D}(\mu^N, \rho).$$

Sketch of the proof

We then compare the N-agent dynamics with the "intermediate" dynamics given by

$$\begin{cases} \dot{X}_i^f(t) = V_i^f(t) \\ \dot{V}_i^f(t) = W[Sf_t, \nu_t^N](X_i^f, V_i^f), \end{cases} \tag{1}$$

where

$$\nu_t^N = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k^f(t)} \delta_{V_k^f(t)}$$

is the empirical measure. The initial datum is $\nu_0^N = \mu_0^N$.

By the triangular inequality,

$$\mathcal{W}(f_t, \mu_t^N) \leq \mathcal{W}(f_t, \nu_t^N) + \mathcal{W}(\nu_t^N, \mu_t^N).$$

The first term is easily bounded since in this case the interaction $K(M[Sf_t](x, |x-y|))$ is locally Lipschitz in both the variables:

$$|M[\rho](x, r_1) - M[\rho](x, r_2)| \leq c\|\rho\|_\infty \left| r_1^d - r_2^d \right| .$$

$$|M[\rho](x_1, r) - M[\rho](x_2, r)| \leq c\|\rho\|_\infty r^{d-1}|x_1 - x_2|.$$

from which the continuity w.r.t the initial datum follows as in the Dobrushin's proof. Hence

$$\mathcal{W}(f_t, \mu_t^N) \leq C(T)\mathcal{W}(f_0, \mu_0^N) + \mathcal{W}(\nu_t^N, \mu_t^N).$$

$$\mathcal{W}(f_t, \mu_t^N) \leq C(T)\mathcal{W}(f_0, \mu_0^N) + \mathcal{W}(\nu_t^N, \mu_t^N).$$

For the second term we have that $\mathcal{W}(\nu_t^N, \mu_t^N) \leq \delta(t)$ where

$$\delta(t) = \max_{i=1,\ldots,N} \Big(|X_i^f(t) - X_i^N(t)| + |V_i^f(t) - V_i^N(t)| \Big)$$

In order to estimate $\delta(t)$, we write $\delta(t)$ in terms of the time integral of $\delta(s)$ and the difference of the interaction terms.

The hard term is

$$|W[Sf_s, \mu_s^N] - W[S\mu_s^N, \mu_s^N]| \leq c\mathcal{D}(Sf_s, S\mu_s^N).$$

Since

$$\mathcal{D}(Sf_s, S\mu_s^N) \leq \mathcal{D}(Sf_s, S\nu_s^N) + \mathcal{D}(S\nu_s^N, S\mu_s^N),$$

by the proximity Lemma we get

$$\mathcal{D}(S\nu_s^N, S\mu_s^N) \leq c\delta(s) + c\mathcal{D}(Sf_s, S\nu_s^N).$$

and using the Gronwall lemma, we readily get the estimate

$$\delta(t) \leq C(T) \int_0^t \mathcal{D}(Sf_s, S\nu_s^N)\,\mathrm{d}s,$$

valid for $0 \leq t \leq T$.

We conclude the proof by using the weak-strong continuity of $\mathcal{D}$ w.r.t to $\mathcal{W}$:

$$\delta(t) \leq C(T) \int_0^t \sqrt{\mathcal{W}(Sf_s, S\nu_s^N)}\,\mathrm{d}s$$

and using the estimate done in the first term

$$\delta(t) \leq C(T) \int_0^t \sqrt{\mathcal{W}(f_0, S\mu_0^N)}\,\mathrm{d}s$$

Collecting all togheter the first and the second estimate we get the thesis.

Conclusions

Extending to more dimensions a technique used by Trocheris, we deduced the mean-field limit for a topological Cucker-Smale model. Here the classical approach fails.

Possible future directions:

- Studying other interesting models with TI.
- Generalizations to more singular mean-field equations.
- General framework for the MFL outside of the Lipschitz case.

Conclusions

Extending to more dimensions a technique used by Trocheris, we deduced the mean-field limit for a topological Cucker-Smale model. Here the classical approach fails.

Possible future directions:

- Studying other interesting models with TI.
- Generalizations to more singular mean-field equations.
- General framework for the MFL outside of the Lipschitz case.

Minimal bibliography

- M. Ballerini et al. *Proc. Nat. Acad. Sci. USA* **105** (2008) 1232–1237.
- D. Benedetto, E. Caglioti, S. Rossi *Mathematics and Mechanics of Complex Systems* **9**, 4 (2021) 421–438.
- R. Dobrushin *Funct. Anal. Appl.* **13** (1979) 115–123
- M. Hauray *Sem. Laurent Schwatz - EDP et appl.* **21** (2012)
- M. Trocheris *Transport Theory and Statistical Physics* **15**, 5 (1986) 597–628.

Thank you for the attention!