Propagation of chaos for a stochastic system modelling epidemics via a coupling approach

Mario Pulvirenti

Dipartimento di Matematica, Universita' di Roma, Sapienza
and International Research Center on the Mathematics and Mechanics of Complex Systems MeMoCS, University of L'Aquila, Italy

IAC-CNR Roma, September 27-29 2022

Works in collaboration with A. Ciallella and S. Simonella

The well known SIR eq.ns (W. O. Kermack and A. G. McKendrick, 1927). The simplest version for the evolution of the fraction of the three species $\mathcal{S}, \mathcal{I}, \mathcal{R}$

$$\begin{cases} \dot{\mathcal{S}} = -\beta \mathcal{I} \mathcal{S} \\ \dot{\mathcal{I}} = \beta \mathcal{I} \mathcal{S} - \gamma \mathcal{I} \\ \dot{\mathcal{R}} = \gamma \mathcal{I} \end{cases}$$

$\mathcal{S}$ susceptible, $\mathcal{I}$ infected and $\mathcal{R}$ recovered. $\beta, \gamma > 0$.
Looking for a more detailed picture describing the time evolution of spatial patterns.

We do not pretend to explain the COVID process !

N particles (agents). Positions are in the two-dimensional torus $\mathcal{T}^2 = (0, D)^2$. Velocities are in S^1. $Z_N = \{z_i\}_{i=1}^N$ with $z_i = (x_i, v_i) \in \mathcal{T}^2 \times S^1$. Particles have a label. Set $A_N = \{a_i\}_{i=1}^N$. $a_i \in \{S, I, R\} = L$. Therefore a state of the system (Z_N, A_N). The system evolves according to a stochastic process whose generator is:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_d + \mathcal{L}_i^N, \tag{1}$$

where $\mathcal{L}_0 = \sum v_i \cdot \nabla_{x_i}$ is the generator of the free motion.

$$\mathcal{L}_1 \Phi(Z_N) = \sum_i \frac{1}{2\pi} \int_{S^1} dw [\Phi(z_1, \dots, x_i, w, \dots, z_N) - \Phi(Z_N)]$$

$v_i \to w$.

$\mathcal{L}_0 + \mathcal{L}_1$ generates N independent random flights. Here the dynamics does not involve labels.

$$\mathcal{L}_d \Phi(Z_N; A_N) = \gamma \sum_{i=1}^{N} [\Phi(Z_N; a_1, \dots, \tilde{a}_i, \dots, a_N) - \Phi(Z_N; a_1, \dots, a_i, \dots, a_N)].$$

The transition $a_i \to \tilde{a}_i$ is defined as

$$\tilde{a}_i = R \quad \text{iff} \quad a_i = I; \qquad \tilde{a}_i = a_i \quad \text{otherwise.}$$

Up to now the agents behave independently. The interaction:

$$\mathcal{L}_i^N \Phi(Z_N; A_N) = \lambda \frac{1}{N} \sum_{\substack{i \le N, j \le N \\ i < j}} \big\{ \Phi(Z_N; a_1, \dots, a'_i, \dots, a'_j, \dots, a_N)$$

$$- \Phi(Z_N; a_1, \dots, a_N) \big\}.$$

The transition $(a_i, a_j) \to (a'_i, a'_j)$ is defined as

$$\begin{cases} \text{If } \; a_i = I, a_j = S \; \text{ or } \; a_j = I, a_i = S \; \text{ and } \; \chi_{i,j} = 1, \; \text{ then } \; a'_i = a'_j = I \\ a'_i = a_i, \; a_j = a'_j \; \text{ otherwise.} \end{cases}$$

Here $\chi_{i,j}$ denotes the characteristic function

$$\chi_{i,j} = \chi\{x_i, x_j || x_i - x_j| < R_0\}, \quad R_0 > 0$$

and $\lambda > 0$ is given.

Statistical description given by an initial symmetric probability measure on the phase space $W_0^N(Z_N)$ normalization

$$\sum_{A_N} \int dZ_N W_0^N(Z_N; A_N) = 1.$$

The time evolved measure $W_t^N(Z_N; A_N), t > 0$ is given by

$$\sum_{A_N} \int dZ_N W_t^N(Z_N; A_N)\Phi(Z_N; A_N) =$$

$$\sum_{A_N} \int dZ_N W_0^N(Z_N; A_N)\mathbb{E}[\Phi(Z_N(t); A_N(t))],$$

where Φ is a test function, $(Z_N; A_N) \to (Z_N(t); A_N(t))$ is the process and $\mathbb{E} = \mathbb{E}_{(Z_N, A_N)}$ is the expectation conditioned to the initial value $(Z_N; A_N)$. The j-particle marginals, $j = 1, \ldots, N$, are defined by

$$f_j^N(Z_j; A_j; t) = \sum_{A_{N-j} \in L^{N-j}} \int dZ_{N-j} W_t^N(Z_j, Z_{N-j}; A_j, A_{N-j}; t)$$

the probability density of finding j agents with labels A_j in the configuration Z_j.

The notation is $Z_N = (Z_j, Z_{N-j}), Z_j = \{z_i\}_{i=1}^{j}, Z_{N-j} = \{z_i\}_{i=j+1}^{N}$, $z_i = (x_i, v_i)$, $A_N = (A_j, A_{N-j}), A_j = \{a_i\}_{i=1}^{j}, A_{N-j} = \{a_i\}_{i=j+1}^{N}$.
Full independence at time zero

$$W_0^N(Z_N; A_N) = \prod_{i=1}^{N} f_0(x, v, a_i)$$

where f_0 is a one-particle (normalized) distribution density.

We derive heuristically the kinetic equations we expect to be valid in the limit $N \to \infty$. We choose a test function of the type $\Phi(Z_N; A_N) = \phi(z_1; a_1)$.

$$\frac{d}{dt} \sum_a \int dz f_1^N(z;a) \phi(z;a) = \frac{d}{dt} \sum_{A_N} \int dZ_N W^N \Phi(Z_N; A_N)$$

$$= \sum_a \int dz f_1^N(z,a) (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_d) \phi(x,a)$$

$$+ \sum_{A_N} \int dZ_N W^N \mathcal{L}_i^N \Phi(Z_N; A_N).$$

The last term in the above expression is

$$\frac{\lambda}{N} \sum_{1<j} \sum_{A_N} \int dZ_N W^N (Z_N; A_N) [\phi(z_1, a_1') - \phi(z_1, a_1)] =$$

$$\lambda \frac{N-1}{N} \sum_{a_1, a_2} \int dz_1 dz_2 f_2^N (z_1, a_1; z_2; a_2) [\phi(z_1, a_1') - \phi(z_1, a_1)]$$

$$= \lambda \frac{N-1}{N} \int dz_1 dz_2 f_2^N(z_1, S; z_2; I) \chi_{1,2}[\phi(z_1, I) - \phi(z_1, S)]$$

We just used the symmetry of W^N and the fact that the only surviving transition is $a_1 = S, a_1' = a_2 = a_2' = I$. Assuming the propagation of chaos at time t in the limit $N \to \infty$, we obtain for the interaction term

$$\lambda \int dz_1 dz_2 f(z_1, S) f(z_2; I) \chi_{1,2}[\phi(z_1, I) - \phi(z_1, S)]$$

($f = \lim_N f_1^N$).

Therefore we conclude that the triple $f(S;t), f(I;t) f(R;t)$ satisfies the following kinetic equations ($z = (x, v)$),

$$(\partial_t + v \cdot \nabla_x - \mathcal{L}_1)\, f(z;S) = -\lambda f(z;S) \int f(z_1;I)\chi(|x - x_1| < R_0) dz_1$$

$$(\partial_t + v \cdot \nabla_x - \mathcal{L}_1)\, f(z;I) = +\lambda f(z;S) \int f(z_1;I)\chi(|x - x_1| < R_0) dz_1 - \gamma f(z;I)$$

$$(\partial_t + v \cdot \nabla_x \mathcal{L}_1)\, f(z;R) = +\gamma f(z;I).$$

Note that the sum

$$f(z,t) = \sum_{a \in L} f(z;a;t)$$

does satisfy the very simple random flight equation

$$(\partial_t + v \cdot \nabla_x)\, f(z,t) = \mathcal{L}_1 f(z,t).$$

As usual in kinetic theory, the dynamics creates correlations so that the measure is not factorized anymore at any positive times. However we hope to recover such independence in the limit $N \to \infty$, due to the mean-field nature of the interaction. Indeed the agents move independently and, given a pair of particles, say 1 and 2, the probability that the label of 2 influences the label of 1 is $O(\frac{1}{N})$ so that we expect that propagation of chaos does hold in the limit, provided it is fulfilled at time zero. As we shall see later on this does not always happen . We recall that by propagation of chaos we mean the statistical independence of any given finite group of particles at time $t > 0$. More explicitly the fact that any j-particle marginal factorizes.
At equilibrium, which is the uniform distribution on x and v, the above equations recover the SIR model

Hierarchies Vs Coupling.
The (nonlinear, one-particle) process associated with the kinetic equations.

$$\tilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_d + \tilde{\mathcal{L}}_i$$

where $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_d$ as before and

$$\tilde{\mathcal{L}}_i \phi(z,b) = \lambda \mathcal{N}_f \{\phi(z,b') - \phi(z,b)\},$$

and

$$\mathcal{N}_f(z;t) = (f(I;t) * \chi_{R_0})(z) = \int f(z_1, I; t)\chi(|x - x_1| < R_0) dz_1,$$

b' is taking value I if $b = S$, while in the other cases $b' = b$.

The N–particle process defined as N independent copies of the one–particle process. The generator is

$$\tilde{\mathcal{L}}^N = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_d + \tilde{\mathcal{L}}_i^N \quad ,$$

where

$$\tilde{\mathcal{L}}_i^N \Phi(Z_N, B_N) = \lambda \sum_{i=1}^{N} \mathcal{N}_f(z_i; t) \left\{ \Phi(Z_N, B_N^i) - \Phi(Z_N, B_N) \right\} .$$

and the i superscript has the meaning

$$\begin{cases} B_N^i = (b_1, \ldots, b_{i-1}, I, b_{i+1}, \ldots, b_N) \text{ if } B_N = (b_1, \ldots, b_{i-1}, S, b_{i+1}, \ldots, b_N) \\ B_N^i = B_N \text{ otherwise} \end{cases}$$

Comparison between $\mathcal{L}_i^N$ and $\tilde{\mathcal{L}}_i^N$.

$$\mathcal{L}_i^N \Phi(Z_N, A_N) = \lambda \sum_{i=1}^{N} \mathcal{J}_{emp}^i(Z_N; A_N) \left\{ \Phi(Z_N, A_N^i) - \Phi(Z_N, A_N) \right\},$$

$$\tilde{\mathcal{L}}_i^N \Phi(Z_N, B_N) = \lambda \sum_{i=1}^{N} \mathcal{N}_f(z_i; t) \left\{ \Phi(Z_N, B_N^i) - \Phi(Z_N, B_N) \right\},$$

$$\mathcal{J}_{emp}^i(Z_N; A_N) = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \delta_{a_j, I} \chi_{i,j}.$$

Define

$$\mathcal{E}^i = \mathcal{N}_f(z_i; t) - \mathcal{J}_{emp}^i(Z_N, B_N)$$

Then $\mathcal{E}^i = O(\frac{1}{N})$. L.l.n..

We remark that the positions and velocities are the same, so that the difference is only in the distributions of the labels. We consider as the coupling the process $t \to (Z_N(t), A_N(t), B_N(t))$ with generator Q^N. $R^N(t)$ is the law at time t for the coupled process, and we consider as the initial distribution

$$R^N(0) = f_0^{\otimes N}(Z_N, A_N)\delta_{A_N,B_N}.$$

We define $D_N(t)$ that takes into account the fraction of particles having different labels a_i, b_i, $\Phi(Z_N, A_N, B_N) = \frac{1}{N}\sum_{i=1}^N d(a_i, b_i)$, where $d(a,b) = 1 - \delta_{a,b}$. Thanks to the symmetry

$$D_N(t) = \int dR^N(t)\frac{1}{N}\sum_{i=1}^N d(a_i, b_i) = \int dR^N(t) d(a_1, b_1).$$

We notice that $D_N(0) = 0$ and $D_N(t)$ is positive.

In computing $\dot{D}_N(t)$ we need to specify the coupling as regards the interaction. We do it in words. We perform simultaneous jumps $(A_N, B_N) \to (A^i_N, B^i_N)$ when possible ($a_i = b_i = S$ and the partner particle is close and infected. This given no contribution to $\dot{D}_N(t)$. Then we perform independent jumps
$(A_N, B_N) \to (A^i_N, B_N), (A_N, B_N) \to (A_N, B^i_N)$ when $a_i \neq b_i$. This contribution is bounded by $\lambda D_N(t)$ itself. Finally there is also an error due to the change of rate in the B part, yielding

$$\int dR^N \mathcal{E}^i \leq \frac{1}{N} \sum_a \int dz dz_1 f(z, a, t) \chi_{R_0}(|x - x_1|) f(z_1, I, t) \leq \frac{1}{N}$$

In conclusion since $D_N(0) = 0$, we find that

$$\frac{d}{dt} D_N(t) \leq \lambda D_N(t) + \frac{\lambda}{2N},$$

hence, by the Gronwall Lemma:

$$D_N(t) \leq \frac{C(t)}{2N}.$$

A second model based on the stochastic process with generator:

$$\mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_d + \mathcal{L}_i^{2,N}$$

$$\mathcal{L}_i^{2,N}\Phi(Z_N; A_N) = \frac{\lambda}{N}\sum_{i=1}^{N}\left\{\Phi(Z_N; A'_N) - \Phi(Z_N; A_N)\right\},$$

and

$$\begin{cases} A'_N = A'_N(i) = \{a'_j\}_{j=1}^N \\ a'_j = I \ \text{ iff } \ a_j = S, a_i = I \ \text{ and } \ \chi_{i,j} = 1 \\ a'_j = a_j \ \text{ otherwise.} \end{cases}$$

As before but the infection mechanism is now different. We first choose an agent (say i) with uniform probability. Then with a rate λ all the susceptible agents around i (namely at distance less than R_0) become immediately infected. May be more realistic in some circumstances.

Same kinetic equations formally. Actually this is not the case, at least for a suitable choice of the parameters. Indeed, for instance, choosing R_0 larger than the side of the whole domain after the first jump all the susceptible agents disappear. On the other hand, as we shall see, in case of an homogeneous datum, the evolution of the labels follow the evolution of the SIR model and this contrasts with the vanishing of all the susceptible agents. Hence propagation of chaos does not hold.

Numerics.