# On the Lax–Wendroff theorem

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# Lax theorems

Two well known Lax theorems on consistency :

- Lax-Ritchmyer 1954 $\longrightarrow$ linear equations
- Lax-Wendroff 1960 $\longrightarrow$ nonlinear conservation laws

## Theorem (Lax–Richtmyer (LR))

*Consider a linear PDE for which the initial data problem is well posed, and a consistent finite difference scheme for its approximation;*

*then this scheme is convergent if and only if it is stable.*

This is often called the "Lax equivalence theorem". However it does **NOT** state

"consistency + stability $\Longleftrightarrow$ convergence"

see e.g. [Faille Gallouët H, Matapli 1991], [Eymard Gallouët H. 2000], [Despres 2004], [Eymard, Gallouët, H, Latché, Matapli November 2021]...

# The Lax-Wendroff Theorem

## Theorem (Lax–Wendroff (LW))

*Consider a numerical scheme for a system of nonlinear (hyperbolic) conservation laws; if the scheme is conservative, with consistent fluxes, and converges boundedly almost everywhere towards a limit as $\delta t$ and $h$ tend to 0, then*

*this limit is necessarily a weak solution of the system.*

System of nonlinear conservation laws : $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$, $f \in C^1(\mathbb{R}^p, \mathbb{R}^p)$

$$\partial_t U(x,t) + \partial_x f(U(x,t)) = 0, \; x \in \mathbb{R}, \; t \in ]0, T[.$$

Conservative numerical scheme :

$$h \frac{U_i^{n+1} - U_i^n}{\delta t} + F_{i+1/2}^n - F_{i-1/2}^n \text{ with } F_{i+1/2}^n : \text{ approximation of } f(U(x_{i+1/2}, t_n)).$$

$\iff$ (explicit-in-time) FV scheme for

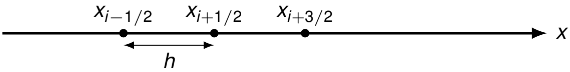
$$\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t U(x, t^n) + f(U(x_{i+1/2}, t^n)) - f(U(x_{i-1/2}, t^n)) = 0$$

Consistent (two-point) flux : $F_{i+1/2}^n = g(U_i^n, U_{i+1}^n)$ with $g(U, U) = f(U)$.

Note: conservative finite difference scheme = finite volume (FV) scheme

# LW theorem: proof in the 1D uniform scalar case, I

Sequence of space-time discretizations, with $h^{(m)} \to 0$, $\delta t^{(m)} \to 0$ as $m \to +\infty$




$u^{(m)}$ piecewise constant function $u^{(m)}(x,t) = u_i^n$ for $x \in K_i = ]x_{i-1/2}, x_{i+1/2}[$ and $t \in ]t_n, t_{n+1}[$, given by a conservative numerical scheme (or FV scheme)

$$h\frac{u_i^{n+1} - u_i^n}{\delta t} + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) = 0 \qquad (FV)$$

with consistent numerical fluxes $g(u,u) = f(u)$ :

Assume $\|u^{(m)}\| \leq C$ and $u^{(m)} \to u$ in $L^1$ as $m \to +\infty$.

We want to show that $u$ is a weak solution, i.e.

$$u \in L^\infty(\mathbb{R} \times \mathbb{R}_+^*);\ \forall \varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}),$$
$$\int_0^T\int_{\mathbb{R}} u(x,t)\partial_t\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}} u(x,t)\partial_x\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x = 0.$$

# LW theorem: sketch of proof, 1D uniform scalar case, II

Let $\varphi$ be a smooth function, and $\varphi_i^n = \frac{1}{h}\int_{K_i} \varphi(\cdot, t_n)$. Multiply

$$h\frac{u_i^{n+1} - u_i^n}{\delta t} + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) = 0$$

by $\delta t\ \varphi_i^n$, sum over $i$ and $n$, sum by parts :

- Conservativity $\rightsquigarrow$

$$-\sum_n \delta t \sum_i h u_i^n \frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t} - \sum_i h u_i^0 \varphi_i^0 - \sum_n \delta t \sum_i h \frac{\varphi_{i+1}^n - \varphi_i^n}{h} g(u_i^n, u_{i+1}^n) = 0$$

- Flux consistency $\rightsquigarrow$

$$-\sum_n \delta t \sum_i h u_i^n \frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t} - \sum_i h u_i^0 \varphi_i^0 - \sum_n \delta t \sum_i h \frac{\varphi_{i+1}^n - \varphi_i^n}{h} \underbrace{g(u_i^n, u_i^n)}_{f(u_i^n)} = R^{(m)}$$

$$|R^{(m)}| = \left| \sum_n \delta t \sum_i h \frac{\varphi_{i+1}^n - \varphi_i^n}{h} (g(u_i^n, u_i^n) - g(u_i^n, u_{i+1}^n)) \right|$$

$$\leq C_\varphi \delta g^{(m)} \text{ with } \delta g^{(m)} = \sum_n \delta t \sum_i h |g(u_i^n, u_i^n) - g(u_i^n, u_{i+1}^n)|$$

- $g$ “Lip-diag” : $|g(u, v) - g(u, u)| \leq L_g |u - v|$ $\rightsquigarrow$

$$\delta g^{(m)} \leq L_g \sum_n \delta t \sum_i h |u_{i+1}^n - u_i^n|$$

# Lax-Wendroff Theorem: sketch of proof, 1D uniform scalar case, III

- $|\delta g^{(m)}| \leq L_g \int\int |u^{(m)}(x+h,t) - u^{(m)}(x,t)| \,\mathrm{d}x\,\mathrm{d}t$

  Convergence in $L^1$ + Kolmogorov theorem $\rightsquigarrow \Longrightarrow \delta g^{(m)}$ and $R^{(m)} \to 0$ as $m \to +\infty$

- pass to the limit as $m \to +\infty$

$$\underbrace{-\sum_n \delta t \sum_i h u_i^n \overbrace{\frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t}}^{\to \partial_t \varphi}}_{\to\ -\int\int u \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t} \quad \underbrace{-\sum_i h u_i^0 \varphi_i^0}_{-\int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x\,\mathrm{d}t} \quad \underbrace{-\sum_n \delta t \sum_i h \overbrace{\frac{\varphi_{i+1}^n - \varphi_i^n}{h}}^{\to \partial_x \varphi} f(u_i^n)}_{-\int\int f(u)\partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t} = \underbrace{R^{(m)}}_{\to 0}$$

**Remark** No need for BV estimates (because of the assumption $u^{(m)} \to u$ in $L^1$).

# Lax-Wendroff Theorem: strong and weak points

- weak point:
  - ”IF” theorem: needs coompactness, hard to obtain for general non structured meshes, even in the scalar case.
- strong points

  **# 1** introduces two crucial notions: conservativity, and consistency of the fluxes crucial for the mathematical analysis of FV schemes, for hyperbolic conservation laws, but also for elliptic and parabolic equations.

  **# 2** tool to design the schemes for complex systems

  LW consistency: if the scheme converges strongly and boundedly to some limit, then this limit is a weak solution.

# Strong point # 1 convergence of FV schemes (I)

Discretization on a 2D triangular mesh of the linear transport equation

$$\partial_t u + \mathrm{div}(\boldsymbol{v} u) = 0$$

**FD** scheme : if consistent and stable, (LR) $\rightsquigarrow$ convergence.
Needs the regularity of the exact solution. Not much used in industrial codes

**FV** scheme [Champier Gallouët, RAIRO 1992]. Conservative and consistent fluxes, upwinding $\rightsquigarrow$ stability.

- not consistent in the FD sense. (LR) does not apply
- $||u^{(m)}||_{L^\infty} \leq C \rightsquigarrow u^{(m)} \to u$ in $L^\infty$ weak $\star$.

# Strong point # 1 convergence of FV schemes (II)

**FV** scheme for $\partial_t u + \mathrm{div}(\boldsymbol{v}u) = 0$, $u^{(m)} \to u$ in $L^\infty$ weak $\star$.

- show that $u$ is a weak solution by passing to the limit in the scheme: $h^{(m)}$ and $\delta t^{(m)} \to 0$ as $m \to +\infty$.
- Use LW idea of the conservativity of the fluxes: mutiply scheme by test function, sum by parts so that the discrete derivatives are on the test function

$$\rightsquigarrow -\int_{\mathbb{R}_+}\int_{\mathbb{R}^2} (u^{(m)}\varphi_t + u^{(m)}\boldsymbol{v}\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}_+} u_0(x)\varphi(x,0)\,\mathrm{d}x = R^{(m)}.$$

- $R^{(m)} \to 0$ as $m \to +\infty$?
  1. Assume $u^{(m)} \to u$ in $L^1$, then $R^{(m)} \to 0$ as in (LW).
     $u^{(m)} \to u$ in $L^1$ true on Cartesian meshes, thanks to $BV$ estimate if $u_0 \in BV$, and Helly's theorem $\rightsquigarrow$ huge litterature on "TVD" schemes. .

     BUT: Upwind FV scheme is not TVD on triangles [[Champier Gallouët, RAIRO 1992]]
  2. Show a "weak $BV$ estimate": $\displaystyle\sum_{\sigma=K|L} v_\sigma |u_K^{(m)} - u_L^{(m)}| \le C\frac{1}{\sqrt{h^{(m)}}}$
     Does not yield any compactness in $L^1$, but yields that the limit is a weak solution.

LW theorem not used in the proof of convergence, but crucial concepts from LW are.

# Strong point 2, “Lax-Wendroff consistency”

Assuming estimates and compactness, show that the scheme converges to a weak solution of the continuous problem.

Example: Compressible Euler equations, not much known!

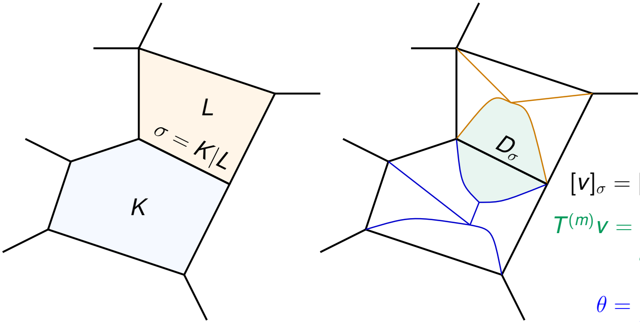
LW consistency: if the scheme converges strongly and boundedly to some limit, then this limit is a weak solution.

Multidimensional setting?

# Lax-Wendroff Theorem: possible generalizations

- time implicit scheme OK
- convergence to the entropy weak solution (if discrete entropy inequality available)
- non constant space step [Eymard, Gallouët H. 2000], $h_i \geq \alpha h$.
- Multidimensional meshes, colocated schemes:
  - Triangular meshes, [Kröner Rokyta Wierse 1996] [Godlewski Raviart 1996]
  - Quasi-uniform meshes, [Elling, 2007] by relating to a Cartesian mesh.
  - “lip-diag” numerical flux, general meshes [Gallouët, H., Latché. 2019] adaptation of Kolmogorov’s theorem on the limit of the space translates
- Multidimensional meshes, staggered schemes [Gallouët, H., Latché 2021] adaptation of the flux condition $g(u, u) = f(u)$

# Space translates for multiD meshes




$$[v]_\sigma = |v_K - v_L|$$

$$T^{(m)} v = \sum_{\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L} |D_\sigma| [v]_\sigma$$

$$\theta = \max_{K \in \mathcal{M}} \max_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma|}{|K|}$$

## Lemma (Space translates, multi-D)

*Sequence of meshes s.t.* $h_m \to 0$ *as* $m \to +\infty$,

- *with number of faces of each cell bounded by* $N_{\text{faces}}$, *independent of m.*
- *Regularity condition* $\theta^{(m)} \leq \Theta$, $\forall m \in \mathbb{N}$
- $(v_p)_{p \in \mathbb{N}} \subset L^1(\Omega)$, $v_p \to v$ *in* $L^1(\Omega)$ *as* $p \to +\infty$, $(v_p)_K = \dfrac{1}{|K|} \displaystyle\int_K v_p \, \mathrm{d}x$

*then* $T^{(m)} v_p \to 0$ *as* $m \to +\infty$ *uniformly w.r.t. p.*

For a 1D uniform mesh, $Tu = \int_\Omega |u(x+h) - u(x)| \, \mathrm{d}x$.

# Space translates, multi-D meshes: main ideas of the proof

As in the 1D case: decompose in two steps:

- Step 1 - Continuity in mean for a given piecewise constant function:
  $v \in L^1(\Omega)$, $v_K = \dfrac{1}{|K|}\displaystyle\int_K v \,\mathrm{d}x$, $T^{(m)} v = \displaystyle\sum_{\sigma=K|L} |D_\sigma| [v]_\sigma$. Then

$$\lim_{m\to+\infty} T^{(m)} v = 0.$$

- Step 2 - Convergence of the difference of a translate for a sequence of piecewise functions on a sequence of meshes.
  $(v_p)_{p\in\mathbb{N}} \subset L^1(\Omega)$, $v_p \to v$ in $L^1(\Omega)$ as $p \to +\infty$, $(v_p)_K = \dfrac{1}{|K|}\displaystyle\int_K v_p \,\mathrm{d}x$

$$T^{(m)} v_p = \sum_{\sigma=K|L} |D_\sigma| [v_p]_\sigma$$

  Then $T^{(m)} v_p \to 0$ as $m \to +\infty$ uniformly with respect to $p \in \mathbb{N}$

# Space translates, multi-D meshes: proof of Step 1

- Bound on $T^{(m)}v$

$$T^{(m)}v = \sum_{\sigma=K|L} |D_\sigma| [v]_\sigma \leq \sum_{\sigma=K|L} |D_\sigma| \left(|v_K| + |v_L|\right)$$
$$\leq \theta^{(m)} \sum_{\sigma=K|L} \left(|K|\,|v_K| + |L|\,|v_L|\right) \leq N_{\text{faces}}\,\theta^{(m)}\,||v||_{L^1(\Omega)}.$$

- Then, for any $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$,

$$T^{(m)}v \leq T^{(m)}(v-\varphi) + T^{(m)}\varphi \leq \underbrace{N_{\text{faces}}\,\theta^{(m)}\,||v-\varphi||_{L^1(\Omega)}}_{\leq \varepsilon \text{ by density}} + \underbrace{T^{(m)}\varphi}_{= \sum_{\sigma=K|L} |D_\sigma||\varphi_K - \varphi_L|}.$$

$$|\varphi_K - \varphi_L| = \frac{1}{|K|\,|L|}\Big|\int_K \int_L (\varphi(x) - \varphi(y))\,\mathrm{d}y\,\mathrm{d}x\Big| \leq \frac{1}{|K|\,|L|}\int_K \int_L \underbrace{|\varphi(x) - \varphi(y)|}_{\leq M_\varphi |x-y|}\,\mathrm{d}y\,\mathrm{d}x$$

$$\bar{K} \cap \bar{L} \neq \emptyset \rightsquigarrow |x-y| \leq h_K + h_L\,,$$

$$T^{(m)}\varphi \leq 2M_\varphi\,h^{(m)} \sum_{\sigma=K|L} |D_\sigma| \leq 2M_\varphi\,h^{(m)}|\Omega| \leq \varepsilon \text{ for } m \geq m_0.$$

Finally, $T^{(m)}v \leq 2\varepsilon \quad \text{for } m \geq m_0.$

# The Lax-Wendroff theorem for multidimensional meshes

- $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$, $\theta^{(m)} \le \Theta$, $h_{\mathcal{M}^{(m)}}$, $\delta t^{(m)} \to 0$ as $m \to +\infty$, and,
- $(u^{(m)})_{m\in\mathbb{N}}$ converges to in $\bar{u}$ $L^1(\Omega \times (0,T))$ to a function $\bar{u}$.
- $f(u^{(m)})) \to f(\bar{u})$ in $L^1(\Omega \times (0,T))$,
- $\delta g^{(m)} \to 0$ as $m \to +\infty$, replacing

$$\text{(1D)} \quad \delta g^{(m)} = \sum_n \delta t \sum_i \; h \; \left(g(u_i^n, u_{i+1}^n) - f(u_i^n)\right).$$

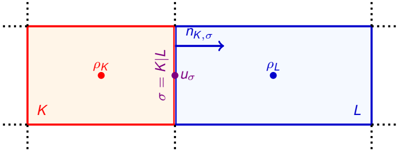
by

$$\text{(MD)} \quad \delta g^{(m)} = \sum_n \delta t \sum_K \mathrm{diam}(K) \sum_{\sigma \subset \partial K} |\sigma| \left| \left( (g^{(m)})_\sigma^n - f(u_K^n) \right) \cdot n_{K,\sigma} \right| \mathrm{d}x \, \mathrm{d}t$$

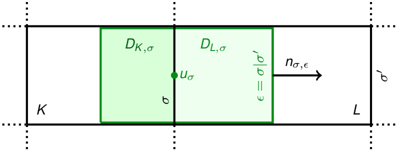
Then $\bar{u}$ is a weak solution of $\partial_t u + \mathrm{div}(f(u)) = 0$ + C.I.

# Staggered meshes: the isentropic Euler equations

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = 0,$$
$$p = \rho^\gamma$$



Primal mesh



Dual mesh for the first component of the velocity

- Density on the primal grid. $\rho_{\mathrm{app}} = \sum_K \rho_K \mathbb{1}_K$
- Velocity on the dual grids. $u_{\mathrm{app}} = \begin{bmatrix} \sum_{\sigma \perp \mathrm{e}_1} u_{1,\sigma} \mathbb{1}_{D_\sigma} \\ \sum_{\sigma \perp \mathrm{e}_2} u_{2,\sigma} \mathbb{1}_{D_\sigma} \end{bmatrix}$
- Mass equation on primal grid: $u$ is not constant on $K$
- Momentum equation on dual grids: $\rho$, $u_{2,\sigma}$, not constant on $D_\sigma$ $\sigma \perp e_1$,

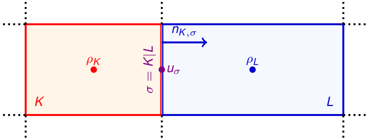
In both cases, the numerical flux depends on $U = (\rho_{\mathrm{app}}, u_{\mathrm{app}}) (= (\rho_K)_K, (u_{i,\sigma})_\sigma)$

# General meshes: generalization of the flux condition

- Example: isentropic Euler, unknown $U = (\rho, u_1, u_2)$, mass equation $\partial_t \beta(U) + \mathrm{div}(f(U)) = 0$ with $\beta(U) = \rho$, $f(U) = \rho u$

$$|K| \frac{\rho_K^{(n+1)} - \rho_K^{(n)}}{\delta t} + \sum_{\sigma \subset \partial K} g_\sigma \cdot n_{K,\sigma} = 0$$

- Flux consistency colocated case: $g_\sigma(U, U) = f(U)$
- Staggered case: more difficult to write for a general scheme . Here: $g_\sigma(\rho, \rho, u) = f(\rho, u) = \rho u$




- Flux condition $\delta g^{(m)} \to 0$ as $m \to +\infty$.

$$\text{(MD)} \qquad \delta g^{(m)} = \sum_n \delta t \sum_K \mathrm{diam}(K) \sum_{\sigma \subset \partial K} |\sigma| \left| \left( (g^{(m)})^n_\sigma - f(U^n_K) \right) \cdot n_{K,\sigma} \right| \mathrm{d}x \, \mathrm{d}t$$

replaced by $\widetilde{\text{(MD)}}$:

$$\delta g^{(m)} = \sum_n \delta t \sum_K \mathrm{diam}(K) \sum_{\sigma \subset \partial K} |\sigma| \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \frac{1}{|K|} \int_K \left| \left( (g^{(m)})^n_\sigma - f(U^m)(x, t) \right) \cdot n_{K,\sigma} \right| \mathrm{d}x \, \mathrm{d}t$$

$$U^{(m)} = (\rho^{(m)}, u_1^{(m)}, u_2^{(m)})$$

# Generalized Lax-Wendroff theorem: the setting

- Multi-dimensional conservative convection operator $\Omega \times (0,T) \to \mathbb{R}$ :

Continuous setting $\mathcal{C}(\mathcal{U})$

$(x,t) \in \Omega \times (0,T) \mapsto$

$$\partial_t(\beta(\mathcal{U}))(x,t) + \mathrm{div}(f(\mathcal{U}))(x,t)$$

Discrete setting $C(U)$

$(x,t) \in K \times [t_n, t_{n+1}[ \mapsto C(U)_K^n$

$$= (\eth_t \beta)_K^n + \frac{1}{|K|} \sum_{\sigma \subset \partial K} |\sigma| \; g_\sigma^n \cdot n_{K,\sigma}$$

- $U_0 \in L^\infty(\Omega, \mathbb{R}^p)$
- sequence of meshes and time steps $h^{(m)} \to 0$ and $\delta t^{(m)} \to 0$ as $m \to +\infty$,
- $(U^{(m)})_{m \in \mathbb{N}}$ sequence of functions such that

$$\|U^{(m)}\|_\infty \leq C^u,$$
$$U^{(m)} \to \mathcal{U} \in L^\infty(\Omega \times (0,T), \mathbb{R}^p) \text{ in } L^1(\Omega \times (0,T), \mathbb{R}^p) \text{ as } m \to +\infty.$$

- $|\beta^{(m)})_K^n| \leq C^\beta$

# Generalized Lax-Wendroff theorem: the theorem

Assume that as $m \to +\infty$,

$$\sum_K \int_K ((\beta^{(m)})_K^0 - \beta(U_0)(x))\ \varphi(x)\,\mathrm{d}x \to 0, \text{ for any } \varphi \in C_c^\infty(\Omega),$$

$$\sum_n \sum_K \int_{t_{n-1}}^{t_n} \int_K \Big((\beta^{(m)})_K^n - \beta(U^{(m)})(x,t)\Big)\ \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \to 0, \text{ for any } \varphi \in C_c^\infty(\Omega \times [0,T)),$$

$$\delta g^{(m)} = \sum_n \delta t \sum_K \mathrm{diam}(K) \sum_{\sigma \subset \partial K} |\sigma| \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \frac{1}{|K|} \int_K \Big| \Big((g^{(m)})_\sigma^n - f(U^m)(x,t)\Big) \cdot n_{K,\sigma} \Big| \,\mathrm{d}x\,\mathrm{d}t \to 0,$$

Then, for any $\varphi \in C_c^\infty(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega \mathcal{C}^{(m)}(U^{(m)})\, \mathcal{I}^{(m)}(\varphi)(x,t)\,\mathrm{d}x\,\mathrm{d}t \to -\int_\Omega \beta(U_0)(x)\ \varphi(x,0)\,\mathrm{d}x$$
$$-\int_0^T \int_\Omega \Big(\beta(\mathcal{U})(x,t)\ \partial_t\varphi(x,t) + f(\mathcal{U})(x,t) \cdot \boldsymbol{\nabla}\varphi(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } m \to +\infty,$$

$\mathcal{I}^{(m)}(\varphi)$ : interpolate of $\varphi$

$$\mathcal{I}^{(m)}(\varphi)(x,t) = \varphi_K^n = \frac{1}{|K|} \int_K \varphi(x,t_n)\,\mathrm{d}x \text{ for } x \in K \text{ and } t \in (t_n, t_{n+1}),$$

# LW Consistency of a staggered scheme, I

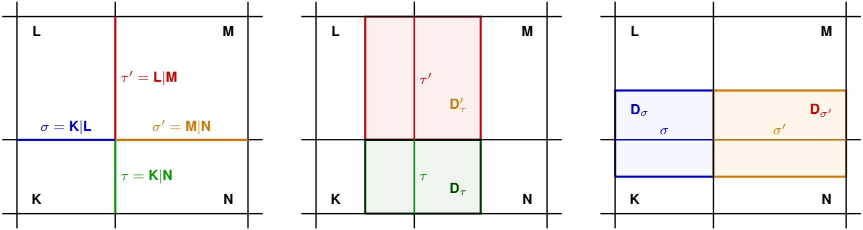
Example: non linear hyperbolic equation on a MAC grid.

$$\partial_t \rho + \mathrm{div}(\psi(\rho) u) = 0$$

$$\mathcal{U} = (\rho, u_1, u_2)$$

$$\partial_t \beta(\mathcal{U}) + \mathrm{div}(f(\mathcal{U})) = 0$$

$$\rightsquigarrow \begin{cases} |K| \dfrac{\rho_K^{(n+1)} - \rho_K^{(n)}}{\delta t} + \displaystyle\sum_{\substack{\sigma \subset \partial K \\ \sigma = K|L}} g_\sigma^n(U_K^n, U_L^n) \cdot n_{K,\sigma} = 0 \\ g_\sigma^n(U_K^n, U_L^n) = \psi(\rho_\sigma^n) u_\sigma^n, \\ \rho_\sigma^n \text{ : convex combination of } \rho_K^n, \rho_L^n \end{cases}$$



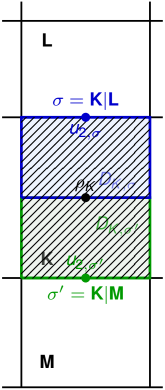

$$\delta g^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_K \frac{\mathrm{diam}(K)}{|K|} \int_{t_n}^{t_{n+1}} \int_K \sum_{\sigma \subset \partial K} |\sigma| \left| \Big( (g^{(m)})_\sigma(U_K, U_L) - f(U^m)(x,t) \Big) \cdot n_{K,\sigma} \right| \mathrm{d}x \, \mathrm{d}t$$

$\delta g^{(m)} \to 0$ as $m \to +\infty$ ??

# LW Consistency of a staggered scheme, II

$$\delta g^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_K \frac{\mathrm{diam}(K)}{|K|} \sum_{\sigma \subset \partial K} |\sigma| \int_{t_n}^{t_{n+1}} \int_K r_{K,\sigma}^{(m)}(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

$$r_{K,\sigma}^{(m)} = \Big| \Big( (g^{(m)})_\sigma(U_K^{(m)}, U_L^{(m)}) - f(U^{(m)})(x,t) \Big) \cdot n_{K,\sigma} \Big|$$




Example: bound on $r_{K,\sigma}^{(m)}$ for $K$ and $\sigma = K|L$ on the figure.

$$(g^{(m)})_\sigma^n \cdot n_{K,\sigma} = \psi(\rho_\sigma^n)\, u_{2,\sigma}^n, \quad f(U^{(m)})(x,t) = \begin{cases} \psi(\rho_K^n)\, u_{2,\sigma}^n, & x \in D_{K,\sigma} \\ \psi(\rho_K^n)\, u_{2,\sigma'}^n, & x \in D_{K,\sigma'} \end{cases}$$

$$\Longrightarrow r_{K,\sigma}^{(m)} = \begin{cases} \left|\psi(\rho_\sigma^n)\, u_{2,\sigma}^n - \psi(\rho_K^n)\, u_{2,\sigma}^n\right| \text{ if } x \in D_{K,\sigma}, \\ \left|\psi(\rho_\sigma^n)\, u_{2,\sigma}^n - \psi(\rho_K^n)\, u_{2,\sigma'}^n\right| \text{ if } x \in D_{K,\sigma'}, \end{cases}$$

$$\Big| \Big( (g^{(m)})_\sigma^n - f(U^{(m)})(x,t) \Big) \cdot n_{K,\sigma} \Big| \leq C\, \big( |\rho_K^n - \rho_L^n| + |u_{2,\sigma} - u_{2,\sigma'}^n| \big),$$

# LW Consistency of a staggered scheme, III

$$\delta g^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_{K} \frac{\mathrm{diam}(K)}{|K|} \sum_{\sigma \subset \partial K} |\sigma| \int_{t_n}^{t_{n+1}} \int_K r_{K,\sigma}^{(m)}(x,t)\, \mathrm{d}x\, \mathrm{d}t$$

$$r_{K,\sigma}^{(m)} = \Big| \Big( (g^{(m)})_\sigma (U_K^{(m)}, U_L^{(m)}) - f(U^{(m)})(x,t) \Big) \cdot n_{K,\sigma} \Big|$$

Finally

$$\delta g^{(m)} \leq C\, (\delta g_1^{(m)} + \delta g_2^{(m)}),$$

$$\delta g_1^{(m)} = \sum_n \delta t^{(m)} \sum_K \mathrm{diam}(K) \sum_{\substack{\sigma \subset \partial(K), \\ \sigma = K|L}} |\sigma|\, |\rho_K^n - \rho_L^n|,$$

$$\delta g_2^{(m)} = \sum_n \delta t^{(m)} \sum_K \mathrm{diam}(K) \sum_{i=1,2} \sum_{\substack{\sigma,\sigma' \subset \partial K \\ \sigma,\sigma' \perp e_i}} (|\sigma| + |\sigma'|)\, |u_{i,\sigma}^n - u_{i,\sigma'}^n|$$

$\delta g^{(m)} \to 0$ by Lemma on the translates (generalized to space time functions).

Example of application : LW-consistency of RK2 in time + MAC-MUSCL in space scheme for the shallow water eqns. [H. Latché Nasseri Therme 2021]

# Conclusion

- 1960, Lax and Wendroff
  - two crucial notions for FV theory: conservativity and consistency of the flux
  - LW consistency for nonlinear hyperbolic conservation laws: *if a sequence of approximate solutions converges a.e. and boundedly, then the limit is a weak solution.*
  - In the proof of consistency (uniform 1D mesh) (silent) use of the Kolmogorov result on the space translates
  - no need for *BV* bound.
- Generalization to piecewise constant functions on general, possibly staggered meshes:
  - Space translates: [Gallouët, H., Latché, SeMa 2019]
  - Flux condition and application to staggered schemes, [Gallouët, H., . Latché, SeMa 2021]
  - Application to the MAC discretization of the shallow water equations, H., Latché, Nasseri, Therme, IMAJNA 2022
- Applicable to implicit or semi-implicit schemes and wide stencils (higher order schemes)...
- ... but not always applicable, e.g. ongoing work for the projection algorithm for incompressible NS [Gallouet-H.-Latché-Maltese]