Quantum Topology Beyond Semisimplicity

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A motivational toy model: Gauge theory with finite structure group

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To a surface Σ, it assigns the state space (or *conformal block*)

$$\begin{aligned} Z_G(\Sigma) &= k[\pi_0(\mathrm{PBun}_G(\Sigma))] \\ &= k[\text{isomorphism classes of } G\text{-bundles over } \Sigma] \ , \end{aligned}$$

where k is our ground field (always algebraically closed).

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A ‘richer’ state space is the chain complex of vector spaces

$$C_*(\mathsf{PBun}_G(\Sigma); k) \simeq \bigoplus_{[P]\in\pi_0(\mathsf{PBun}_G(\Sigma))} C_*(\mathrm{Aut}(P); k)$$

given by the chains on the groupoid of G-bundles over Σ.

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given by the chains on the groupoid of G-bundles over Σ.
If $|G|$ divides the characteristic of k (i.e. if $k[G]$ is not semisimple), this contains more information.

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An account in the context of algebraic quantum field theory is: M. Benini, A. Schenkel, R. Szabo. Homotopy Colimits and Global Observables in Abelian Gauge Theory. *Lett. Math. Physics* 105:1193–1222, 2015.

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Sources of examples

Modular categories arise as categories of modules over a ribbon factorizable Hopf algebra or a suitable vertex operator algebra.

A reminder on modular categories (Moore-Seiberg data)

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- A *ribbon structure* on a braided finite tensor category $\mathcal{C}$ is a natural automorphism θ of $\mathrm{id}_{\mathcal{C}}$ which satisfies

$$\begin{aligned} \theta_{X \otimes Y} &= c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y) \ , \\ \theta_I &= \mathrm{id}_I \ , \\ \theta_{X^\vee} &= \theta_X^\vee \end{aligned}$$

for all $X, Y \in \mathcal{C}$.

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The semisimple case

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Moreover, semisimple modular categories are essentially equivalent to 3-2-1-dimensional topological field theories [Bartlett-Douglas-Schommer-Pries-Vicary 15].

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We will not assume semisimplicity!

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  - a fixed point on every boundary component
  - and an orientation of the boundary (leading to *incoming* and *outgoing* boundary components)

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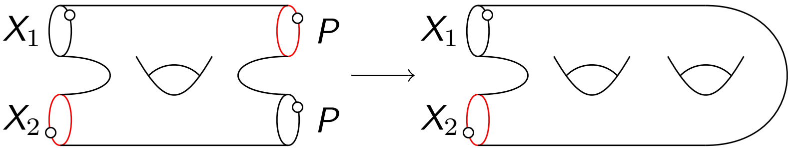
- and whose morphisms are generated by mapping classes (twisted by a cocycle coming from the framing anomaly) and sewings compatible with labels.

Disjoint union provides a symmetric monoidal structure.

Example of a sewing (incoming boundary in red):



Differential graded modular functor of a modular category

Definition

A *differential graded modular functor* $M : \mathcal{C}\text{-}\mathsf{Surf}^c \longrightarrow \mathsf{Ch}_k$ for a linear category $\mathcal{C}$ is a symmetric monoidal functor whose cylinder category is equivalent to $\operatorname{Proj}\mathcal{C}$ and which satisfies excision (compatibility with sewing expressed through homotopy coends).

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Theorem [Schweigert-W. 20]

Any modular category $\mathcal{C}$ gives rise in a canonical way to a differential graded modular functor $\mathfrak{F}_{\mathcal{C}} : \mathcal{C}\text{-Surf}^c \longrightarrow \mathrm{Ch}_k$.

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In homology, this recovers mapping class group actions given by Lentner-Mierach-Schweigert-Sommerhäuser (and provides a marking-independent descriptions of these actions). In zeroth homology, this recovers Lyubashenko's mapping class group actions.

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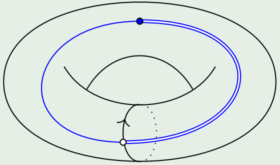
The torus



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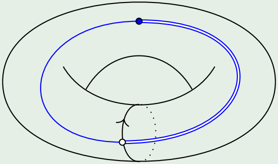

After choosing this marking, there is a canonical equivalence

$$\int_{\mathbb{L}}^{X \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X,X) \xrightarrow{\simeq} \mathfrak{F}_\mathcal{C}(\mathbb{T}^2) \ ,$$

where the homotopy coend on the left hand side is actually the Hochschild complex of $\mathcal{C}$

$$\ldots \quad \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0,X_1 \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X_1,X_0) \otimes \mathcal{C}(X_0,X_1) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0 \in \operatorname{Proj} \mathcal{C}} \mathcal{C}(X_0,X_0) \ .$$

The Verlinde formula

For a semisimple modular category $\mathcal{C}$, the conformal block for the torus

$$\mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2) = k\{x_0 = I, x_1, \ldots, x_n\}$$

is spanned by the simple objects of $\mathcal{C}$ and comes with a commutative product

$$[x_i] \otimes [x_j] = \sum_{\ell=0}^{n} N_{ij}^{\ell}[x_\ell] \ , \quad \textit{(Verlinde algebra)}$$

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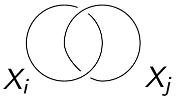
where the non-negative integers N_{ij}^{ℓ} are the *fusion coefficients*.The *Verlinde formula* tells us

$$N_{ij}^{\ell} = \sum_{p=0}^{n} \frac{S_{ip} S_{jp} \left(S^{-1}\right)_{p\ell}}{S_{0p}} \quad \text{with the } S\text{-matrix} \quad S_{ij} =$$



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The Verlinde formula

The Verlinde formula can be rephrased by saying that the mapping class group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathsf{SL}(2,\mathbb{Z})$ acts as an algebra isomorphism

$$\underset{\textit{fusion product}}{\big(\mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2), \otimes\big)} \quad \xrightarrow{S} \quad \underset{\textit{diagonal product}}{\big(\mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2), \star\big)} \;,$$

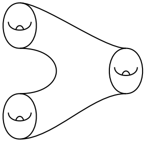
where

$$[x_i] \star [x_j] = d_i^{-1} \delta_{ij} [x_i]$$

with d_i being the quantum dimension of x_i.
In other words: *S diagonalizes the fusion!*

The Verlinde formula

The fusion product $\otimes$ with structure constants N_{ij}^{ℓ} is the evaluation on the three-dimensional bordism:



$$= \quad \text{pair of pants} \times \mathbb{S}^1 \quad : \quad \mathbb{T}^2 \sqcup \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \ .$$

The Verlinde formula can be regarded as a consequence of the breaking of symmetry in the definition of this bordism.

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Theorem (tentative incorrect version)

Let $\mathcal{C}$ be a braided finite tensor category.

1. *The Hochschild chain complex of $\mathcal{C}$ comes with structure of a commutative (?!?) algebra induced by the monoidal product.*
2. *The Hochschild cochain complex of $\mathcal{C}$ comes with the structure of a commutative (?!?) algebra induced by the monoidal product.*

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We have seen that, in the differential graded setting, the conformal block of the torus is the Hochschild (co)chain complex.

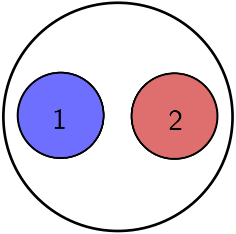
Theorem [Schweigert-W. 19/21]

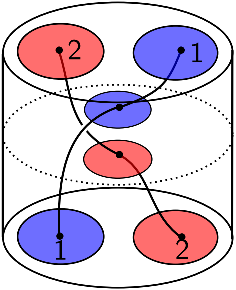
Let $\mathcal{C}$ be a braided finite tensor category.

1. *The Hochschild chain complex of $\mathcal{C}$ comes with structure of a non-unital E_2-algebra induced by the monoidal product.*
2. *Let $\mathcal{C}$ be additionally unimodular. The Hochschild cochain complex of $\mathcal{C}$ comes with the structure of an E_2-algebra induced by the monoidal product.*

Commutativity of the differential graded Verlinde algebra

An E_2-algebra is an algebra whose commutativity behaviour is parametrized by the *braid group*:





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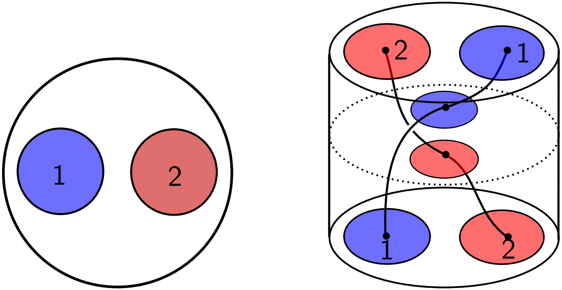

The homology of an E_2-algebra is a *Gerstenhaber algebra*. For *framed* E_2-algebras, one obtains a *Batalin-Vilkovisky algebra*.

Deligne's Conjecture

Theorem [Gerstenhaber 63]

The Hochschild cohomology of an algebra naturally comes with a bracket of degree -1. (Today this is called a Gerstenhaber algebra.)

Deligne conjectured in 1993 that this structure comes in fact from an E_2-structure on Hochschild cochains. This was later proven by Tamarkin, McClure-Smith, Berger-Fresse, Costello and many others.

The differential graded Verlinde formula

- At the heart of our argument leading to the differential graded Verlinde formula lies a new ‘quantum algebra adapted’ solution to Deligne’s Conjecture on $\mathcal{C}(I, \mathbb{A}^\bullet)$, where $\mathbb{A}^\bullet$ is an injective resolution of the canonical end $\mathbb{A} = \int_{X \in \mathcal{C}} X \otimes X^\vee$.

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- The end $\mathbb{A} = \int_{X \in \mathcal{C}} X \otimes X^\vee$, together with the so-called *non-crossing half braiding*, becomes a braided commutative algebra in the Drinfeld center of $\mathcal{C}$ (this is a well-known argument due to Davydov-Müger-Nikshych-Ostrik). This makes $\mathbb{A}^\bullet$ an algebra over an acyclic braided operad, and $\mathcal{C}(I, \mathbb{A}^\bullet)$ passes to the symmetrization of this acyclic braided operad, which is E_2!

The differential graded Verlinde formula

Theorem [Schweigert-W. 21]

Let $\mathcal{C}$ be a modular category. The mapping class group element S acts on the Hochschild (co)chain complex as an equivalence of E_2-algebras

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(Through the action of the mapping class group element S, the differential graded Verlinde algebra (both on the Hochschild chain and cochain complex) is transformed into the E_2-algebra afforded by the cyclic Deligne Conjecture applied to the Calabi-Yau structure coming from the modified trace on the tensor ideal of projective objects.)

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differential graded Verlinde algebra $\xrightarrow{S}$ *Deligne's E_2-structure* .

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2. In the chain version, the differential graded Verlinde algebra has a product supported in homological degree zero. In degree zero, it is, however, rather non-trivial. We achieve in degree zero a block diagonalization of the Verlinde algebra.
3. In the cochain version version, we recover in zeroth cohomology a non-semisimple generalization of the linear Verlinde formula given by Gainutdinov-Runkel. The E_2-structure, however, has strictly more information and, for instance, a non-trivial Gerstenhaber bracket.

Important tool: The trace field theory

The *modified trace* [Geer, Kujawa, Patureau-Mirand, Turaev, Virelizier, . . .]

$$t : \mathcal{C}(P, P) \longrightarrow k \ , \quad P \in \mathrm{Proj}\,\mathcal{C}$$

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It can be described through a trivialization of the Nakayama functor of $\mathcal{C}$ as module functor relative to the pivotal structure [Shibata-Shimizu, Schweigert-W.].

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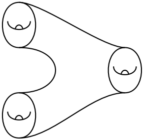
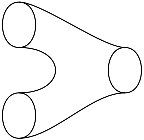
From the Calabi-Yau structure on $\operatorname{Proj}\mathcal{C}$, we can build, through a construction of Costello, an *open closed topological conformal field theory* $\Phi_{\mathcal{C}} : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$, the *trace field theory*. Note that $\Phi_{\mathcal{C}}$ assigns to the circle the Hochschild complex of $\mathcal{C}$.

Important tool: The trace field theory

With the trace field theory $\Phi_{\mathcal{C}} : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$, the differential graded Verlinde formula is

‘ $\mathfrak{F}_{\mathcal{C}} \Bigg($  $\Bigg) \simeq S^{-1} \circ \Phi_{\mathcal{C}} \Bigg($  $\Bigg) \circ (S \otimes S)$ ’

There are generalizations to higher genus.

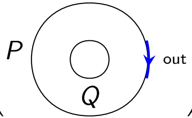
The block diagonalization

The mapping class group element S transforms the product $\otimes$ on $HH_0(\mathcal{C})$ to a product $\star$ which, for identity morphisms id_P and id_Q with $P, Q \in \mathrm{Proj}\,\mathcal{C}$ (we will just write $[P]$ and $[Q]$), satisfies

$$[P] \star [Q] \simeq \xi_{P,Q} \ ,$$

where

$$\xi_{P,Q} = \Phi_{\mathcal{C}} \left(\quad \right) \in \mathcal{C}(P, P)$$



is the handle element of the trace field theory.

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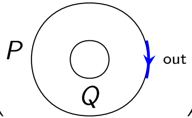
$$\xi_{P,Q} = \Phi_{\mathcal{C}} \Bigg($$



$$\Bigg) \in \mathcal{C}(P,P)$$

is the handle element of the trace field theory. It has modified trace

$$t_P \xi_{P,Q} = \dim \mathcal{C}(P,Q) \ .$$

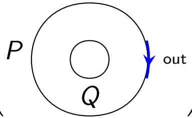
The block diagonalization

The mapping class group element S transforms the product $\otimes$ on $HH_0(\mathcal{C})$ to a product $\star$ which, for identity morphisms id_P and id_Q with $P, Q \in \mathrm{Proj}\,\mathcal{C}$ (we will just write $[P]$ and $[Q]$), satisfies

$$[P] \star [Q] \simeq \xi_{P,Q} \ ,$$

where

$$\xi_{P,Q} = \Phi_{\mathcal{C}} \left(\quad \right) \in \mathcal{C}(P,P)$$



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If P is simple,

$$[P] \star [Q] \simeq (d^m_P)^{-1} \dim \mathcal{C}(P,Q) \cdot \mathrm{id}_P$$

with the *modified dimension* d^m_P *of* P.

More applications / directions . . .

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- Factorization homology construction of (differential graded) conformal blocks (with A. Brochier).