Colloque 2015 du GDR 2875
Topologie Algébrique et Applications

21-23 octobre 2015, Institut de Mathématiques de Toulouse

Generalized Quillen rational homotopy and its applications

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Lie algebras in algebraic topology

Definition A *Lie algebra* is a vector space L with a bilinear operation

$$[-,-]\colon L \times L \to L$$

satisfying:

1. $[a,b] = -[b,a], \ a,b \in L,$
2. $\big[a,[b,c]\big] = \big[[a,b],c\big] + \big[b,[a,c]\big], \ a,b,c \in L.$

Lie algebras in algebraic topology

Definition A *differential graded Lie algebra* is a graded vector space $L = \oplus_{p\in\mathbb{Z}} L_p$

- with a bilinear operation

$$[-,-]\colon L \times L \to L$$

such that $[L_p, L_q] \subset L_{p+q}$, satisfying:

1. $[a,b] = -(-1)^{pq}[b,a],\ \ a \in L_p, b \in L_q,$
2. $\big[a,[b,c]\big] = \big[[a,b],c\big] + (-1)^{pq}\big[b,[a,c]\big],\ \ a \in L_p, b \in L_q, c \in L.$

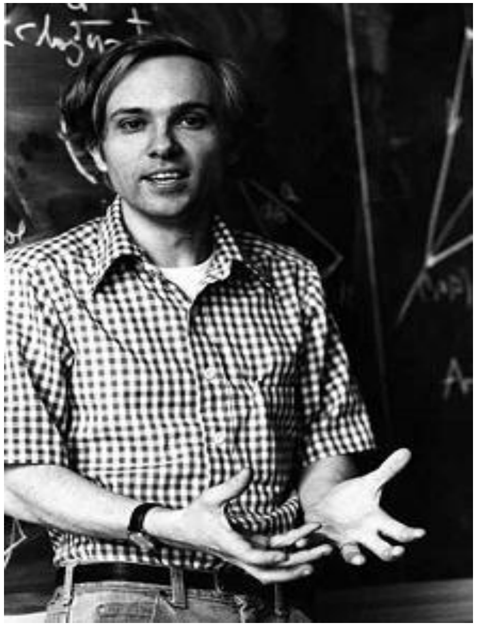
Lie algebras in algebraic topology

Definition A *differential graded Lie algebra* is a graded vector space $L = \oplus_{p \in \mathbb{Z}} L_p$

- and a linear map $\partial \colon L \to L$ such that $\partial L_p \subset L_{p-1}$, satisfying:
  1. $\partial \circ \partial = 0$,
  2. $\partial[a, b] = [\partial a, b] + (-1)^p [a, \partial b], \ a \in L_p, b \in L.$

Lie algebras in algebraic topology

At the end of the 60's Daniel Quillen introduced Rational Homotopy Theory



If $f\colon X \to Y$ is a continuous map between simply connected CW-complexes, the following properties are equivalent:

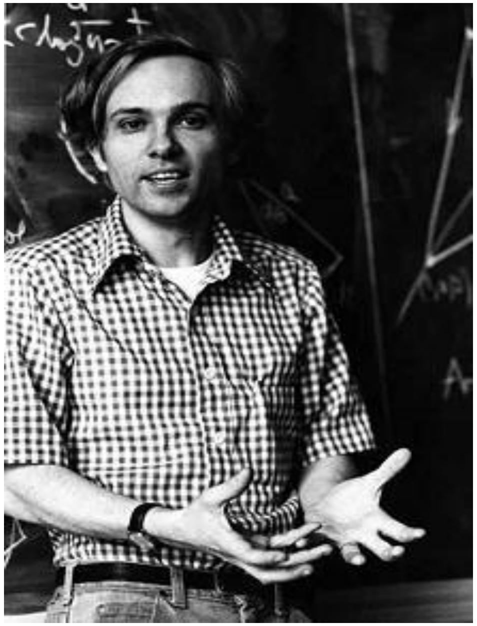
1. $\pi_n(f) \otimes \mathbb{Q}\colon \pi_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_n(Y) \otimes \mathbb{Q},\ n \geq 2.$
2. $H_n(f) \otimes \mathbb{Q}\colon H_n(X;\mathbb{Q}) \xrightarrow{\cong} H_n(Y;\mathbb{Q}),\ n \geq 2.$

Such a map is called a *rational homotopy equivalence*.

Lie algebras in algebraic topology

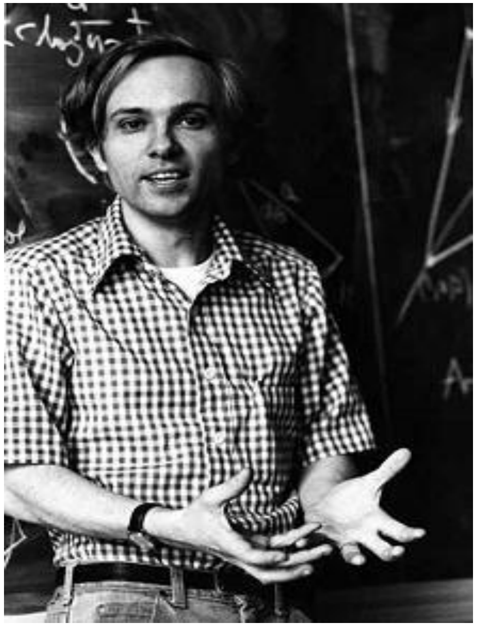
Definition
X is *rational* if its homotopy groups are $\mathbb{Q}$-vectors spaces.
A *rationalisation* of X is a pair $(X_{\mathbb{Q}}, f)$, with $X_{\mathbb{Q}}$ a rational space and $f\colon X \to X_{\mathbb{Q}}$ a rational homotopy equivalence.

The study of the rational homotopy type of X is the study of the homotopy type of $X_{\mathbb{Q}}$.



Lie algebras in algebraic topology

The importance of Quillen work is that it associates to any simply connected space X a differential graded Lie algebra (DGL).



$$\left\{ \begin{array}{l} \text{Simply connected} \\ \text{spaces} \end{array} \right\} \underset{\langle - \rangle_Q}{\overset{\lambda}{\rightleftarrows}} \mathrm{DGL}_+$$

$$X \longmapsto \lambda(X)$$

$$\langle L \rangle_Q \longleftarrow\!\shortmid L$$

$$\langle \lambda(X) \rangle_Q \simeq X_{\mathbb{Q}}$$

$(L, [-,-], \partial)$ is a DGL model of X if

$$\lambda(X) \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} L$$

Lie algebras in algebraic topology

Example $$H_{n-1}(L,\partial) \cong \pi_n(X) \otimes \mathbb{Q}.$$

$$S^n \rightsquigarrow (\mathbb{L}(x), 0), \ \ |x| = n-1.$$

In 1951 J.P. Serre showed that homotopy groups of spheres are all finite except those of the form

$$\pi_m(S^m) \ \text{ or } \ \pi_{4m-1}(S^{2m}).$$

n **odd**. Then $|x| = n-1$ is even.

As graded vector space $\mathbb{L}(x) = \langle x \rangle$, since $[x,x] = 0$ by antisymmetry.

$$\mathbb{Q} = H_{n-1}(\mathbb{L}(x), 0) \cong \pi_n(S^n) \otimes \mathbb{Q}.$$

Lie algebras in algebraic topology

Example $$H_{n-1}(L,\partial) \cong \pi_n(X)\otimes \mathbb{Q}.$$

$$S^n \rightsquigarrow (\mathbb{L}(x),0),\ \ |x| = n-1.$$

In 1951 J.P. Serre showed that homotopy groups of spheres are all finite except those of the form

$$\pi_m(S^m) \ \text{ or } \ \pi_{4m-1}(S^{2m}).$$

n **even.** Then $|x| = n-1$ is odd. As graded vector space

$$\mathbb{L}(x) = \underbrace{\langle x\rangle}_{n-1} \oplus \underbrace{\langle [x,x]\rangle}_{2n-2}, \text{ since } [x,[x,x]] = 0 \text{ by Jacobi.}$$

$$\mathbb{Q} = H_{n-1}(\mathbb{L}(x),0) \cong \pi_n(S^n)\otimes\mathbb{Q}.$$

$$\mathbb{Q} = H_{2n-2}(\mathbb{L}(x),0) = \langle [x,x]\rangle$$

$$\cong \pi_{2n-1}(S^n)\otimes\mathbb{Q} = \pi_{4m-1}(S^{2m})\otimes\mathbb{Q}.$$

Lie algebras in deformation theory

Let A be an associative algebra.

$$A \times A \to A \qquad \text{bilinear and associative}$$

$$(a, b) \mapsto a \cdot b$$

$$A[\![t]\!] = \{\sum_{i=0}^{\infty} a_i t^i, \quad a_i \in A\ \}$$

A *deformation* of A in $A[\![t]\!]$

is a new bilinear and associative product

$$*\colon A[\![t]\!] \times A[\![t]\!] \to A[\![t]\!]$$

$$\left(\sum_{i=0}^{\infty} a_i t^i\right) * \left(\sum_{i=0}^{\infty} b_i t^i\right) = a_0 \cdot b_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

$$c_i \in A.$$

Lie algebras in deformation theory

We can deform any algebraic structure in a vector space A.

Instead of $A[\![t]\!]$, we can consider a local ring R with a unique maximal ideal $\mathfrak{M}$ with $R/\mathfrak{M} \cong \mathbb{K}$.

A *deformation* of A on $A \otimes R$ is an operation

$$*\colon (A \otimes R) \times (A \otimes R) \to (A \otimes R)$$

satisfying the same properties of the original such that we recover the original operation taking quotient by $\mathfrak{M}$

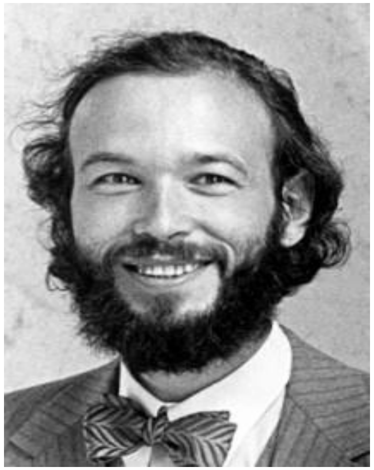
$$A \times A \to A.$$

Lie algebras in deformation theory

We denote by $\mathrm{Def}(A, R)$ the set of equivalence classes of deformations of A on $A \otimes R$.

Deligne principle

"in characteristic 0 every deformation problem is governed by a differential graded Lie algebra"



There exists a differential graded Lie algebra $L = L(A, R)$ such that we have a bijection

$$\mathrm{Def}(A, R) \cong \mathrm{MC}(L)/\mathcal{G}.$$

Lie algebras in deformation theory

Let L be a DGL. $a \in L_{-1}$ is a *Maurer-Cartan element* if satisfies the equation

$$\partial a = -\frac{1}{2}[a, a].$$

Two Maurer-Cartan elements $a, b \in \mathrm{MC}(L)$ are *gauge-equivalent*, $a \sim_{\mathcal{G}} b$ if there is an element $x \in L_0$ such that

$$b = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - \mathrm{id}}{\mathrm{ad}_x}(\partial x).$$

Deligne principle can be written as

$$\mathrm{Def}(A; R) \cong \mathrm{MC}(L)/ \sim_{\mathcal{G}} .$$

Lie algebras in deformation theory

What is the connection between differential graded Lie algebras of Quillen rational homotopy and Deformation theory?

DGL_+	DGL
$\updownarrow$	$\updownarrow$
$\left\{ \begin{array}{l} \text{Simply connected} \\ \text{spaces} \end{array} \right\}$	$\mathrm{Def}(A, R)$
There is only one Maurer-Cartan element $\partial(0) = 0 = -\frac{1}{2}[0, 0]$	$\mathrm{Def}(A; R) \cong \mathrm{MC}(L)/ \sim_{\mathcal{G}}$.

Sullivan vs Quillen rational homotopy theory

$$\left\{\begin{array}{l}\text{Simply connected}\\ \text{spaces}\end{array}\right\} \rightleftarrows \text{sGp} \rightleftarrows \text{sCHA} \rightleftarrows \text{sLA} \rightleftarrows \text{DGL}_+$$

$$X \longmapsto \lambda(X) \qquad \langle L\rangle_Q \longleftarrow\!\shortmid L$$

$$\left\{\begin{array}{l}\text{Simply connected}\\ \text{finite type spaces}\end{array}\right\} \quad \leadsto \quad \mathcal{L} \upharpoonleft\downharpoonright \mathcal{C}^*$$

$$\left\{\begin{array}{l}\text{Nilpotent, finite}\\ \text{type spaces}\end{array}\right\} \underset{\langle - \rangle_S}{\overset{A_{PL}}{\rightleftarrows}} \text{CDGA}_+$$

1. $\langle A\rangle_S = \text{CDGA}(A, \Omega_\bullet)$

   where $\Omega_\bullet$ is a simplicial CDGA

$$X \longmapsto A_{PL}(X) \qquad \langle A\rangle_S \longleftarrow\!\shortmid A$$

2. $\langle A\rangle_S$ has sense also for A a $\mathbb{Z}$-graded CDGA, but $\langle L\rangle_Q$ is not defined if L is $\mathbb{Z}$-graded DGL.

Free Lie model of an n-simplex

For each $n \geq 0$, consider the standard n-simplex Δ^n

$\Delta^n_p = \{(i_0, \ldots, i_p) \mid 0 \leq i_0 \leq \cdots < i_p \leq n\ \}$, if $p \leq n$

and $\Delta^n_p = \emptyset$ if $p > 0$.

Let $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ be the complete free DGL on the desuspended rational simplicial chain complex on Δ^n

$$da_{i_0 \ldots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0 \ldots \widehat{i_j} \ldots i_p}$$

where $a_{i_0 \ldots i_p}$ denotes the generator of degree $p-1$ represented by the p-simplex $(i_0, \ldots, i_p) \in \Delta^n_p$.

Free Lie model of an n-simplex

Problem Define a differential ∂ in $\widehat{\mathbb{L}}(s^{-1}\Delta^n)$ such that:

1. For each $i = 0, \ldots, n$ the generator $a_i \in \Delta_0^n$ is a Maurer-Cartan element

$$\partial a_i = -\frac{1}{2}[a_i, a_i].$$

2. The linear part ∂_1 of ∂ is precisely d.

Free Lie model of an n-simplex

Examples

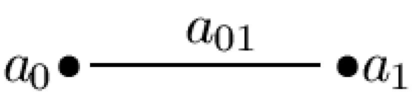
$n = 0$ $\bullet a_0$

$(\widehat{\mathbb{L}}(s^{-1}\Delta^0), d) = (\widehat{\mathbb{L}}(a_0), da_0 = 0).$

The first condition implies $(\widehat{\mathbb{L}}(a_0), \partial a_0 = -\frac{1}{2}[a_0, a_0]).$

Free Lie model of an n-simplex

$n = 1$



$$(\widehat{\mathbb{L}}(s^{-1}\Delta^1), d) = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d),$$

$$da_0 = da_1 = 0$$
$$da_{01} = a_0 - a_1$$

The first condition implies $\quad \partial a_0 = -\frac{1}{2}[a_0, a_0], \quad \partial a_1 = -\frac{1}{2}[a_1, a_1].$

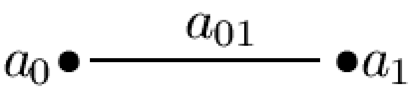
The second condition implies $\quad \partial_1 a_{01} = a_0 - a_1\,.$

Is this a differential? **NO!**

$$\partial^2 a_{01} = \partial(a_0 - a_1) = -\frac{1}{2}[a_0, a_0] + \frac{1}{2}[a_1, a_1] \neq 0.$$

Free Lie model of an n-simplex

$n = 1$



$$\partial a_{01} = a_0 - a_1 \quad + \lambda_1 [a_{01}, a_0] + \mu_1 [a_{01}, a_1]$$
$$+\lambda_2 \Big[a_{01}, [a_{01}, a_0]\Big] + \mu_2 \Big[a_{01}, [a_{01}, a_1]\Big]$$
$$+\lambda_3 \Big[a_{01}, \Big[a_{01}, [a_{01}, a_0]\Big]\Big] + \mu_3 \Big[a_{01}, \Big[a_{01}, [a_{01}, a_1]\Big]\Big]$$
$$\cdots$$

The Lawrence-Sullivan interval

$(\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d),$

$$\partial a_{01} = [a_{01}, a_0] + \sum_{i \geq 0} \frac{B_i}{i!} \mathrm{ad}^i_{a_{01}} (a_0 - a_1)$$

Free Lie model of an n-simplex

$n = 2$

$(\widehat{\mathbb{L}}(s^{-1}\Delta^2), d)$

$= (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d)$

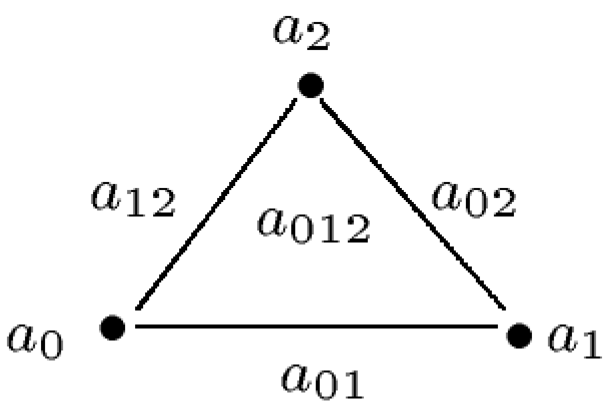
$da_0 = da_1 = da_2 = 0$

$da_{01} = a_0 - a_1$

$da_{02} = a_0 - a_2$

$da_{12} = a_1 - a_2$

$da_{012} = a_{12} - a_{02} + a_{01}$



Free Lie model of an n-simplex

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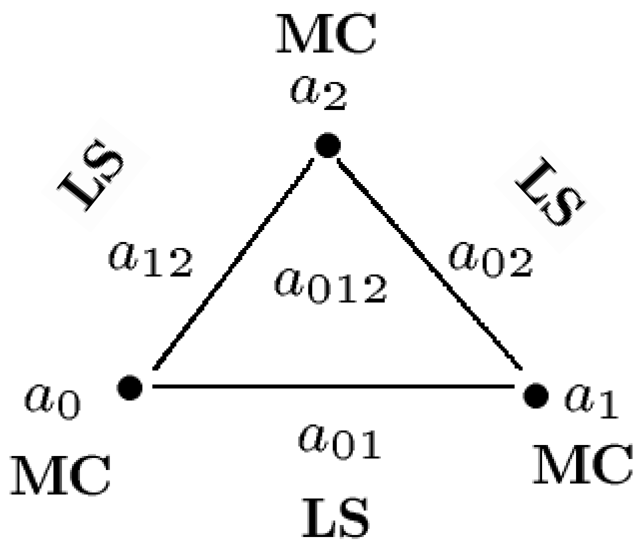

Free Lie model of an n-simplex

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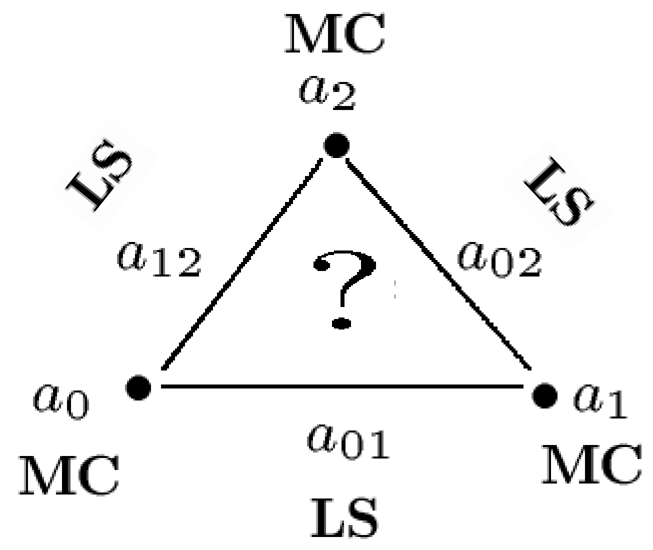
$da_{012} = a_{12} - a_{02} + a_{01}$



$\partial a_{012} = a_{12} - a_{02} + a_{01} + \; ?$

Free Lie model of an n-simplex

The Baker-Campbell-Hausdorff formula

Let x, y be two non-commuting variables. $\widehat{T}(x,y)$

The *Baker-Campbell-Hausdorff formula* $x * y$ is the solution to $z = \log\big(\exp(x)\exp(y)\big)$.

Explicitely

$$x * y = \sum_{n=0}^{\infty} z_n(x,y)$$

$$z_n(x,y) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum \frac{x^{p_1} y^{q_1} \cdots x^{p_k} y^{q_k}}{p_1! q_1! \cdots p_k! q_k!}$$

$p_1 + q_1 > 0; \ldots ; p_k + q_k > 0.$ $p_1 + q_1 + \cdots + p_k + q_k = n.$

Free Lie model of an n-simplex

The Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff formula satisfies the following properties:

- $*$ is an associative product: $(x * y) * z = x * (y * z)$.
- $x * y$ can be written as a linear combination of nested commutators

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\Big[x, [x, y]\Big] - \frac{1}{12}\Big[y, [x, y]\Big] + \cdots$$

$$x * y \in \widehat{\mathbb{L}}(x, y) \subset \widehat{T}(x, y).$$

Free Lie model of an n-simplex

$n = 2$

$(\widehat{\mathbb{L}}(s^{-1}\Delta^2), d)$

$= (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d)$

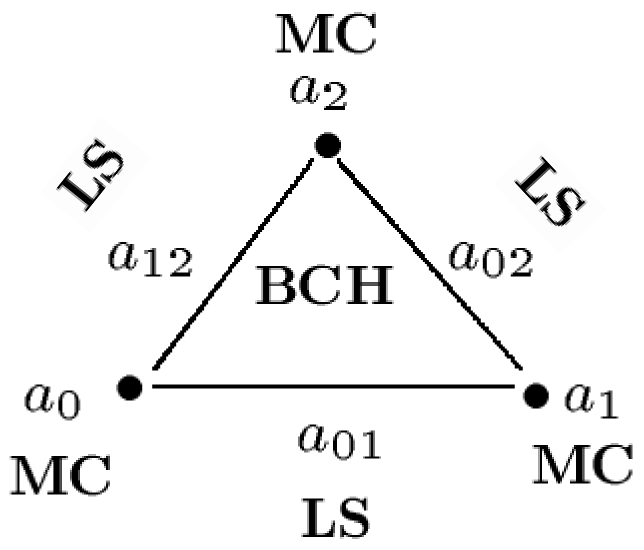
$da_0 = da_1 = da_2 = 0$

$da_{01} = a_0 - a_1$

$da_{02} = a_0 - a_2$

$da_{12} = a_1 - a_2$

$da_{012} = a_{12} - a_{02} + a_{01}$



$\partial a_{012} = a_{12} - a_{02} + a_{01} + \;?$

$\partial a_{012} = a_{01} * a_{12} * (-a_{02}) - [a_{012}, a_0]$

The realization functor

$$(\mathbb{L}(s^{-1}\Delta^0),\partial) \;\underset{\delta_0\ \delta_1}{\overset{\sigma_0}{\leftleftarrows\!\!\rightrightarrows}}\; (\mathbb{L}(s^{-1}\Delta^1),\partial) \;\underset{\delta_0\ \delta_1\ \delta_2}{\overset{\sigma_0\ \sigma_1}{\leftleftarrows\!\!\rightrightarrows}}\; (\mathbb{L}(s^{-1}\Delta^2),\partial) \quad \cdots$$

$$\mathfrak{L}_0 = (\mathbb{L}(s^{-1}\Delta^0),\partial), \quad \mathfrak{L}_1 = (\mathbb{L}(s^{-1}\Delta^1),\partial), \quad \mathfrak{L}_2 = (\mathbb{L}(s^{-1}\Delta^2),\partial)$$

Theorem

$\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n\geq 0}$ is a cosimplicial DGL.

B. , Y. Félix, A. Murillo, D. Tanré.
Preprint 2015

Definition

Define a functor from the category of complete DGL's to the category of simplicial sets by

$$\langle - \rangle : \mathbf{DGL} \to \mathbf{SimpSet} \qquad \langle L \rangle_n = \mathbf{DGL}(\mathfrak{L}_n, L)$$

$$\langle - \rangle_S : \mathbf{CDGA} \to \mathbf{SimpSet} \qquad \langle A \rangle_n = \mathbf{CDGA}(A, \Omega_n)$$

where $\Omega_n = A_{PL}(\Delta^n)$.

The realization functor

Proposition

For (L, ∂) any DGL, there is a bijection $\quad \pi_0\langle L\rangle \cong \mathrm{MC}(L)/\sim_{\mathcal{G}}$

Proof:

$\langle L\rangle_0 = \mathbf{DGL}(\mathbb{L}(a_0), L) = \mathrm{MC}(L).$

Two Maurer-Cartan elements $z_0, z_1 \in \mathrm{MC}(L)$ are gauge-equivalent if there is a map $\psi : \mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial) \to L$ with $\psi(a_0) = z_0$ and $\psi(a_1) = z_1$.

$\langle L\rangle_1 = \mathbf{DGL}(\mathbb{L}(a_0, a_1, a_{01}), L).$ $\square$

The realization functor

Let (L, ∂) be a DGL. $\langle L \rangle$ is a non-connected space

$\pi_0 \langle L \rangle \cong \mathrm{MC}(L)/\sim_{\mathcal{G}}$ Let $z \in \mathrm{MC}(L)$

Consider the differential $\partial^z = \partial + \mathrm{ad}_z$.

Define $(L^z, \partial^z) \in \mathrm{DGL}_+$ by

$$(L_p)^z = \begin{cases} 0 & \text{if} \quad p < 0 \\ \mathrm{Ker}\partial^z : L_0 \to L_{-1} & \text{if} \quad p = 0 \\ L_p & \text{if} \quad p > 0 \end{cases}$$

Theorem

$$\langle L \rangle \simeq \bigcup_{[z] \in \mathrm{MC}(L)/\sim} \langle (L^z, \partial^z) \rangle$$

The realization functor

Proposition

Let (L, ∂) be a non-negatively graded DGL. Then $\langle L \rangle$ is a connected simplicial set and there are bijections

$$\pi_n \langle L \rangle \cong H_{n-1}(L, \partial), \quad n \geq 1,$$

which are group isomorphism for $n \geq 2$.

The realization functor

- **Theorem**

  Let (L, ∂) be a finite type DGL positively graded.
  There is a homotopy equivalence of simplicial sets

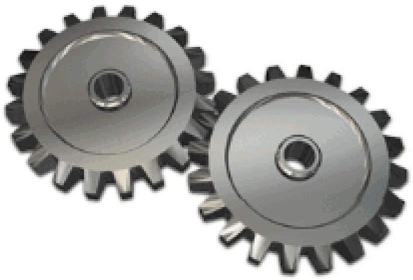
$$\langle L \rangle \simeq \langle \mathcal{C}^*(L) \rangle_S$$

The realization functor

- **Conjecture**

Let (L, ∂) be a DGL. Consider the simplicial set

$$\mathrm{MC}_\bullet(L) = \mathrm{MC}(L \otimes \Omega_\bullet)$$



$$\langle L \rangle \overset{?}{\simeq} \mathrm{MC}_\bullet(L)$$

If L and L' are DGL's conectrated in $[0, \infty)$ and $\psi\colon L \to L'$ is a quasi-isomorphism, then

$$\mathrm{MC}_\bullet(\psi)\colon \mathrm{MC}_\bullet(L) \to \mathrm{MC}_\bullet(L')$$

is a homotopy equivalence.

The realization functor

If L and L' are $\mathbb{Z}$-graded DGL's and $\psi\colon L \to L'$ is a morphism with:

- $\mathrm{MC}(\psi)\colon \mathrm{MC}(L)/\sim_{\mathcal{G}} \xrightarrow{\cong} \mathrm{MC}(L')/\sim_{\mathcal{G}}$
- $\psi\colon L^z \xrightarrow{\simeq} L'^{\psi z}$ for any $z \in \mathrm{MC}(L)$,

Then there is a homotopy equivalence of simplicial sets

$$\langle L\rangle \simeq \langle L'\rangle$$

Proof: $$\langle L\rangle \simeq \bigcup_{[z]\in \mathrm{MC}(L)/\sim} \langle (L^z, \partial^z)\rangle$$

$$\mathrm{MC}_\bullet(L) \simeq \mathrm{MC}_\bullet(L')$$

The realization functor

- **Conjecture**

  Let $(L, \partial) \in \mathrm{DGL}_+$

  $\langle L \rangle$ the realization of L

  $\lambda\langle L \rangle$ the Quillen DGL associated to the above simplicial set

$$\lambda\langle L\rangle \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} L$$

This will show that the original Quillen realization functor is representable.

Topological number theory

$$\mathfrak{L} = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$$

$$\partial^2 = 0$$

$$\Updownarrow$$

$$a_0 \sim_{\mathcal{G}} a_1 \text{ through } a_{01} \in \mathfrak{L}_0$$

$$\Updownarrow$$

For any triple (a, b, c) with $a + b + c = n - 1$

$$\sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0, \quad n \geq 4.$$

$$\lambda_k = \binom{n}{k}\left[(-1)^c \binom{n-k}{c} \sum_{\ell=\max(0,k-b)}^{\min(a,k)} (-1)^\ell \binom{k}{\ell} - (-1)^a \binom{n-k}{a} \sum_{\ell=\max(0,k-b)}^{\min(c,k)} (-1)^\ell \binom{k}{\ell}\right].$$

B. , J.G. Carrasquel, and A. Murillo,
Preprint 2015

Topological number theory

Euler identity

$$-(n+1)B_n = \sum_{k=2}^{n-2} \binom{n}{k} B_k B_{n-k}, \quad n \geq 4,$$

Consider $(a, b, c) = (0, 0, n-1)$.

H. Miki identity

$$2H_n B_n = \sum_{k=2}^{n-2} \frac{n}{k(n-k)} \left(1 - \binom{n}{k}\right) B_k B_{n-k}, \quad n \geq 4,$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Consider $(a, b, c) = (0, b, n-1-b)$ for $b = 1, \ldots, \frac{n}{2}$

Topological number theory

$$(\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), \partial)$$

$$\partial^2 = 0$$

$$\Updownarrow$$

Properties of the **Baker-Campbell-Hausdorff** formula

$$\Updownarrow$$

$$\sum_{k=0}^{n} \lambda_k \mathfrak{B}_w B_k = 0$$

B. , J.G. Carrasquel,
and A. Murillo,
In preparation

where $\mathcal{B}_w$ is the coefficient of the word $w = x^{p_1} y^{q_1} \cdots x^{p_s} y^{q_s}$ in the Baker-Campbell-Hausdorff formula.

Note that $\mathfrak{B}_w$ generalizes the Bernoulli numbers
If $w = x^p y x^q$, then $\mathfrak{B}_w = \frac{B_{p+q}}{p!q!}$.

Topological number theory

What is a 2-dimensional analogue of Miki's identity?

What is a 2-dimensional analogue of $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$?

$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$?

$$\sum_{1 \le i,j \le n} \frac{1}{ij} \ ?$$

Thank you!