On the deformation theory of dg-categories

Anthony Blanc

MPIM Bonn

October 21st, 2015

1 Introduction

2 Formal moduli problems

3 Deformation functor of a dg-category

Let k be a field and B_0 an associative k-algebra.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Associativity of $\star$ implies $\phi \in Z^2(B_0)$.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Associativity of $\star$ implies $\phi \in Z^2(B_0)$.

Two deformations B_ϵ and B'_ϵ are isomorphic iff $\phi - \phi'$ is a Hochschild coboundary.

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Associativity of $\star$ implies $\phi \in Z^2(B_0)$.

Two deformations B_ϵ and B'_ϵ are isomorphic iff $\phi - \phi'$ is a Hochschild coboundary.

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Associativity of $\star$ implies $\phi \in Z^2(B_0)$.

Two deformations B_ϵ and B'_ϵ are isomorphic iff $\phi - \phi'$ is a Hochschild coboundary.

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$Def_{B_0}^{Ass}(A) = \{A - \text{deformations of } B_0\}$$

Let k be a field and B_0 an associative k-algebra.
Let A be a commutative local artinian k-algebra with residue field k.

Definition

An A-deformation of B_0 is an associative A-algebra B_A such that there exists an isomorphism $B_A \otimes_A k \simeq B_0$.

Let $A = k[\epsilon] := k[t]/(t^2)$. Let B_ϵ be a $k[\epsilon]$-deformation of B. The product on B_ϵ is given by:

$$b \star b' = bb' + \phi(b, b').$$

Associativity of $\star$ implies $\phi \in Z^2(B_0)$.

Two deformations B_ϵ and B'_ϵ are isomorphic iff $\phi - \phi'$ is a Hochschild coboundary.

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$Def_{B_0}^{Ass}(A) = \{A - \text{deformations of } B_0\}$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0).$$

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0),$$

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def_{B_0}^{Ass}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def_{B_0}^{Ass}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

$o(B_1) = 0 \quad \Leftrightarrow \quad$ There exists $B_2 \in \pi_0 Def_{B_0}^{Ass}(k[t]/(t^3))$ such that $B_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) \simeq B_1$.

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def_{B_0}^{Ass}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

$o(B_1) = 0 \quad \Leftrightarrow \quad$ There exists $B_2 \in \pi_0 Def_{B_0}^{Ass}(k[t]/(t^3))$ such that $B_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) \simeq B_1$.

In fact:

$$\pi_0 Def_{B_0}^{Ass}(k \oplus k[1]) \simeq HH^3(B_0).$$

where $k \oplus k[1]$ is the trivial square zero extension and is a commutative dg-algebra.

$$Def_{B_0}^{Ass} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def_{B_0}^{Ass}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def_{B_0}^{Ass}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

$o(B_1) = 0 \quad \Leftrightarrow \quad$ There exists $B_2 \in \pi_0 Def_{B_0}^{Ass}(k[t]/(t^3))$ such that $B_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) \simeq B_1$.

In fact:

$$\pi_0 Def_{B_0}^{Ass}(k \oplus k[1]) \simeq HH^3(B_0).$$

where $k \oplus k[1]$ is the trivial square zero extension and is a commutative dg-algebra.

Key idea of Derived Deformation Theory

Enlarge Art_k to $dgArt_k$ commutative artinian connective dg-algebras.

$$Def^{Ass}_{B_0} : Art_k \longrightarrow Gpd$$

$$\pi_0 Def^{Ass}_{B_0}(k[\epsilon]) \simeq HH^2(B_0), \qquad \pi_1(Def_{B_0}(k[\epsilon]), B_\epsilon) \simeq HH^1(B_0).$$

Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def^{Ass}_{B_0}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

$o(B_1) = 0 \quad \Leftrightarrow \quad$ There exists $B_2 \in \pi_0 Def^{Ass}_{B_0}(k[t]/(t^3))$ such that $B_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) \simeq B_1$.

In fact:

$$\pi_0 Def^{Ass}_{B_0}(k \oplus k[1]) \simeq HH^3(B_0).$$

where $k \oplus k[1]$ is the trivial square zero extension and is a commutative dg-algebra.

Key idea of Derived Deformation Theory

Enlarge Art_k to $dgArt_k$ commutative artinian connective dg-algebras.

A commutative dg-algebra A is artinian if

- $H^0(A)$ is a local ring.
- $H^i(A) = 0, \forall i \gg 0$.
- $dim_k(H^i(A)) < \infty$.

If A is a commutative dg-algebra$/k$.

If A is a commutative dg-algebra$/k$.
$dg - Alg(A) =$the ∞-category of A-dg-algebras up to quasi-isomorphism.

If A is a commutative dg-algebra$/k$.
$dg - Alg(A)$ =the ∞-category of A-dg-algebras up to quasi-isomorphism.

Definition

$$Def_{B_0}^{dga} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{B_0}^{dga}(A) := dg - Alg(A)^{\simeq} \times_{dg-Alg(k)^{\simeq}} \{B_0\}$$

If A is a commutative dg-algebra$/k$.
$dg - Alg(A)$ =the ∞-category of A-dg-algebras up to quasi-isomorphism.

Definition

$$Def_{B_0}^{dga} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{B_0}^{dga}(A) := dg - Alg(A)^{\simeq} \times_{dg-Alg(k)^{\simeq}} \{B_0\}$$

$$\begin{array}{ccc} k[t]/(t^3) & \longrightarrow & k[t]/(t^2) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[1] \end{array}$$

is a pullback diagram in $dgArt_k$.

If A is a commutative dg-algebra$/k$.
$dg - Alg(A)$ =the ∞-category of A-dg-algebras up to quasi-isomorphism.

Definition

$$Def_{B_0}^{dga} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{B_0}^{dga}(A) := dg - Alg(A)^{\simeq} \times_{dg-Alg(k)^{\simeq}} \{B_0\}$$

$$\begin{array}{ccc} k[t]/(t^3) & \longrightarrow & k[t]/(t^2) \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[1] \end{array}$$

is a pullback diagram in $dgArt_k$. We can prove that if $dim_k(B_0) < \infty$ the diagram

$$\begin{array}{ccc} Def_{B_0}^{dga}(k[t]/(t^3)) & \longrightarrow & Def_{B_0}^{dga}(k[t]/(t^2)) \\ \downarrow & & \downarrow \text{"obstruction"} \\ * = Def_{B_0}^{dga}(k) & \longrightarrow & Def_{B_0}^{dga}(k \oplus k[1]) \end{array}$$

is a pullback in spaces.

Let k be a field of characteristic zero.

Let k be a field of characteristic zero.

Definition (Lurie, DAGX)

A formal moduli problem F over k is an ∞-functor

$$F : dgArt_k \longrightarrow \mathcal{S}$$

which verifies the two conditions:

Let k be a field of characteristic zero.

Definition (Lurie, DAGX)

A formal moduli problem F over k is an ∞-functor

$$F : dgArt_k \longrightarrow \mathcal{S}$$

which verifies the two conditions:

1. $F(k) \simeq *$.

Let k be a field of characteristic zero.

Definition (Lurie, DAGX)

A formal moduli problem F over k is an ∞-functor

$$F : dgArt_k \longrightarrow \mathcal{S}$$

which verifies the two conditions:

1. $F(k) \simeq *$.
2. For every pullback square $\begin{array}{ccc} B & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$ in $dgArt_k$ the square

$$\begin{array}{ccc} F(B) & \longrightarrow & F(k) \simeq * \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(k \oplus k[n]) \end{array}$$

is a pullback in $\mathcal{S}$.

Let k be a field of characteristic zero.

Definition (Lurie, DAGX)

A formal moduli problem F over k is an ∞-functor

$$F : dgArt_k \longrightarrow \mathcal{S}$$

which verifies the two conditions:

1. $F(k) \simeq *$.
2. For every pullback square $\begin{array}{ccc} B & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$ in $dgArt_k$ the square

$$\begin{array}{ccc} F(B) & \longrightarrow & F(k) \simeq * \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(k \oplus k[n]) \end{array}$$

   is a pullback in $\mathcal{S}$.

We denote by FMP_k the ∞-category of fmps over k. And by $PFMP_k$ the ∞-category of ∞-functors $dgArt_k \longrightarrow \mathcal{S}$ which verifies only conditions 1.

Examples of fmps:

- The formal completion at a point of an Artin stack is an fmp.

Examples of fmps:

- The formal completion at a point of an Artin stack is an fmp.
- If B_0 is a proper k-dg-algebra, then $Def^{dga}_{B_0}$ is an fmp.

Examples of fmps:

- The formal completion at a point of an Artin stack is an fmp.
- If B_0 is a proper k-dg-algebra, then $Def_{B_0}^{dga}$ is an fmp.
- If X_0 is a smooth projective scheme over k. Then Def_{X_0} is an fmp. $\pi_0 Def_{X_0}(k[\epsilon]) \simeq H^1(T_{X_0})$.

Examples of fmps:

- The formal completion at a point of an Artin stack is an fmp.
- If B_0 is a proper k-dg-algebra, then $Def^{dga}_{B_0}$ is an fmp.
- If X_0 is a smooth projective scheme over k. Then Def_{X_0} is an fmp.
  $\pi_0 Def_{X_0}(k[\epsilon]) \simeq H^1(T_{X_0})$.
- If V is an algebraic vector bundle over an algebraic variety X. Then Def_V is an fmp.
  $\pi_0 Def_V(k[\epsilon]) \simeq H^1(X, End(V))$.

Examples of fmps:

- The formal completion at a point of an Artin stack is an fmp.
- If B_0 is a proper k-dg-algebra, then $Def_{B_0}^{dga}$ is an fmp.
- If X_0 is a smooth projective scheme over k. Then Def_{X_0} is an fmp. $\pi_0 Def_{X_0}(k[\epsilon]) \simeq H^1(T_{X_0})$.
- If V is an algebraic vector bundle over an algebraic variety X. Then Def_V is an fmp. $\pi_0 Def_V(k[\epsilon]) \simeq H^1(X, End(V))$.

Property

The inclusion ∞-functor $FMP_k \hookrightarrow PFMP_k$ commutes with limits. It has a left adjoint denoted by

$$(-)^{\wedge} : PFMP_k \longrightarrow FMP_k$$
$$F \longmapsto F^{\wedge}$$

For every $F \in PFMP_k$ we therefore have a canonical map $F \longrightarrow F^{\wedge}$.

A formal moduli problem has a tangent spectrum.

A formal moduli problem has a tangent spectrum. The diagrams

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$.

A formal moduli problem has a tangent spectrum. The diagrams

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$. For every formal moduli problem F, we have

$$F(k \oplus k[n-1]) \simeq \Omega F(k \oplus k[n]).$$

A formal moduli problem has a tangent spectrum. The diagrams

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$. For every formal moduli problem F, we have

$$F(k \oplus k[n-1]) \simeq \Omega F(k \oplus k[n]).$$

Hence a spectrum T_F with $(T_F)_n = F(k \oplus k[n])$.

A formal moduli problem has a tangent spectrum. The diagrams

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$. For every formal moduli problem F, we have

$$F(k \oplus k[n-1]) \simeq \Omega F(k \oplus k[n]).$$

Hence a spectrum T_F with $(T_F)_n = F(k \oplus k[n])$.

The loop space functor is

$$\Omega : PFMP_k \longrightarrow PFMP_k$$

with $\Omega(F)(A) := \Omega(F(A))$ the space of loops at the point $* \simeq F(k) \longrightarrow F(A)$.

A formal moduli problem has a tangent spectrum. The diagrams

$$\begin{array}{ccc} k \oplus k[n-1] & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$. For every formal moduli problem F, we have

$$F(k \oplus k[n-1]) \simeq \Omega F(k \oplus k[n]).$$

Hence a spectrum T_F with $(T_F)_n = F(k \oplus k[n])$.

The loop space functor is

$$\Omega : PFMP_k \longrightarrow PFMP_k$$

with $\Omega(F)(A) := \Omega(F(A))$ the space of loops at the point $* \simeq F(k) \longrightarrow F(A)$. We have

$$T_{\Omega F} \simeq \Omega T_F \simeq T_F[-1],$$

hence, informally, the latter is a dg-Lie-algebra.

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

In the examples:

- B_0 a finite dimensional algebra$/k$. $T_{Def^{dga}_{B_0}}[-1] \simeq HH^\bullet(B_0)[1]$.

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

In the examples:

- B_0 a finite dimensional algebra$/k$. $T_{Def^{dga}_{B_0}}[-1] \simeq HH^\bullet(B_0)[1]$.
- X_0 smooth projective scheme$/k$. $T_{Def_{X_0}}[-1] \simeq T_{X_0}[-1]$.

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

In the examples:

- B_0 a finite dimensional algebra$/k$. $T_{Def_{B_0}^{dga}}[-1] \simeq HH^\bullet(B_0)[1]$.
- X_0 smooth projective scheme$/k$. $T_{Def_{X_0}}[-1] \simeq T_{X_0}[-1]$.
- V vector bundle on an algebraic variety X/k. $T_{Def_V}[-1] \simeq \mathbb{R}\Gamma(X, End(V))$.

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

In the examples:

- B_0 a finite dimensional algebra$/k$. $T_{Def^{dga}_{B_0}}[-1] \simeq HH^{\bullet}(B_0)[1]$.
- X_0 smooth projective scheme$/k$. $T_{Def_{X_0}}[-1] \simeq T_{X_0}[-1]$.
- V vector bundle on an algebraic variety X/k. $T_{Def_V}[-1] \simeq \mathbb{R}\Gamma(X, End(V))$.

The right adjoint Ψ of the equivalence is given by

Theorem (Lurie, Hinich)

If $char(k) = 0$, there exists an ∞-functor

$$FMP_k \longrightarrow dgLie_k$$

given by $F \mapsto T_F[-1]$, which is an equivalence.

In the examples:

- B_0 a finite dimensional algebra$/k$. $T_{Def^{dga}_{B_0}}[-1] \simeq HH^\bullet(B_0)[1]$.
- X_0 smooth projective scheme$/k$. $T_{Def_{X_0}}[-1] \simeq T_{X_0}[-1]$.
- V vector bundle on an algebraic variety X/k. $T_{Def_V}[-1] \simeq \mathbb{R}\Gamma(X, End(V))$.

The right adjoint Ψ of the equivalence is given by

$$\Psi(L)(A) = Map_{dgLie_k}(\mathfrak{D}(A), L)$$

where $\mathfrak{D} : (CAlg_k^{aug})^{op} \longrightarrow dgLie_k$ is the Kozsul duality functor.

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \mathrm{RMod}^{cof}_{A^{op},dg}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \mathrm{RMod}^{cof}_{A^{op},dg}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

The morphisms in $Dg^{cc}(k)$ are given by Morita bimodules.

$$\mathbb{R}\underline{\mathrm{Hom}}_c(T, T') \simeq T^{op} \widehat{\otimes}_k T' \quad \text{(Toën)}$$

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \mathrm{RMod}^{cof}_{A^{op},dg}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

The morphisms in $Dg^{cc}(k)$ are given by Morita bimodules.

$$\mathbb{R}\underline{\mathrm{Hom}}_c(T, T') \simeq T^{op}\widehat{\otimes}_k T' \quad \text{(Toën)}$$

Let $T_0 \in Dg^{cc}(k)$.

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \mathrm{RMod}^{cof}_{A^{op},dg}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

The morphisms in $Dg^{cc}(k)$ are given by Morita bimodules.

$$\mathbb{R}\underline{\mathrm{Hom}}_c(T, T') \simeq T^{op}\widehat{\otimes}_k T' \quad \text{(Toën)}$$

Let $T_0 \in Dg^{cc}(k)$.

Definition

$$Def_{T_0} : dgArt_k \longrightarrow \hat{\mathcal{S}}$$

$$Def_{T_0}(A) := Dg^{cc}(A)^{\simeq} \times_{Dg^{cc}(k)^{\simeq}} \{T_0\}$$

$Dg^{cc}(k)$: the ∞-category of big cocomplete compactly generated k-dg-categories and cocontinuous dg-functors.

The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \mathrm{RMod}^{cof}_{A^{op},dg}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

The morphisms in $Dg^{cc}(k)$ are given by Morita bimodules.

$$\mathbb{R}\underline{\mathrm{Hom}}_c(T, T') \simeq T^{op}\widehat{\otimes}_k T' \quad \text{(Toën)}$$

Let $T_0 \in Dg^{cc}(k)$.

Definition

$$Def_{T_0} : dgArt_k \longrightarrow \hat{\mathcal{S}}$$

$$Def_{T_0}(A) := Dg^{cc}(A)^{\simeq} \times_{Dg^{cc}(k)^{\simeq}} \{T_0\}$$

An point in $Def_{T_0}(A)$ corresponds to a pair (T_A, u) with $T_A \in Dg^{cc}(A)$ and $u : T_A\widehat{\otimes}_A\widehat{k} \simeq T_0$ a Morita equivalence.

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^{\wedge}(k[\epsilon])$$

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^{\wedge}(k[\epsilon])$$

We can show $\pi_0 Def_{T_0}^{\wedge}(k[\epsilon]) \simeq HH^2(T_0)$.

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def^{\wedge}_{T_0}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def^{\wedge}_{T_0}(k[\epsilon])$$

We can show $\pi_0 Def^{\wedge}_{T_0}(k[\epsilon]) \simeq HH^2(T_0)$.

Typical example

$T_0 = k\widehat{[u, u^{-1}]}$ with $deg(u) = 2$ and $d = 0$. $HH^2(T_0) \simeq k$ generated by u. Keller–Lowen showed that $u \notin Im(\Theta_0)$.

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^{\wedge}(k[\epsilon])$$

We can show $\pi_0 Def_{T_0}^{\wedge}(k[\epsilon]) \simeq HH^2(T_0)$.

Typical example

$T_0 = k[\widehat{u, u^{-1}}]$ with $deg(u) = 2$ and $d = 0$. $HH^2(T_0) \simeq k$ generated by u. Keller–Lowen showed that $u \notin Im(\Theta_0)$.

Theorem (Lurie)

Let $T_0 \in Dg^{cc}(k)$. Suppose the following conditions holds:

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^{\wedge}(k[\epsilon])$$

We can show $\pi_0 Def_{T_0}^{\wedge}(k[\epsilon]) \simeq HH^2(T_0)$.

Typical example

$T_0 = k\widehat{[u, u^{-1}]}$ with $deg(u) = 2$ and $d = 0$. $HH^2(T_0) \simeq k$ generated by u. Keller–Lowen showed that $u \notin Im(\Theta_0)$.

Theorem (Lurie)

Let $T_0 \in Dg^{cc}(k)$. Suppose the following conditions holds:

- *For every compact objects $E, F \in T_0$, $Ext^n_{T_0}(E, F) = 0$, $\forall n \gg 0$.*

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^\wedge$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^\wedge$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^\wedge(k[\epsilon])$$

We can show $\pi_0 Def_{T_0}^\wedge(k[\epsilon]) \simeq HH^2(T_0)$.

Typical example

$T_0 = k\widehat{[u, u^{-1}]}$ with $deg(u) = 2$ and $d = 0$. $HH^2(T_0) \simeq k$ generated by u. Keller–Lowen showed that $u \notin Im(\Theta_0)$.

Theorem (Lurie)

Let $T_0 \in Dg^{cc}(k)$. Suppose the following conditions holds:

- *For every compact objects $E, F \in T_0$, $Ext^n_{T_0}(E, F) = 0$, $\forall n \gg 0$.*
- *There exists a set of compact objects $\{E_\alpha\}_\alpha$ of T_0 with $Ext^n_{T_0}(E_\alpha, E_\alpha) = 0$, $\forall n \geq 2$, $\forall \alpha$ and which generates T_0 under colimits.*

Fact

For many interesting dg-categories T_0, the functor Def_{T_0} is not a formal moduli problem.

The map $F \longrightarrow F^{\wedge}$ for $F = Def_{T_0}$ gives a map

$$\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$$

Hence a map on the tangents

$$\Theta_0 : \pi_0 Def_{T_0}(k[\epsilon]) \longrightarrow \pi_0 Def_{T_0}^{\wedge}(k[\epsilon])$$

We can show $\pi_0 Def_{T_0}^{\wedge}(k[\epsilon]) \simeq HH^2(T_0)$.

Typical example

$T_0 = k\widehat{[u, u^{-1}]}$ with $deg(u) = 2$ and $d = 0$. $HH^2(T_0) \simeq k$ generated by u. Keller–Lowen showed that $u \notin Im(\Theta_0)$.

Theorem (Lurie)

Let $T_0 \in Dg^{cc}(k)$. Suppose the following conditions holds:

- *For every compact objects $E, F \in T_0$, $Ext^n_{T_0}(E, F) = 0$, $\forall n \gg 0$.*
- *There exists a set of compact objects $\{E_\alpha\}_\alpha$ of T_0 with $Ext^n_{T_0}(E_\alpha, E_\alpha) = 0$, $\forall n \geq 2$, $\forall \alpha$ and which generates T_0 under colimits.*

Then Def_{T_0} is a formal moduli problem.

Questions

- What is $Def_{T_0}^{\wedge}$?
- Is the map $\Theta : Def_{T_0} \longrightarrow Def_{T_0}^{\wedge}$ an equivalence for a larger nice class dg-categories T_0 ? Or on specific algebras e.g. $A = k[\![t]\!]$.

Questions

- What is $Def^{\wedge}_{T_0}$?
- Is the map $\Theta : Def_{T_0} \longrightarrow Def^{\wedge}_{T_0}$ an equivalence for a larger nice class dg-categories T_0 ? Or on specific algebras e.g. $A = k[\![t]\!]$.

Define $Def_{T_0}(k[\![t]\!]) := lim_{i \geq 1} Def_{T_0}(k[t]/(t^i))$ and $Def^{\wedge}_{T_0}(k[\![t]\!]) := lim_{i \geq 1} Def^{\wedge}_{T_0}(k[t]/(t^i))$.

Questions

- What is $Def^{\wedge}_{T_0}$?
- Is the map $\Theta : Def_{T_0} \longrightarrow Def^{\wedge}_{T_0}$ an equivalence for a larger nice class dg-categories T_0 ? Or on specific algebras e.g. $A = k[\![t]\!]$.

Define $Def_{T_0}(k[\![t]\!]) := lim_{i\geq 1} Def_{T_0}(k[t]/(t^i))$ and $Def^{\wedge}_{T_0}(k[\![t]\!]) := lim_{i\geq 1} Def^{\wedge}_{T_0}(k[t]/(t^i))$.

The following is our main result, which has benefited from discussions with B. Toën (see his 2014 ICM adress).

Theorem (B.–Pandit)

Let $T_0 \in Dg^{cc}(k)$ be a smooth proper dg-category. For every $\{\alpha_i\}_i \in Def^{\wedge}_{T_0}(k[\![t]\!])$ there exists a proper dg-algebra B over $k[\![t]\!]$ such that

$$Def^{dga}_{B_0}(k[\![t]\!]) \longrightarrow Def^{\wedge}_{T_0}(k[\![t]\!])$$

maps B to $\{\alpha_i\}_i$.

Questions

- What is $Def^{\wedge}_{T_0}$?
- Is the map $\Theta : Def_{T_0} \longrightarrow Def^{\wedge}_{T_0}$ an equivalence for a larger nice class dg-categories T_0 ? Or on specific algebras e.g. $A = k[\![t]\!]$.

Define $Def_{T_0}(k[\![t]\!]) := lim_{i\geq 1} Def_{T_0}(k[t]/(t^i))$ and $Def^{\wedge}_{T_0}(k[\![t]\!]) := lim_{i\geq 1} Def^{\wedge}_{T_0}(k[t]/(t^i))$.

The following is our main result, which has benefited from discussions with B. Toën (see his 2014 ICM adress).

Theorem (B.–Pandit)

Let $T_0 \in Dg^{cc}(k)$ be a smooth proper dg-category. For every $\{\alpha_i\}_i \in Def^{\wedge}_{T_0}(k[\![t]\!])$ there exists a proper dg-algebra B over $k[\![t]\!]$ such that

$$Def^{dga}_{B_0}(k[\![t]\!]) \longrightarrow Def^{\wedge}_{T_0}(k[\![t]\!])$$

maps B to $\{\alpha_i\}_i$.

Let's now review the necessary tools for proving this.

Slogan

Let F be a pre-fmp which is not too bad, then $\Omega^n F$ is eventually an fmp for big enough n.

Slogan

Let F be a pre-fmp which is not too bad, then $\Omega^n F$ is eventually an fmp for big enough n.

Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

1. $\Omega^n F$ is an fmp.

Slogan

Let F be a pre-fmp which is not too bad, then $\Omega^n F$ is eventually an fmp for big enough n.

Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

1. $\Omega^n F$ is an fmp.
2. For all pullback squares $\begin{array}{ccc} B & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$ in $dgArt_k$, the map

   $F(B) \longrightarrow F(A) \times_{F(k \oplus k[n])} *$ is $(n-2)$-truncated.

Slogan

Let F be a pre-fmp which is not too bad, then $\Omega^n F$ is eventually an fmp for big enough n.

Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

1. $\Omega^n F$ is an fmp.
2. For all pullback squares $$\begin{array}{ccc} B & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$$ in $dgArt_k$, the map $F(B) \longrightarrow F(A) \times_{F(k \oplus k[n])} *$ is $(n-2)$-truncated.
3. The map $F(A) \longrightarrow F^{\wedge}(A)$ is $(n-2)$-truncated for all $A \in dgArt_k$.

Slogan

Let F be a pre-fmp which is not too bad, then $\Omega^n F$ is eventually an fmp for big enough n.

Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

1. $\Omega^n F$ is an fmp.
2. For all pullback squares $\begin{array}{ccc} B & \longrightarrow & k \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \oplus k[n] \end{array}$ in $dgArt_k$, the map $F(B) \longrightarrow F(A) \times_{F(k \oplus k[n])} *$ is $(n-2)$-truncated.
3. The map $F(A) \longrightarrow F^{\wedge}(A)$ is $(n-2)$-truncated for all $A \in dgArt_k$.

We call such a fmp *n-proximate*.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp. However if T_0 has a left complete t-structure and E_0 is connective, then Def_{E_0} is an fmp.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp. However if T_0 has a left complete t-structure and E_0 is connective, then Def_{E_0} is an fmp.

$$\Omega Def_{E_0}(A) \simeq Aut_{\infty(T_A)}(E_0 \otimes_k A)$$

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp. However if T_0 has a left complete t-structure and E_0 is connective, then Def_{E_0} is an fmp.

$$\Omega Def_{E_0}(A) \simeq Aut_{\infty(T_A)}(E_0 \otimes_k A)$$

It is known that ΩDef_{E_0} is an fmp.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp. However if T_0 has a left complete t-structure and E_0 is connective, then Def_{E_0} is an fmp.

$$\Omega Def_{E_0}(A) \simeq Aut_{\infty(T_A)}(E_0 \otimes_k A)$$

It is known that ΩDef_{E_0} is an fmp.

Proposition (Lurie, Toën–Vaquié)

The pre-fmp Def_{E_0} is 1-proximate.

Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$. The deformation functor of E_0 in T_0 is

$$Def_{E_0} : dgArt_k \longrightarrow \mathcal{S}$$

$$Def_{E_0}(A) := \infty(T_A)^{\simeq} \times_{\infty(T_0)^{\simeq}} \{E_0\}$$

There exists examples of T_0 and E_0 such that Def_{E_0} is not an fmp. However if T_0 has a left complete t-structure and E_0 is connective, then Def_{E_0} is an fmp.

$$\Omega Def_{E_0}(A) \simeq Aut_{\infty(T_A)}(E_0 \otimes_k A)$$

It is known that ΩDef_{E_0} is an fmp.

Proposition (Lurie, Toën–Vaquié)

The pre-fmp Def_{E_0} is 1-proximate.

When T_0 is smooth proper, we even have a locally geometric stack of objects in T_0 (Toën–Vaquié).

$$
\begin{aligned}
\Omega Def_{T_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\} \\
&\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}
\end{aligned}
$$

$$\Omega Def_{T_0}(A) \simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\}$$
$$\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}$$

Proposition (Lurie)

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\begin{aligned}\Omega Def_{T_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\} \\ &\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}\end{aligned}$$

Proposition (Lurie)

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\Omega^2 Def_{T_0}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}(T_A)}(id_{T_A}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{T_0})} \{id_{id_{T_0}}\}$$

$$\Omega Def_{T_0}(A) \simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\}$$
$$\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}$$

Proposition (Lurie)

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\Omega^2 Def_{T_0}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}(T_A)}(id_{T_A}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{T_0})} \{id_{id_{T_0}}\}$$

$$T_{\Omega^2 Def^{\wedge}_{T_0}} \simeq T_{\Omega^2 Def_{T_0}} \simeq End_{End(T_0)}(id_{T_0}) \simeq HH^{\bullet}(T_0)$$

$$\Omega Def_{T_0}(A) \simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\}$$
$$\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}$$

Proposition (Lurie)

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\Omega^2 Def_{T_0}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}(T_A)}(id_{T_A}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{T_0})} \{id_{id_{T_0}}\}$$

$$T_{\Omega^2 Def^{\wedge}_{T_0}} \simeq T_{\Omega^2 Def_{T_0}} \simeq End_{End(T_0)}(id_{T_0}) \simeq HH^{\bullet}(T_0)$$

Corollary

The map $Def_{T_0}(A) \longrightarrow Def^{\wedge}_{T_0}(A)$ is 0-truncated, i.e.

$$\Omega Def_{T_0}(A) \simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{id_{T_0}\}$$
$$\simeq \mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0 \otimes_k A} \times_{\mathrm{RMod}^{inv,\simeq}_{T_0^{op}\otimes_k T_0}} \{id_{T_0}\}$$

Proposition (Lurie)

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\Omega^2 Def_{T_0}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}(T_A)}(id_{T_A}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{T_0})} \{id_{id_{T_0}}\}$$

$$T_{\Omega^2 Def^{\wedge}_{T_0}} \simeq T_{\Omega^2 Def_{T_0}} \simeq End_{End(T_0)}(id_{T_0}) \simeq HH^{\bullet}(T_0)$$

Corollary

The map $Def_{T_0}(A) \longrightarrow Def^{\wedge}_{T_0}(A)$ is 0-truncated, i.e.

- is an isomorphism on π_i for all $i \geq 2$.
- is injective on π_1.

Idea

The n-proximate property is related to the property of being defined on E_n-algebras, and to a description in terms of Koszul duality of E_n-algebras, where eventually $n = \infty$.

Idea

The n-proximate property is related to the property of being defined on E_n-algebras, and to a description in terms of Koszul duality of E_n-algebras, where eventually $n = \infty$.

Proposition (Lurie)

- Def_{E_0} is an E_1-fmp. There exists an equivalence

$$Def^{\wedge}_{E_0}(A) \simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)).$$

- Def_{T_0} is an E_2-fmp. There exists an equivalence

$$Def^{\wedge}_{T_0}(A) \simeq Map_{E_2-Alg_k}(\mathfrak{D}^{(2)}(A), HH^{\bullet}(T_0)).$$

Where $\mathfrak{D}^{(n)}$ is the Koszul duality functor for E_n-algebras.

Idea

The n-proximate property is related to the property of being defined on E_n-algebras, and to a description in terms of Koszul duality of E_n-algebras, where eventually $n = \infty$.

Proposition (Lurie)

- Def_{E_0} is an E_1-fmp. There exists an equivalence

$$Def^{\wedge}_{E_0}(A) \simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)).$$

- Def_{T_0} is an E_2-fmp. There exists an equivalence

$$Def^{\wedge}_{T_0}(A) \simeq Map_{E_2-Alg_k}(\mathfrak{D}^{(2)}(A), HH^{\bullet}(T_0)).$$

Where $\mathfrak{D}^{(n)}$ is the Koszul duality functor for E_n-algebras.

Corollary

$$Def^{\wedge}_{T_0}(A) \simeq Map_{Cat^{\otimes}_{\infty}}(\mathrm{RMod}^{\otimes}_{\mathfrak{D}^{(2)}(A)}, End^{\otimes}(T_0))$$
$$\simeq \{\mathfrak{D}^{(2)}(A)\text{-linear structures on } T_0\}$$

$$
\begin{aligned}
Def^{\wedge}_{E_0}(A) &\simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)) \\
&\simeq (\mathrm{LMod}_{\mathfrak{D}^{(1)}(A)}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\} \\
&\simeq (\mathrm{RMod}^{!}_{A}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\}
\end{aligned}
$$

$$
\begin{aligned}
Def^{\wedge}_{E_0}(A) &\simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)) \\
&\simeq (\mathrm{LMod}_{\mathfrak{D}^{(1)}(A)}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\} \\
&\simeq (\mathrm{RMod}^{!}_{A}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\}
\end{aligned}
$$

If A is an E_1-algebra, $\mathrm{RMod}^{!}_{A} := Ind(\mathrm{RMod}^{sm}_{A})$ where an A-module is small if its underlying k-module has finite dimensional total cohomology.

$$\begin{aligned} Def^{\wedge}_{E_0}(A) &\simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)) \\ &\simeq (\mathrm{LMod}_{\mathfrak{D}^{(1)}(A)}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\} \\ &\simeq (\mathrm{RMod}^{!}_{A}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\} \end{aligned}$$

If A is an E_1-algebra, $\mathrm{RMod}^!_A := Ind(\mathrm{RMod}^{sm}_A)$ where an A-module is small if its underlying k-module has finite dimensional total cohomology.

Let $\underline{\mathrm{Aut}}_{T_0} := \Omega Def_{T_0}$ be the deformation functor of id_{T_0} in $\mathrm{RMod}_{T_0^{op} \otimes_k T_0}$.

$$
\begin{aligned}
Def^{\wedge}_{E_0}(A) &\simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)) \\
&\simeq (\mathrm{LMod}_{\mathfrak{D}^{(1)}(A)}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\} \\
&\simeq (\mathrm{RMod}^{!}_{A}(\infty(T_0))^{\simeq} \times_{\infty(T_0)} \{E_0\}
\end{aligned}
$$

If A is an E_1-algebra, $\mathrm{RMod}^!_A := Ind(\mathrm{RMod}^{sm}_A)$ where an A-module is small if its underlying k-module has finite dimensional total cohomology.

Let $\underline{\mathrm{Aut}}_{T_0} := \Omega Def_{T_0}$ be the deformation functor of id_{T_0} in $\mathrm{RMod}_{T_0^{op}\otimes_k T_0}$.

Corollary

The formal moduli problem $\underline{\mathrm{Aut}}^{\wedge}_{T_0}$ is equivalent to

$$
\underline{\mathrm{Aut}}^{!}_{T_0} : A \longmapsto (\mathrm{RMod}^{!}_{T_0^{op}\otimes_k T_0 \otimes_k A})^{inv,\simeq} \times_{(\mathrm{RMod}^{!}_{T_0^{op}\otimes_k T_0})^{inv,\simeq}} \{id_{T_0}\}
$$

$$FMP_k \subset FMP_k^{(1)} \subset FMP_k^{(2)} \subset \ldots \subset PFMP_k$$

$$FMP_k \subset FMP_k^{(1)} \subset FMP_k^{(2)} \subset \ldots \subset PFMP_k$$

For every integer n,

$$\Omega^n : FMP_k^{(n)} \longrightarrow E_n - Gp(FMP_k)$$

$$FMP_k \subset FMP_k^{(1)} \subset FMP_k^{(2)} \subset \ldots \subset PFMP_k$$

For every integer n,

$$\Omega^n : FMP_k^{(n)} \longrightarrow E_n - Gp(FMP_k)$$

Proposition

Let F be an n-proximate formal moduli problem. Then we have an equivalence:

$$F^{\wedge}(A) \simeq Map_{E_n - Gp(FMP_k)}(\Omega^n h_{Spec(A)}, \Omega^n F)$$

$$FMP_k \subset FMP_k^{(1)} \subset FMP_k^{(2)} \subset \ldots \subset PFMP_k$$

For every integer n,

$$\Omega^n : FMP_k^{(n)} \longrightarrow E_n - Gp(FMP_k)$$

Proposition

Let F be an n-proximate formal moduli problem. Then we have an equivalence:

$$F^{\wedge}(A) \simeq Map_{E_n - Gp(FMP_k)}(\Omega^n h_{Spec(A)}, \Omega^n F)$$

This is based on the

Proposition

The functor Ω^n restricted to FMP_k is an equivalence.

$$\Omega^{(n)}_{|FMP_k} : FMP_k \longrightarrow E_n - Gp(FMP_k)$$

Proposition

Let $T_0 \in Dg^{cc}(k)$, there exists an equivalence:

$$Def^{\wedge}_{T_0}(A) \simeq Map_{E_1-Gp(FMP_k)}(\Omega h_{Spec(A)}, \underline{\mathrm{Aut}}^{!}_{T_0}).$$

Proposition

Let $T_0 \in Dg^{cc}(k)$, there exists an equivalence:

$$Def^{\wedge}_{T_0}(A) \simeq Map_{E_1-Gp(FMP_k)}(\Omega h_{Spec(A)}, \underline{\mathrm{Aut}}^!_{T_0}).$$

Remark

When T_0 is smooth and proper, the canonical map $\underline{\mathrm{Aut}}_{T_0} \longrightarrow \underline{\mathrm{Aut}}^!_{T_0}$ is an equivalence, because $\underline{\mathrm{Aut}}_{T_0}$ is the restriction to artinian algebras of a geometric stack.

Proposition

Let $T_0 \in Dg^{cc}(k)$, there exists an equivalence:

$$Def^{\wedge}_{T_0}(A) \simeq Map_{E_1-Gp(FMP_k)}(\Omega h_{Spec(A)}, \underline{\mathrm{Aut}}^{!}_{T_0}).$$

Remark

When T_0 is smooth and proper, the canonical map $\underline{\mathrm{Aut}}_{T_0} \longrightarrow \underline{\mathrm{Aut}}^{!}_{T_0}$ is an equivalence, because $\underline{\mathrm{Aut}}_{T_0}$ is the restriction to artinian algebras of a geometric stack.

Theorem (B.–Pandit)

Let T_0 be a smooth proper dg-category in $Dg^{cc}(k)$. For every $\{\alpha_i\}_i \in Def^{\wedge}_{T_0}(k[\![t]\!])$ there exists a proper dg-algebra B over $k[\![t]\!]$ such that

$$Def^{dga}_{B_0}(k[\![t]\!]) \longrightarrow Def^{\wedge}_{T_0}(k[\![t]\!])$$

maps B to $\{\alpha_i\}_i$.

Corollary

Let $T_0 \in Dg^{cc}(k)$ be a smooth proper dg-category. Then the map $Def_{T_0}(k[\![t]\!]) \longrightarrow Def^{\wedge}_{T_0}(k[\![t]\!])$ is an equivalence. In other word, the naive definition of deformation works out.