On the deformation theory of dg-categories

Anthony Blanc

MPIM Bonn

October 21st, 2015

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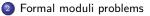




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Two deformations B_{ϵ} and B'_{ϵ} are isomorphic iff $\phi - \phi'$ is a Hochschild coboundary.

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Suppose $dim_k(B_0) < \infty$; let $B_1 \in \pi_0 Def_{B_0}^{Ass}(k[\epsilon])$, then there exists a $o(B_1) \in HH^3(B_0)$ which verifies the property:

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$$\pi_0 Def_{B_0}^{Ass}(k \oplus k[1]) \simeq HH^3(B_0).$$

where $k \oplus k[1]$ is the trivial square zero extension and is a commutative dg-algebra.

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Key idea of Derived Deformation Theory

Enlarge Art_k to $dgArt_k$ commutative artinian connective dg-algebras.

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Key idea of Derived Deformation Theory

Enlarge Art_k to $dgArt_k$ commutative artinian connective dg-algebras.

A commutative dg-algebra A is artinian if

- $H^0(A)$ is a local ring.
- $H^i(A) = 0, \forall i \gg 0.$
- $dim_k(H^i(A)) < \infty$.

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If A is a commutative dg-algebra/k. dg - Alg(A) =the ∞ -category of A-dg-algebras up to quasi-isomorphism.

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Definition

$$Def_{B_0}^{dga}: dgArt_k \longrightarrow S$$

$$Def_{B_0}^{dga}(A) := dg - Alg(A)^{\simeq} imes_{dg-Alg(k)^{\simeq}} \{B_0\}$$

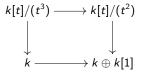
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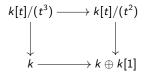
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is a pullback diagram in $dgArt_k$. We can prove that if $dim_k(B_0) < \infty$ the diagram

is a pullback in spaces.

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Definition (Lurie, DAGX)

A formal moduli problem F over k is an ∞ -functor

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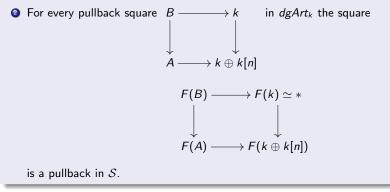
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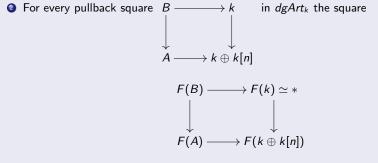
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is a pullback in \mathcal{S} .

We denote by FMP_k the ∞ -category of fmps over k. And by $PFMP_k$ the ∞ -category of ∞ -functors $dgArt_k \longrightarrow S$ which verifies only conditions $1 < 0 > \langle B \rangle < B \rangle < B \rangle < B \rangle > \langle B$

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- If X_0 is a smooth projective scheme over k. Then Def_{X_0} is an fmp. $\pi_0 Def_{X_0}(k[\epsilon]) \simeq H^1(T_{X_0})$.
- If V is an algebraic vector bundle over an algebraic variety X. Then Def_V is an fmp. $\pi_0 Def_V(k[\epsilon]) \simeq H^1(X, End(V)).$

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Property

The inclusion ∞ -functor $FMP_k \hookrightarrow PFMP_k$ commutes with limits. It has a left adjoint denoted by

$$(-)^{\wedge}: PFMP_k \longrightarrow FMP_k$$

$$F \longmapsto F^{\wedge}$$

For every $F \in PFMP_k$ we therefore have a canonical map $F \longrightarrow F^{\wedge}$.

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A formal moduli problem has a tangent spectrum.

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$$\begin{array}{c} k \oplus k[n-1] \longrightarrow k \\ \downarrow \qquad \qquad \downarrow \\ k \longrightarrow k \oplus k[n] \end{array}$$

are pullback squares in $dgArt_k$.

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$$F(k \oplus k[n-1]) \simeq \Omega F(k \oplus k[n]).$$

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The loop space functor is

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with $\Omega(F)(A) := \Omega(F(A))$ the space of loops at the point $* \simeq F(k) \longrightarrow F(A)$.

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$$T_{\Omega F} \simeq \Omega T_F \simeq T_F[-1],$$

hence, informally, the latter is a dg-Lie-algebra.

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• B_0 a finite dimensional algebra/k. $T_{Def_{B_0}^{dga}}[-1] \simeq HH^{\bullet}(B_0)[1].$

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$$\Psi(L)(A) = Map_{dgLie_k}(\mathfrak{D}(A), L)$$

where $\mathfrak{D} : (CAlg_k^{aug})^{op} \longrightarrow dgLie_k$ is the Kozsul duality functor.

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The objects of $Dg^{cc}(k)$ are dg-categories of the form $\widehat{A} := \operatorname{RMod}_{A^{op}, dg}^{cof}$ for A a small k-dg-category. Equivalences are quasi-equivalences.

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Definition $Def_{T_0}: dgArt_k \longrightarrow \hat{S}$ $Def_{T_0}(A) := Dg^{cc}(A)^{\simeq} imes_{Dg^{cc}(k)^{\simeq}} \{T_0\}$

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An point in $Def_{T_0}(A)$ corresponds to a pair (T_A, u) with $T_A \in Dg^{cc}(A)$ and $u : T_A \widehat{\otimes}_A \widehat{k} \simeq T_0$ a Morita equivalence.

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 $T_0 = k[u, u^{-1}]$ with deg(u) = 2 and d = 0. $HH^2(T_0) \simeq k$ generated by u. Keller-Lowen showed that $u \notin Im(\Theta_0)$.

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Then Def_{T_0} is a formal moduli problem.

- What is $Def_{T_0}^{\wedge}$?
- Is the map Θ : Def_{T0} → Def[∧]_{T0} an equivalence for a larger nice class dg-categories T₀ ? Or on specific algebras e.g. A = k[[t]].

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Define $Def_{T_0}(k\llbracket t \rrbracket) := \lim_{i \ge 1} Def_{T_0}(k\llbracket t]/(t^i))$ and $Def^{\wedge}_{T_0}(k\llbracket t \rrbracket) := \lim_{i \ge 1} Def^{\wedge}_{T_0}(k\llbracket t]/(t^i)).$

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The following is our main result, which has benefited from discussions with B. Toën (see his 2014 ICM adress).

Theorem (B.–Pandit)

Let $T_0 \in Dg^{cc}(k)$ be a smooth proper dg-category. For every $\{\alpha_i\}_i \in Def^{\wedge}_{T_0}(k[t])$ there exists a proper dg-algebra B over k[t] such that

$$Def_{B_0}^{dga}(k\llbracket t
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maps *B* to $\{\alpha_i\}_i$.

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Let's now review the necessary tools for proving this.

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Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

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Lemma–Definition (Lurie)

Let F be a pre-fmp over k. Then the following are equivalent:

- $\Omega^n F$ is an fmp.
- So For all pullback squares B → k in dgArt_k, the map $\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad A \rightarrow k \oplus k[n]$ F(B) → F(A) ×_{F(k⊕k[n])} * is (n 2)-truncated.

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- For all pullback squares $B \longrightarrow k$ in $dgArt_k$, the map $\downarrow \qquad \downarrow$ $A \longrightarrow k \oplus k[n]$ $\sum_{k \to k} \sum_{k \to k} \sum$
 - $F(B) \longrightarrow F(A) \times_{F(k \oplus k[n])} *$ is (n-2)-truncated.
- **3** The map $F(A) \longrightarrow F^{\wedge}(A)$ is (n-2)-truncated for all $A \in dgArt_k$.

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• The map $F(A) \longrightarrow F^{\wedge}(A)$ is (n-2)-truncated for all $A \in dgArt_k$.

We call such a fmp *n-proximate*.

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Let $E_0 \in T_0$ be an object. Set $T_A := T_0 \widehat{\otimes}_k \widehat{A}$.

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 $Def_{E_0} : dgArt_k \longrightarrow S$

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Proposition (Lurie, Toën–Vaquié)

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Proposition (Lurie, Toën–Vaquié)

The pre-fmp Def_{E_0} is 1-proximate.

When T_0 is smooth proper, we even have a locally geometric stack of objects in T_0 (Toën–Vaquié).

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$$\begin{split} \Omega Def_{T_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{ id_{T_0} \} \\ &\simeq \operatorname{RMod}_{T_0^{op} \otimes_k T_0 \otimes_k A}^{inv, \simeq} \times_{\operatorname{RMod}_{T_0^{op} \otimes_k T_0}} \{ id_{T_0} \} \end{split}$$

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$$\begin{split} \Omega Def_{\tau_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{ id_{\tau_0} \} \\ &\simeq \operatorname{RMod}_{T_0^{op} \otimes_k T_0 \otimes_k A}^{inv, \simeq} \times_{\operatorname{RMod}_{T_0^{op} \otimes_k T_0}} \{ id_{\tau_0} \} \end{split}$$

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$$\Omega^{2} Def_{T_{0}}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}(T_{A})}(id_{T_{A}}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{T_{0}})} \{id_{id_{T_{0}}}\}$$

$$\begin{split} \Omega Def_{T_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{ id_{T_0} \} \\ &\simeq \operatorname{RMod}_{T_0^{op} \otimes_k T_0 \otimes_k A}^{inv, \simeq} \times_{\operatorname{RMod}_{T_0^{op} \otimes_k T_0}} \{ id_{T_0} \} \end{split}$$

The pre-fmp ΩDef_{T_0} is 1-proximate, hence Def_{T_0} is 2-proximate.

$$\Omega^2 Def_{T_0}(A) \simeq Aut_{Aut_{Dg^{cc}(A)}}(\tau_A)(id_{\tau_A}) \times_{Aut_{Aut_{Dg^{cc}(k)}}(id_{\tau_0})} \{id_{id_{\tau_0}}\}$$

$$\mathcal{T}_{\Omega^2 Def_{\mathcal{T}_0}^{\wedge}} \simeq \mathcal{T}_{\Omega^2 Def_{\mathcal{T}_0}} \simeq \mathit{End}_{\mathit{End}(\mathcal{T}_0)}(\mathit{id}_{\mathcal{T}_0}) \simeq \mathit{HH}^{ullet}(\mathcal{T}_0)$$

$$\begin{split} \Omega Def_{\tau_0}(A) &\simeq Aut_{Dg^{cc}(A)}(T_A) \times_{Aut_{Dg^{cc}(k)}(T_k)} \{ id_{\tau_0} \} \\ &\simeq \operatorname{RMod}_{\tau_0^{op} \otimes_k \tau_0 \otimes_k A}^{inv, \simeq} \times_{\operatorname{RMod}_{\tau_0^{op} \otimes_k \tau_0}} \{ id_{\tau_0} \} \end{split}$$

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Corollary

The map $Def_{\mathcal{T}_0}(A) \longrightarrow Def_{\mathcal{T}_0}^{\wedge}(A)$ is 0-truncated, i.e.

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$$\Omega^2 \mathsf{Def}_{\mathcal{T}_0}(A) \simeq \mathsf{Aut}_{\mathsf{Aut}_{\mathsf{Dg}^{\mathsf{cc}}(A)}(\mathcal{T}_A)}(\mathsf{id}_{\mathcal{T}_A}) \times_{\mathsf{Aut}_{\mathsf{Aut}_{\mathsf{Dg}^{\mathsf{cc}}(k)}}(\mathsf{id}_{\mathcal{T}_0})} \{\mathsf{id}_{\mathsf{id}_{\mathcal{T}_0}}\}$$

$$T_{\Omega^2 Def_{T_0}^{\wedge}} \simeq T_{\Omega^2 Def_{T_0}} \simeq End_{End(T_0)}(id_{T_0}) \simeq HH^{\bullet}(T_0)$$

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The map $Def_{\mathcal{T}_0}(A) \longrightarrow Def_{\mathcal{T}_0}^{\wedge}(A)$ is 0-truncated, i.e.

- is an isomorphism on π_i for all $i \ge 2$.
- is injective on π_1 .

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Idea

The *n*-proximate property is related to the property of being defined on E_n -algebras, and to a description in terms of Koszul duality of E_n -algebras, where eventually $n = \infty$.

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Proposition (Lurie)

• Def_{E_0} is an E_1 -fmp. There exists an equivalence

$$Def_{E_0}^{\wedge}(A) \simeq Map_{E_1-Alg_k}(\mathfrak{D}^{(1)}(A), End_{T_0}(E_0)).$$

• Def_{T_0} is an E_2 -fmp. There exists an equivalence

$$\mathsf{Def}^{\wedge}_{T_0}(\mathsf{A})\simeq \mathsf{Map}_{\mathsf{E}_2-\mathsf{Alg}_k}(\mathfrak{D}^{(2)}(\mathsf{A}),\mathsf{HH}^{ullet}(T_0)).$$

Where $\mathfrak{D}^{(n)}$ is the Koszul duality functor for E_n -algebras.

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Idea

The *n*-proximate property is related to the property of being defined on E_n -algebras, and to a description in terms of Koszul duality of E_n -algebras, where eventually $n = \infty$.

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Corollary

$$Def^{\wedge}_{T_0}(A) \simeq Map_{Cat^{\otimes}_{\infty}}(\mathrm{RMod}^{\otimes}_{\mathfrak{D}^{(2)}(A)}, End^{\otimes}(T_0))$$

 $\simeq \{\mathfrak{D}^{(2)}(A)\text{-linear structures on } T_0\}$

Anthony Blanc (MPIM Bonn)

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If A is an E_1 -algebra, $\operatorname{RMod}_A^! := \operatorname{Ind}(\operatorname{RMod}_A^{sm})$ where an A-module is small if its underlying k-module has finite dimensional total cohomology.

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Let $\underline{\operatorname{Aut}}_{\mathcal{T}_0} := \Omega Def_{\mathcal{T}_0}$ be the deformation functor of $id_{\mathcal{T}_0}$ in $\operatorname{RMod}_{\mathcal{T}_0^{op} \otimes_k \mathcal{T}_0}$.

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Corollary

The formal moduli problem $\underline{\operatorname{Aut}}_{T_0}^{\wedge}$ is equivalent to

$$\underline{\operatorname{Aut}}^!_{\mathcal{T}_0}: A \longmapsto \left(\operatorname{RMod}^!_{\mathcal{T}_0^{op} \otimes_k \mathcal{T}_0 \otimes_k A} \right)^{inv, \simeq} \times_{\left(\operatorname{RMod}^!_{\mathcal{T}_0^{op} \otimes_k \mathcal{T}_0} \right)^{inv, \simeq}} \{ id_{\mathcal{T}_0} \}$$

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$FMP_k \subset FMP_k^{(1)} \subset FMP_k^{(2)} \subset \ldots \subset PFMP_k$

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For every integer *n*,

$$\Omega^n : FMP_k^{(n)} \longrightarrow E_n - Gp(FMP_k)$$

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Proposition

Let F be an *n*-proximate formal moduli problem. Then we have an equivalence:

$$F^{\wedge}(A) \simeq Map_{E_n-Gp(FMP_k)}(\Omega^n h_{Spec(A)}, \Omega^n F)$$

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This is based on the

Proposition

The functor Ω^n restricted to FMP_k is an equivalence.

$$\Omega^{(n)}_{|FMP_k}:FMP_k\longrightarrow E_n-Gp(FMP_k)$$

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Proposition

Let $T_0 \in Dg^{cc}(k)$, there exists an equivalence:

$$Def_{T_0}^{\wedge}(A) \simeq Map_{E_1 - Gp(FMP_k)}(\Omega h_{Spec(A)}, \underline{\operatorname{Aut}}_{T_0}^!).$$

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Remark

When \mathcal{T}_0 is smooth and proper, the canonical map $\underline{\operatorname{Aut}}_{\mathcal{T}_0} \longrightarrow \underline{\operatorname{Aut}}_{\mathcal{T}_0}^!$ is an equivalence, because $\underline{\operatorname{Aut}}_{\mathcal{T}_0}$ is the restriction to artinian algebras of a geometric stack.

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Theorem (B.–Pandit)

Let T_0 be a smooth proper dg-category in $Dg^{cc}(k)$. For every $\{\alpha_i\}_i \in Def^{\wedge}_{T_0}(k[t])$ there exists a proper dg-algebra B over k[t] such that

$$\mathsf{Def}^{dga}_{B_0}(k\llbracket t
rbracket) \longrightarrow \mathsf{Def}^\wedge_{T_0}(k\llbracket t
rbracket)$$

maps B to $\{\alpha_i\}_i$.

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Corollary

Let $T_0 \in Dg^{cc}(k)$ be a smooth proper dg-category. Then the map $Def_{T_0}(k[t]) \longrightarrow Def_{T_0}^{\wedge}(k[t])$ is an equivalence. In other word, the naive definition of deformation works out.