Shape approximate controllability of the heat equation through Fenchel duality

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Context

Linear control problem

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0$$

- $y(t) \in X$, $u(t) \in U$ Hilbert spaces, $E := L^2(0, T; U)$
- $(A, D(A))$ operator generating a C_0 semigroup over X, denoted $(S_t)_{t \geq 0}$
- $B \in L(U, X)$

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Goal: given a constraint set $\mathcal{U} \subset E$, investigate constrained approximate controllability : for $T > 0$, $y_0, y_f \in X$, through $\mathcal{U}$-valued controls.

$$\text{for any } \varepsilon > 0, \text{ find } u_\varepsilon \in \mathcal{U} \text{ such that } \|y(T) - y_f\|_X \leq \varepsilon.$$

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$$\text{for any } \varepsilon > 0, \text{ find } u_\varepsilon \in \mathcal{U} \text{ such that } \|y(T) - y_f\|_X \leq \varepsilon.$$

Notations:

$$y(T) = L_T u + S_T y_0, \quad L_T u := \int_0^T S_{T-t} B u(t)\, dt,$$

With $\tilde{y}_T = y_f - S_T y_0$, above rewrites

$$\text{for any } \varepsilon > 0, \text{ find } u_\varepsilon \in \mathcal{U} \text{ such that } L_T u \in \overline{B}(\tilde{y}_T, \varepsilon).$$

Look for abstract dual necessary and sufficient conditions.

Standing assumption: $\mathcal{U}$ closed and convex.

Getting to know support functions

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$$\text{No } \mathcal{U}\text{-approximate controllability} \quad \Longleftrightarrow \quad \exists \varepsilon > 0, \quad L_T \mathcal{U} \cap B(\tilde{y}_T, \varepsilon) = \emptyset.$$

If latter holds, separate disjoint convex closed and convex open set

$$\exists p_f \neq 0, \ \forall u \in \mathcal{U}, \ \forall y \in B(\tilde{y}_T, \varepsilon), \quad \langle u, L_T^* p_f \rangle_E = \langle L_T u, p_f \rangle_X \leq \langle y, p_f \rangle_X.$$

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Equivalently

$$\underbrace{\sup_{u \in \mathcal{U}} \langle u, L_T^* p_f \rangle_E}_{\sigma_{\mathcal{U}}(L_T^* p_f)} \leq \inf_{y \in B(\tilde{y}_T, \varepsilon)} \langle y, p_f \rangle_X.$$

Support function of a set $\mathcal{U} \subset E$ is defined by

$$\forall p \in E, \quad \sigma_U(p) := \sup_{v \in \mathcal{U}} \langle p, v \rangle.$$

Look for abstract dual necessary and sufficient conditions.

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Theorem

Approximate controllability from y_0 to y_f in time $T > 0$ under constraints $\mathcal{U}$ holds if and only if

$$\forall p_f \in X, \quad \sigma_{\mathcal{U}}(L_T^* p_f) \geq \langle \tilde{y}_T, p_f \rangle_X .$$

Note: unconstrained case $\mathcal{U} = E$ is covered; injectivity of L_T^* is sufficient for the above to hold for all y_0, y_f.

Internal controllability of the heat equation

$y(t,x)$: temperature at time $t \geq 0$ and position $x \in \Omega$.

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega \boldsymbol{u} \\ y(0,\cdot) = y_0, \\ y_{|\partial\Omega} = 0, \end{cases} \tag{1}$$

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Cast into the form

$$\dot{y}(t) = Ay(t) + Bu(t),\ y(0) = y_0$$

with state space $X = L^2(\Omega)$, control space $E = L^2(0,T; L^2(\Omega))$.

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For any $\omega \subset \Omega$ (with positive measure),

- the heat equation (4) is approximately controllable

$$\forall y_0, y_f \in X, \forall T > 0, \; \forall \varepsilon > 0, \quad \exists u \in E \text{ tel que } \|y(T) - y_f\|_X \leq \varepsilon.$$

- the heat equation (4) is **not** exactly controllable

Dual equation, $p_f \in X$

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Dual functional

$$\begin{aligned} J(p_f) &= \frac{1}{2} \int_0^T \int_\omega p^2(t,x)\, dx\, dt - \langle \tilde{y}_T, p_f \rangle_X + \varepsilon \|p_f\|_X \\ &= \frac{1}{2} \|\chi_\omega p\|_E^2 - \langle \tilde{y}_T, p_f \rangle_X + \varepsilon \|p_f\|_X \end{aligned}$$

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Key result: coercivity thanks to Holmgren's uniqueness theorem:

$$\Big(\forall (t,x) \in (0,T) \times \omega, \quad p(t,x) = 0 \Big) \quad \Longrightarrow \quad p_f = 0.$$

Functional admits a unique minimiser $p_f^\star$, and the control

$$u^\star := \chi_\omega p^\star$$

steers y_0 to the ball $\overline{B}(\tilde{y}_T, \varepsilon)$ in the unconstrained case.

$$\begin{cases} \partial_t y - \Delta y = u \\ y(0,\cdot) = y_0, \\ y_{|\partial\Omega} = 0, \end{cases} \qquad \text{under constraints } \forall t \in (0,T),\ u(t) \in \mathcal{U}(\subset L^2(\Omega)) \tag{2}$$

with

$$\mathcal{U} \subset \{u \geq 0\}.$$

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First obstruction: monotonicity

$$\forall u \geq 0, \quad \forall t > 0, \quad y(t) \geq S_t y_0.$$

Nonnegative controllability and obstructions in arbitrary time

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Relevant notion of approximate controllability in the nonnegative setting:

Definition

For $\mathcal{U} \subset \{u \geq 0\}$, the system (2) is said to be *nonnegatively approximatively controllable under the constraints* $\mathcal{U}$ in time T if

$$\forall \varepsilon > 0, \forall y_f \geq S_T y_0, \quad \exists u \in \mathcal{U},\ \|y(T) - y_f\|_X \leq \varepsilon.$$

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega u \\ y(0,\cdot) = y_0, \\ y_{|\partial\Omega} = 0, \end{cases} \qquad \text{under constraints } \forall t \in (0,T),\ u(t) \in \mathcal{U} \tag{2}$$

with

$$\mathcal{U} = \{u \geq 0\}.$$

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Theorem (P.-Trélat-Zhang)

If there exists a subdomain $\omega_0 \subset (\Omega \setminus \omega)$, then the system (2) is not nonnegatively approximately controllable in small time T.

Note: see also Pighin-Zuazua '18.

Idea of proof (inspired by Pighin-Zuazua '18)

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$$\frac{d}{dt}\langle y(t), p(t)\rangle_X = \langle p(t), u(t)\rangle_X.$$

Nonnegative controllability and obstructions in small time (2)

Idea of proof (inspired by Pighin-Zuazua '18)

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Case $y_0 = 0 \quad \Longrightarrow \quad y_f \geq S_T y_0 = 0,\ y_f \neq 0.$

$$\langle y(T), p_f\rangle_X = \int_0^T \langle p(t), u(t)\rangle_X \, dt.$$

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Dual equation, $p_f \in X$

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$$\langle y(T), p_f\rangle_X = \int_0^T \langle p(t), u(t)\rangle_X \, dt.$$

Choose y_f such that $\mathrm{supp}(y_f) \subset\subset \omega_0$, then build p_f such that

(i) $p_f < 0$ over $\mathrm{supp}(y_f)$,

(ii) $p \geq 0$ over $(0, T^\star) \times (\Omega \setminus \omega_0)$, where p solves (2).

$$\Longrightarrow \qquad \langle y_f, p_f\rangle_X < 0 \quad \text{and} \quad \forall T < T^\star,\ \int_0^T \langle p(t), u(t)\rangle_X \, dt \geq 0.$$

Constraints:

$$\begin{cases} \partial_t y - \Delta y = \chi_{\omega(t)} u \\ y(0,\cdot) = y_0, \\ y_{|\partial\Omega} = 0, \end{cases}$$

with

$$\forall t \in (0,T), \quad |\omega(t)| \leq m_L, \qquad m_L = L|\Omega|, \quad L \in (0,1).$$

Control by shapes

Constraints:

$$\begin{cases} \partial_t y - \Delta y = M(t)\chi_{\omega(t)} \\ y(0,\cdot) = y_0, \\ y_{|\partial\Omega} = 0, \end{cases}$$

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$$\forall t \in (0,T), \quad |\omega(t)| \leq m_L, \qquad m_L = L|\Omega|, \quad L \in (0,1).$$

$$\mathcal{U}^L_{\text{shape}} := \{M\chi_\omega, \quad \omega \subset \Omega, \quad |\omega| \leq m_L,\ M > 0\}.$$

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Theorem (P.-Trélat-Zhang)

For all $L \in (0,1)$, $T > 0$, the heat equation is nonnegatively approximately controllable under the constraints $\mathcal{U}^L_{\mathrm{shape}}$, in time T.

(Approximate) controllability through optimal control

Dual approach to constrained controllability: similar ideas found in works by Kunisch-Wang ('13), Berrahmoune ('14 and '19), Ervedoza ('20), Biccari-Zuazua ('22) with

- ⋄ dual functional directly introduced
- ⋄ most often, differentiable objective functions

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Without constraints, convex analytic point of view (Lions '88): find $u \in E$ such that $\|y(T) - y_f\|_X \leq \varepsilon$ is equivalent to proving, for a given $F : E \to \mathbb{R} \cup \{+\infty\}$, that

$$\pi := \inf_{u \in E,\ \|y(T)-y_f\|_X \leq \varepsilon} F(u) < +\infty.$$

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Rewriting

$$\inf_{u \in E,\ \|y(T)-y_f\|_X \leq \varepsilon} F(u) = \inf_{u \in E} F(u) + G(L_T u).$$

where

$$G(L_T u) = \begin{cases} 0 & \text{if } \|y(T) - y_f\|_X \leq \varepsilon \\ +\infty & \text{else} \end{cases}$$

i.e. $G = \delta_{\overline{B}(\tilde{y}_T, \varepsilon)}$.

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Idea: if $\pi < +\infty$, choose F that imposes the constraints

- ⋄ directly? i.e. $F(u) < +\infty \implies u \in \mathcal{U}$
- ⋄ subtly? i.e. u optimal $\implies u \in \mathcal{U}$.

H Hilbert space, $f : H \to [-\infty, +\infty]$.

Fenchel conjugate

$$f^*(y) := \sup_{x \in H} \langle y, x \rangle - f(x), \quad y \in H.$$

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$$\frac{1}{2}|x|^2 \quad \longleftrightarrow \quad \frac{1}{2}|x|^2$$

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$$\frac{1}{p}|x|^p \quad \longleftrightarrow \quad \frac{1}{q}|x|^q \qquad 1 < p < +\infty, \;\; q = p^*$$

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$$|x| \quad \longleftrightarrow \quad \delta_{[-1,1]}$$

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$$\|x\|_H \quad \longleftrightarrow \quad \delta_{\overline{B}(0,1)}.$$

Convex analysis in a nutshell: Fenchel conjugate

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Notation for non-empty convex closed set C

$$\delta_C = \begin{cases} 0 & \text{over C} \\ +\infty & \text{outside of C} \end{cases} \quad \longleftrightarrow \quad \sigma_C := (\delta_C)^*.$$

i.e.

$$\sigma_C(y) = \sup_{x \in H} \langle y, x \rangle - \delta_C(x) = \sup_{x \in C} \langle y, x \rangle.$$

H Hilbert space, $f : H \to [-\infty, +\infty]$.

Fenchel conjugate

$$f^*(y) := \sup_{x \in H} \langle y, x \rangle - f(x), \quad y \in H.$$

$$\Gamma_0(H) := \{f : H \to]-\infty, +\infty], \text{ proper, convex, lower semicontinuous}\}.$$

Fenchel-Moreau theorem: if $f \in \Gamma_0(H)$, then

- $f^* \in \Gamma_0(H)$,
- $f^{**} = f$.

Convex analysis in a nutshell: subdifferential

H Hilbert space, $f : H \to [-\infty, +\infty]$.

Subdifferential at $x \in H$

$$\partial f(x) = \{p \in H, \quad \forall y \in X,\ f(y) \geq f(x) + \langle p, y - x \rangle_H\}.$$

Example: $f(x) = |x|$,

$$\partial f(x) = \begin{cases} \{\mathrm{sgn}(x)\} & \text{if } x \neq 0, \\ [-1, 1] & \text{if } x = 0 \end{cases}.$$

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First order optimality condition for convex f

$$x \text{ minimises } f \text{ over } H \quad \Longleftrightarrow \quad 0 \in \partial f(x).$$

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Link with **Fenchel** conjugate: for $f \in \Gamma_0(H)$

$$p \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(p).$$

$F \in \Gamma_0(E)$, $G \in \Gamma_0(X)$, $L_T \in L(E, X)$.

Primal problem

$$\pi = \inf_{u \in E} F(u) + G(L_T u)$$

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Primal problem

$$\pi = \inf_{u \in E} F(u) + G(L_T u)$$

admits the dual problem

$$d = -\inf_{p_f \in X} F^*(L_T^* p_f) + G^*(-p_f)$$

Weak duality

$$\pi \geq d.$$

Fenchel-Rockafellar theorem

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Weak duality

$$\pi \geq d.$$

Fenchel-Rockafellar theorem: under one weak hypothesis for F, G, L_T i.e.

$$\exists u \in E, \quad F(u) < +\infty \text{ and } L_T u \in B(\tilde{y}_T, \varepsilon).$$

- strong duality holds $\pi = d$,
- d is attained *if finite*;

Fenchel-Rockafellar theorem

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Weak duality

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Fenchel-Rockafellar theorem: under one weak hypothesis for F^*, G^*, L_T^*, i.e.

$$\exists p_f \in X, \quad F^* \text{ is continuous at } L_T^* p_f.$$

- ◇ strong duality holds $\pi = d$,
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Saddle-point view

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Equivalence between

- $u^\star$ is primal optimal, $p_f^\star$ is dual optimal and strong duality $\pi = d$ holds,
- $(u^\star, -p_f^\star)$ is a saddle point of the Lagrangian, i.e.,

$$(u, q) \in E \times X \longmapsto \langle q, L_T u \rangle_X + F(u) - G^*(q).$$

Saddle-point view

$F \in \Gamma_0(E)$, $G \in \Gamma_0(X)$, $L_T \in L(E, X)$

Primal problem

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$$u^\star \in \partial F^*(L_T^* p_f^\star) \quad \text{and} \quad p_f^\star \in -\partial G(L_T u^\star)$$

Back to controllability

Controlled equation

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0.$$

$$\pi = \inf_{u \in E} F(u) + G(L_T u), \quad G = \delta_{\overline{B}(\tilde{y}_T, \varepsilon)} \in \Gamma_0(X)$$

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Dual problem (Lions '88)

$$-d = \inf_{p_f \in X} F^*(L_T^* p_f) + G^*(-p_f) = \inf_{p_f \in X} \underbrace{F^*(L_T^* p_f) - \langle \tilde{y}_T, p_f \rangle + \varepsilon \|p_f\|_X}_{J(p_f)}$$

Back to controllability

Controlled equation

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0.$$

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Heat equation:

Adjoint $L_T^* \in L(X, E)$ given by $\quad \forall p_f \in X, \quad L_T^* p_f(t) = B^* p(t)$

$$\begin{cases} \partial_t p + \Delta p = 0 \\ p(T, \cdot) = p_f, \\ p_{|\partial\Omega} = 0, \end{cases}$$

Unconstrained approximate controllability: "$B = \chi_\omega$" + $F = \frac{1}{2}\|\cdot\|_E^2$.

Methodology

Reasonable $F \in \Gamma_0(E)$: there must exist $p_f \in X$ such that F^* is continuous at $L_T^* p_f$.
Then strong duality $\pi = d$ holds... which may be $\pi = d = +\infty$.

- ◇ show that $d < +\infty$: coercivity of the dual functional J,
- ◇ then, if $(F(u) < +\infty \implies u \in \mathcal{U})$, we are done

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Tension between

- directly imposing constraints through a term of the form $\delta_{\mathcal{U}}$
- the coercivity requirement for the dual functional J (as well as the sufficient condition for strong duality)

Conic **nonconvex** constraints

$$\mathcal{U}^L_{\text{shape}} := \{M\chi_\omega, \quad \omega \subset \Omega, \quad |\omega| \le m_L,\, M > 0\} = \bigcup_{M>0} (M\,\mathcal{U}_L),$$

with

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Constraints associated to shapes

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Relaxation of constraints

$$\overline{\mathcal{U}_L} := \left\{ u \in L^2(\Omega),\ 0 \leq u \leq 1,\ \int_\Omega u \leq m_L \right\}.$$

With extremality

$$\mathrm{ext}(\overline{\mathcal{U}_L}) = \mathcal{U}_L$$

Choosing the cost

$J(p_f) = F^*(L_T^* p_f) - \langle \tilde{y}_T, p_f \rangle + \varepsilon \|p_f\|_X \ldots$ how to enforce coercivity?

$$F^*(p) \sim \|p\|^2 \quad \text{homogeneity of degree 2,}$$

$$F^*(p) := \frac{1}{2} \int_0^T \Big(f^*(p(t)) \Big)^2 \, dt, \quad f^* \text{ homogeneous of degree 1.}$$

Then

$$u^\star \in \partial F^*(p) \quad \iff \quad \forall t \in (0, T), \; u^\star(t) \in M(t)\, \partial f^*(p(t)), \; M(t) = f^*(p(t)).$$

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$$\forall p \in X, \quad \partial f^*(p) \subset \mathcal{U}_L = \mathrm{ext}(\overline{\mathcal{U}_L}).$$

How can one (hope to) catch extremal points?

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How can one (hope to) catch extremal points?

By maximising linear functions:

$$\partial f^*(p) = \arg\max_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle,$$

amounts to

$$f^\star = \sigma_{\overline{\mathcal{U}_L}} = \delta^*_{\overline{\mathcal{U}_L}}, \quad \text{i.e.,} \quad f^*(p) = \sup_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle_X.$$

Bathtub principle

Optimisation problem

$$p \in L^2(\Omega) \text{ fixed }, \quad \sup_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle_X = \sup_{v \in \overline{\mathcal{U}_L}} \int_\Omega p(x)v(x)\,dx.$$

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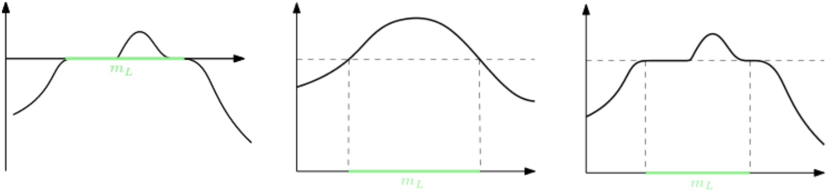

Figure: Let us take a bath.

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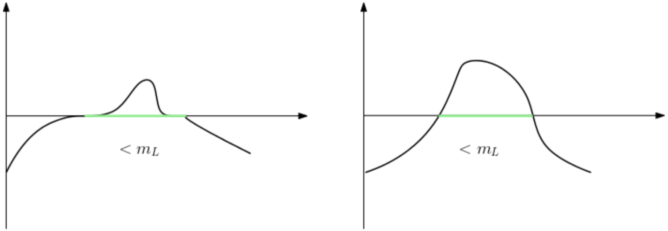

Figure: Let us take a second bath.

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Theorem (Bathtub principle)

Let $p \in L^2(\Omega)$ have 0-measure level sets, $r(p) := \max(0, \Phi_p^{-1}(m_L))$, où Φ_p^{-1} pseudo-inverse of $r \mapsto \Phi_p(r) := |\{p > r\}|$.

- *maximum equals $\int_{\{p>r(p)\}} p$*
- *and is uniquely attained by $\chi_{\{p>r(p)\}}$.*

Chosen function $f^*(p) = \sup_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle_X,\ p \in L^2(\Omega)$,

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Chosen function $f^*(p) = \sup_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle_X$, $p \in L^2(\Omega)$,

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We are left with studying

- the coercivity of the dual functional J
- the optimality condition $u^\star \in \partial F^*(L_T^* p_f^\star)$

Coercivity

Try and show that

$$\liminf_{\|p_f\|_X \to \infty} \frac{J(p_f)}{\|p_f\|_X} > 0.$$

By homogeneity, with $q_f = \frac{p_f}{\|p_f\|}$,

$$\frac{J(p_f)}{\|p_f\|_X} = \|p_f\|_X F^*(L_T^* q_f) - \langle \tilde{y}_T, q_f \rangle_X + \varepsilon.$$

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After taking a sequence (p_f^n) and extracting subsequences $q_f^n \rightharpoonup q_f$, worst case is

$$\liminf_{n\to\infty} F^*(L_T^* q_f^n) = 0 \quad \Longleftrightarrow \quad F^*(L_T^* q_f) = 0 \quad \Longrightarrow \quad q_f \leq 0$$

Thanks to $\tilde{y}_T = y_f - S_T y_0 \geq 0$

$$\liminf_{n\to\infty} \frac{J(p_f^n)}{\|p_f^n\|_X} \geq -\langle \tilde{y}_T, q_f \rangle_X + \varepsilon \geq \varepsilon > 0.$$

Optimality condition

Facts

◇ Strong duality $\pi = d < +\infty$ and existence of primal and dual optimal variables.

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- $u^\star$ optimal control, there exists $p_f^\star$ dual optimal such that

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with $M^\star(t) = \int_{\{p^\star(t) > r(p^\star(t))\}} p(t)$, and $p^\star$ solves

$$\begin{cases} \partial_t p^\star + \Delta p^\star = 0 \\ p^\star(T,\cdot) = p_f^\star, \\ p^\star_{|\partial\Omega} = 0, \end{cases}$$

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Only interesting case $y_f \notin \overline{B}(S_T y_0, \varepsilon)$: any dual optimal variable satisfies $p_f^\star \neq 0$.

$t \in (0,T)$ fixed: solution $p^\star(t)$ of (2) is (real) analytic, hence has level sets of measure 0... Unless it is constant, then it equals 0 by the boundary conditions, and then $p_f^\star = 0$ by the maximum principle ..., which cannot be.

Conclusion with the bathtub principle: any optimal control is a shape

$$u^\star(t) = M^\star(t)\, \chi_{\{p^\star(t) > r(p^\star(t))\}}.$$

Uniqueness for dual optimal variables.

Proof in two steps, very general

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$$\{L_T u^\star,\ u^\star \text{ optimal.}\}$$

is reduced to a singleton $y^\star$

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Proof in two steps, very general

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(ii) Second optimality condition
$$p_T^\star \in -\partial G(L_T u^\star) = -\partial \delta_{\overline{B}(\tilde{y}_T,\varepsilon)}(L_T u^\star) = -\partial \delta_{\overline{B}(\tilde{y}_T,\varepsilon)}(y^\star).$$
reduces the search for $p_T^\star$ to a half-line.

Remark: implies uniqueness for optimal controls.

Series of refinements (2)

Amplitude independent of time t

Slight change of cost, $f^*(p) = \sup_{v \in \overline{\mathcal{U}_L}} \langle p, v \rangle_X, \; p \in L^2(\Omega)$,

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Same proof strategy applies

$$u^\star(t) = M^\star \, \chi_{\{p^\star(t) > r(p^\star(t))\}}, \quad M = \int_0^T \int_{\{p^\star(t) > r(p^\star(t))\}} p(t)\, dt$$

Series of refinements (3)

Amplitude as a function of T and obstructions

Study of the amplitude $T \mapsto M^\star(T)$, y_0, y_f, ε, L all fixed. $\lambda_1 > 0$, first eigenvalue of the Dirichlet Laplacian operator.

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Study of the amplitude $T \mapsto M^\star(T)$, y_0, y_f, ε, L all fixed. $\lambda_1 > 0$, first eigenvalue of the Dirichlet Laplacian operator.

$$M^\star(T) \geq \lambda_1 \frac{\|\tilde{y}_T\|_X - \varepsilon}{\sqrt{m_L}(1 - e^{-\lambda_1 T})}.$$

Leads to

$$\lim_{T \to 0} M^\star(T) = +\infty \text{ at least as } \frac{1}{T},$$

and

$$\liminf_{T \to +\infty} M^\star(T) \geq \lambda_1 \frac{\|y_f\|_X - \varepsilon}{\sqrt{m_L}} > 0$$

yields new obstructions if one constrains the amplitude from above

- in small time,
- in arbitrary time.

◇ In the case of the heat equation

- Dirichlet or Neumann boundary control,
- numerically computing optimal controls

◇ Generalisation in terms of

- the control system
- general convex constraints
- non-convex constraints and extremality after relaxation