Reachable spaces for perturbed heat equations

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Program

Talk based on:

- Reachability results for perturbed heat equations, 2021, *hal-03380745* (S. Ervedoza, K. Le Balc'h, M. Tucsnak).

Plan

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A linear control system given by $y' = Ay + Bu$

- H is an Hilbert space, the space of the states.
- A is a linear (possibly unbounded) operator, with domain $D(A) \subset H$, generating a C^0 semi-group $(\mathbb{T}_t)_{t\geq 0}$ on H.
- U is an Hilbert space, the space of the controls.
- $B \in \mathcal{L}(U; D(A^*)')$ is an admissible control operator, i.e. for $t > 0$, the input map $\Phi_t : u \in L^2(0, +\infty; U) \mapsto \int_0^t \mathbb{T}_{t-s} Bu(s) ds \in H$.
- The linear control system is given by

$$\begin{cases} y'(t) = Ay(t) + Bu(t) & t \geq 0, \\ y(0) = y_0 \in H. \end{cases} \tag{L}$$

- At time $t \geq 0$, $y(t)$ is the state, $u(t)$ is the control.

Theorem

$\forall y_0 \in H,\ u \in L^2(0, +\infty; U)$, $\exists !\ y \in C^0([0, +\infty); H) \cap H^1(0, +\infty; D(A^*)')$ *to* (L):

$$y(t) = \mathbb{T}_t y_0 + \int_0^t \mathbb{T}_{t-s} Bu(s) ds,\ t \geq 0. \tag{Duhamel}$$

Alternative definition: $\Sigma = (\mathbb{T}, \Phi)$

A well-posed linear control system with state space H and input space U is a couple $\Sigma = (\mathbb{T}, \Phi)$ of families of operators such that

1. $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a C^0 semi-group of bounded linear operators on H;
2. $\Phi = (\Phi_t)_{t \geqslant 0}$ is a family of bounded linear operators from $L^2([0,\infty); U)$ to H, called input maps, such that

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v \qquad (t, \tau \geqslant 0,\ u, v \in L^2([0,\infty); U)),$$

where the $\underline{\tau\text{-concatenation}}$ of two signals u and v, denoted $u \underset{\tau}{\diamondsuit} v$, is the function

$$u \underset{\tau}{\diamondsuit} v = \begin{cases} u(t) & \text{for } \ t \in [0, \tau), \\ v(t-\tau) & \text{for } \ t \geqslant \tau. \end{cases}$$

Remark: A is the generator of $\mathbb{T}$ and $B\mathrm{v} = \lim_{t \to 0+} \frac{1}{t} \Phi_t (1_{[0,1]} \cdot \mathrm{v})$ for $v \in U$.

More informations in [Tucsnak, Weiss, Observation and Control for Operator Semigroups, 2009].

The reachable space

Let $y' = Ay + Bu$, or alternatively $\Sigma = (\mathbb{T}, \Phi)$ a well-posed linear control system.

The main objective is to describe the **reachable space**.

Definition

The reachable space at time $T > 0$ from $y_0 \in H$ is the affine space

$$\mathcal{R}_{T,y_0} := \{y(T)\ ;\ y' = Ay + Bu,\ y(0) = y_0,\ u \in L^2(0,T;U)\}$$
$$\mathcal{R}_{T,y_0} = \mathbb{T}_T y_0 + \mathrm{Ran}\,\Phi_T.$$

Several notions of controllability

Definition

Let $T > 0$ and let the pair $(\mathbb{T}, \Phi)$ define a well-posed control system.

- The pair $(\mathbb{T}, \Phi)$ is exactly controllable in time T if $\mathrm{Ran}\, \Phi_T = X$.
- The pair $(\mathbb{T}, \Phi)$ is approximately controllable in time T if $\overline{\mathrm{Ran}\, \Phi_T} = H$.
- The pair $(\mathbb{T}, \Phi)$ is null-controllable in time T if $\mathrm{Ran}\, \Phi_T \supset \mathrm{Ran}\, \mathbb{T}_T$.

Null-controllability in time $T \Leftrightarrow \forall y_0$, $\exists u \in L^2(0, T; U)$ such that

$$\begin{cases} y'(t) = Ay(t) + Bu(t) & t \in [0, T], \\ y(0) = y_0 \end{cases} \Rightarrow y(T) = 0.$$

Null-controllability in time $T \Leftrightarrow$ Exact controllability to trajectories in time T, i.e. $\forall$ trajectory $\bar{y}' = A\bar{y} + B\bar{u},\ t \in [0, T]$, $\bar{y}(0) = \bar{y}_0$, $\forall y_0 \in H$, $\exists u \in L^2(0, T; U)$ s.t.

$$\begin{cases} y'(t) = Ay(t) + Bu(t) & t \in [0, T], \\ y(0) = y_0 \end{cases} \Rightarrow y(T) = \bar{y}(T).$$

Kalman's condition

Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.
$y' = Ay + Bu$, the state space is $H = \mathbb{R}^n$ and the control space is $U = \mathbb{R}^m$.

Theorem (Kalman, Ho, Narendra (1963))

For every $T > 0$,

$$\mathrm{Ran}\ \Phi_T = \mathrm{Ran}(B|AB|A^2B|\dots|A^{n-1}B),$$
$$\mathcal{R}_{T,y_0} = e^{TA}y_0 + \mathrm{Ran}(B|AB|A^2B|\dots|A^{n-1}B).$$

- $\mathrm{Ran}\ \Phi_T = \mathrm{Ran}(B|AB|A^2B|\dots|A^{n-1}B) =: \mathcal{R}$ does not depend on $T > 0$.
- The notions of controllability do not depend on $T > 0$.
- Exact controllability $\Leftrightarrow$ Approximate controllability $\Leftrightarrow$ Null-controllability.
- If $\mathcal{R} = \mathbb{R}^n$, $y' = \tilde{A}y + \tilde{B}u$ is controllable for $(\tilde{A}, \tilde{B})$ closed enough to (A, B).
- If $\mathcal{R} \neq \mathbb{R}^n$, then $A_{\mathcal{R}} := A|_{\mathcal{R}} \in \mathcal{L}(\mathcal{R})$, $B \in \mathcal{L}(U, \mathcal{R})$, $y' = A_{\mathcal{R}} + Bu$ is (exactly) controllable on $\mathcal{R}$.

Specificities of the finite-dimensional setting

- Cayley Hamilton's theorem.
- Every vector subspace is closed.
- Time reversibility.

Question: What happen for infinite dimensional systems?

A typical example would be the heat equation with Neumann boundary controls:

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T)\times(0,\pi), \\ \partial_x y(t,0) = u_0(t),\ \partial_x y(t,\pi) = u_\pi(t) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi). \end{cases}$$

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STNCLS

$\Sigma = (\mathbb{T}, \Phi)$ a well-posed linear control system.

Assumption

Σ is small-time null-controllable, i.e. is null-controllable for every $T > 0$.

Examples:

- Heat equation with internal control

$$\begin{cases} \partial_t y - \Delta y = u 1_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{1}$$

  - $N = 1$: Fattorini, Russell (1971).
  - $N \geq 1$: Lebeau, Robbiano, Fursikov, Imanuvilov (1995-1996).
- Parabolic coupled system with internal control under a Kalman's condition on the coupling matrix and the control matrix.
  - Ammar-Khodja, Benabdallah, Dupaix, Gonzalez-Burgos (2009).
- ...

Several properties of the reachable space

Let Σ be a STNCLS.

$$\mathcal{R}_{T,y_0} := \{y(T)\ ;\ y' = Ay + Bu,\ y(0) = y_0,\ u \in L^2(0,T;U)\}$$
$$\mathcal{R}_{T,y_0} = \mathbb{T}_T y_0 + \operatorname{Ran}\Phi_T.$$

Theorem (Seidman, ... (1979))

- $\mathcal{R}_{T,y_0}$ *does not depend on $T > 0$ and $y_0 \in H$, now simply denoted by $\mathcal{R}$.*
- $\mathcal{R}$ *is an Hilbert space when endowed with the norm*

$$\|\eta\|_{\mathcal{R}_T} = \inf\{\|u\|_{L^2(0,T;U)}\ ;\ \eta = \Phi_T u\}.$$

- *For every $T_1, T_2 > 0$, $\|\cdot\|_{\mathcal{R}_{T_1}}$ and $\|\cdot\|_{\mathcal{R}_{T_2}}$ define equivalent norms on $\mathcal{R}$.*

Every STNCLS is a STECLS

Let Σ be a STNCLS.

$$\mathcal{R}_{T,y_0} := \{y(T)\ ;\ y' = Ay + Bu,\ y(0) = y_0,\ u \in L^2(0,T;U)\}$$
$$\mathcal{R}_{T,y_0} = \mathbb{T}_T y_0 + \operatorname{Ran}\Phi_T.$$

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

For $\tau > 0$, we set

$$\tilde{\mathbb{T}}_t = \mathbb{T}_t|_{\mathcal{R}_\tau}, \qquad (t \geq 0).$$

Then $\tilde{\mathbb{T}} = (\tilde{\mathbb{T}})_{t\geq 0}$

- *does not depend on the choice of $\tau > 0$,*
- *is a C^0 semi-group on $\mathcal{R}_\tau$,*
- *has generator $\tilde{A}$ defined by $D(\tilde{A}) = D(A) \cap \mathcal{R}_\tau$ and $\tilde{A}z = Az\ \forall z \in D(\tilde{A})$.*

Finally, $\tilde{\Sigma} = (\tilde{T}, \Phi)$ (or $y' = \tilde{A}y + Bu$) is a small-time exact controllable system in $\mathcal{R}_\tau$, i.e. is exactly controllable in $\mathcal{R}_\tau$ for every time $T > 0$.

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Small perturbations

Let $y' = Ay + Bu$ be a STNCLS.
Let $P \in \mathcal{L}(H)$ then $A + P$ generates a C^0 semi-group $\mathbb{T}^P$ on H.

$$y' = (A+P)y + Bu, \text{ state space } H, \text{ control space } U \text{ and input map } \Phi^P$$

.

$$\mathcal{R}^P_{T,y_0} := \{y(T)\ ;\ y' = (A+P)y + Bu,\ y(0) = y_0,\ u \in L^2(0,T;U)\}$$
$$\mathcal{R}^P_{T,y_0} = \mathbb{T}^P_T y_0 + \mathrm{Ran}\,\Phi^P_T.$$

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

For all $\tau > 0$, *there exists* $\varepsilon_\tau > 0$ *such that if* $P \in \mathcal{L}(\mathcal{R}_\tau)$ *with*

$$\|P\|_{\mathcal{L}(\mathcal{R}_\tau)} \leq \varepsilon_\tau,$$

then

$$\mathcal{R}^P_{\tau,0} (= \mathrm{Ran}\ \Phi^P_\tau) = \mathrm{Ran}\ \Phi_\tau = \mathcal{R}.$$

Remarks:

- Here, $\mathcal{R}^P_{T,y_0}$ can depend on T and y_0.
- The smallness assumption on P in $\mathcal{R}_\tau$ crucially depends on τ.

Compact perturbations

Let $y' = Ay + Bu$ be a STNCLS with reachable space $\mathcal{R}$.
Assume that A is self-adjoint, negative and has compact resolvents.
$B \in \mathcal{L}(U, H_{-\alpha})$ for $\alpha \in [0, 1/2]$.
Let $P \in \mathcal{L}(H)$ then $A + P$ generates a C^0 semi-group $\mathbb{T}^P$ on H.
$y' = (A + P)y + Bu$, state space H, control space U and input map Φ^P.

$$\mathcal{R}^P_{T,y_0} := \{y(T) \ ; \ y' = (A+P)y + Bu, \ y(0) = y_0, \ u \in L^2(0, T; U)\}$$
$$\mathcal{R}^P_{T,y_0} = \mathbb{T}^P_T y_0 + \operatorname{Ran} \Phi^P_T.$$

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

Assume that

- $P \in \mathcal{L}(H_{1-\alpha-\varepsilon}, \mathcal{R}) \qquad \varepsilon \in (0, 1-\alpha]$,
- *the pair* $(A + P, B)$ *satisfies the Hautus type condition*

$$\operatorname{Ker}(sI - A - P^*) \cap \operatorname{Ker} B^* = \{0\} \qquad (s \in \mathbb{C}).$$

Then for every $\tau > 0$, $\operatorname{Ran} \Phi^P_\tau = \operatorname{Ran} \Phi_\tau = \mathcal{R}$, *and* $\operatorname{Ran} \mathbb{T}^P_\tau \subset \operatorname{Ran} \Phi^P_\tau$.

Proof: Compactness-uniqueness method.

Linear systems with source terms

Let $y' = Ay + Bu$ be a STNCLS.

Proposition (Ervedoza, Le Balc'h, Tucsnak (2021))

Let $\tau > 0$. There exists a continuous linear map

$$\mathcal{L} : \mathrm{Ran}\,\Phi_\tau \times L^1([0,\tau]; \mathrm{Ran}\,\Phi_\tau) \to L^2([0,\tau]; U)$$

such that for every $\eta \in \mathrm{Ran}\,\Phi_\tau$ and $g \in L^1([0,\tau]; \mathrm{Ran}\,\Phi_\tau)$, for $u = \mathcal{L}(\eta, g)$,

$$\begin{cases} y'(t) = Ay(t) + Bu(t) + g & t \in [0,\tau], \\ y(0) = 0 \end{cases} \Rightarrow y(\tau) = \eta,$$

and

$$\|y\|_{C^0([0,\tau];\mathrm{Ran}\,\Phi_\tau)} + \|u\|_{L^2([0,\tau];U)} \leq C\left(\|\eta\|_{\mathrm{Ran}\,\Phi_\tau} + \|g\|_{L^1([0,\tau];\mathrm{Ran}\,\Phi_\tau)}\right).$$

Semi-linear equations

Let $y' = Ay + Bu$ be a STNCLS.

Corollary (Ervedoza, Le Balc'h, Tucsnak (2021))

Let $\tau > 0$.
$f : C^0([0,\tau]; \mathrm{Ran}\,\Phi_\tau) \to L^1([0,\tau]; \mathrm{Ran}\,\Phi_\tau)$, $f(0) = 0$, *and*
$\forall y_1, y_2 \in C^0([0,\tau]; \mathrm{Ran}\,\Phi_\tau)$,

$$\begin{aligned}&\|f(y_1) - f(y_2)\|_{L^1([0,\tau];\mathrm{Ran}\,\Phi_\tau)}\\ &\qquad\qquad \leq \|y_1 - y_2\|_{C^0([0,\tau];\mathrm{Ran}\,\Phi_\tau)} \left(\varepsilon + C\|(y_1, y_2)\|_{(C^0([0,\tau];\mathrm{Ran}\,\Phi_\tau))^2}\right).\end{aligned}$$

$\exists \delta > 0$, $\forall \eta \in \mathrm{Ran}\,\Phi_\tau$ $\|\eta\|_{\mathrm{Ran}\,\Phi_\tau} \leq \delta$, *there exists* $u \in L^2([0,\tau]; U)$ *such that*

$$\left\{\begin{array}{ll} y'(t) = Ay(t) + Bu(t) + f(y)(t) & t \in [0,\tau], \\ y(0) = 0 & \end{array}\right. \Rightarrow y(\tau) = \eta.$$

Proof: Banach fixed-point argument.

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Framework for the one-dimensional heat equation

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T) \times (0,\pi), \\ \partial_x y(t,0) = u_0(t),\ \partial_x y(t,\pi) = u_\pi(t) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi). \end{cases}$$

- $A = \partial_x^2$ is an unbounded operator on $H = L^2(0,\pi)$, whose domain $D(A) = \{y \in H^2(0,\pi)\ ;\ \partial_x y(0) = \partial_x y(\pi) = 0\}$.
- $B(u_0, u_\pi) = -u_0 \delta_0 + u_\pi \delta_\pi$, $B \in \mathcal{L}(\mathbb{C}^2; D(A)')$.
- $y' = Ay + Bu$ is small-time null-controllable (Fattorini, Russell, 1971).

Characterization of the reachable space

$A = \partial_x^2$ on $H = L^2(0,\pi)$, $D(A) = \{y \in H^2(0,\pi)\ ;\ \partial_x y(0) = \partial_x y(\pi) = 0\}$.
$B(u_0, u_\pi) = -u_0\delta_0 + u_\pi\delta_\pi$.

Theorem (Hartmann, Orsoni (2021))

The reachable space of $y' = Ay + Bu$ is given by

$$\mathcal{R} = A^{1,2}(S),$$

where $S = \{a + ib \in \mathbb{C}\ ;\ |b| < a \text{ and } |b| < \pi - a\}$ and $A^{1,2}(S) = \mathrm{Hol}(S) \cap H^1(S)$.

Several attempts lead to the complete characterization

- Fattorini, Russell (1971).
- Martin, Rosier, Rouchon (2016): $\mathrm{Hol}(B) \subset \mathcal{R} \subset \mathrm{Hol}(S)$ with $S \subset\subset B$.
- Dardé, Ervedoza (2018): $\mathrm{Hol}(S_\varepsilon) \subset \mathcal{R} \subset \mathrm{Hol}(S)$.
- Hartmann, Kellay, Tucsnak (2020): $E^{1,2}(S) \subset \mathcal{R} \subset A^{1,2}(S)$.
- Kellay, Normand, Tucsnak (2020): $\mathcal{R} = A^{1,2}(\Delta) + A^{1,2}(\pi - \Delta)$.
- Orsoni (2020): $\mathcal{R} = A^{1,2}(\Delta) + A^{1,2}(\pi - \Delta)$.
- Hartmann, Orsoni (2021): $A^{1,2}(S) = A^{1,2}(\Delta) + A^{1,2}(\pi - \Delta)$.

First implication: a well-posedness result

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

The heat equation

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T)\times(0,\pi), \\ \partial_x y(t,0) = 0,\ \partial_x y(t,\pi) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi), \end{cases}$$

is well-posed in $A^{1,2}(S)$.

Small regular potentials

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

There exists $\varepsilon > 0$ such that for $p \in \mathrm{Hol}(B) \cap W^{1,\infty}(S)$ with

$$\|p\|_{W^{1,\infty}(S)} \leq \varepsilon, \tag{2}$$

the reachable set of the parabolic equation

$$\begin{cases} \partial_t y - \partial_x^2 y = p(x) y & (0,T) \times (0,\pi), \\ \partial_x y(t,0) = u_0(t),\ \partial_x y(t,\pi) = u_\pi(t) & (0,T), \\ y(0,\cdot) = y_0 & (0,\pi), \end{cases} \tag{3}$$

is independent of $T > 0$, y_0, and coincides with $A^{1,2}(S)$.

To be compared with [Laurent, Rosier (2021)]:

- Allows first order terms without any smallness condition.
- Requires stronger analyticity conditions on the coefficients in y.
- Reachable states in $\mathrm{Hol}(B)$ for $S \subset\subset B$.

Non-local perturbations

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

Let $K = K(x,\xi) \in L^2([0,\pi]^2) \cap L^2_\xi([0,\pi]; W^{1,2}_x(S))$.
Assume the Fattorini-Hautus test: $\forall s \in \mathbb{C}$,

$$\left\{\begin{array}{l} -\psi''(x) - s\psi(x) = \displaystyle\int_0^\pi \overline{K(\xi,x)}\psi(\xi)\,\mathrm{d}y, \quad (x \in [0,\pi]), \\ \psi(0) = \psi'(0) = 0, \\ \psi(\pi) = \psi'(\pi) = 0, \end{array}\right. \qquad \Rightarrow \psi = 0.$$

then the reachable set of the parabolic equation

$$\left\{\begin{array}{ll} \partial_t y - \partial_x^2 y = \int_0^\pi K(x,\xi) y(\xi)\,\mathrm{d}y & \text{in } (0,T)\times(0,\pi), \\ \partial_x y(t,0) = u_0(t),\ \partial_x y(t,\pi) = u_\pi(t) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi), \end{array}\right.$$

is independent of $T > 0$, y_0, *and coincides with* $A^{1,2}(S)$.

The reachable set for smooth controls

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{in } (0,T) \times (0,\pi), \\ \partial_x y(t,0) = u_0(t),\ \partial_x y(t,\pi) = u_\pi(t) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi), \end{cases}$$

for $u_0, u_\pi \in H^1_L(0,T) = \{v \in H^1(0,T)\ ;\ v(0) = 0\}$.
The reachable space is now defined by

$$\mathcal{R}_{T,y_0,L} := \{y(T)\ ;\ y' = Ay + Bu,\ y(0) = y_0,\ u \in H^1_L(0,T;U)\}.$$

Theorem (Kellay, Normand, Tucsnak (2021))

For $T > 0$, $y_0 \in H$,

$$\mathcal{R}_{T,y_0,L} = A^{3,2}(S) = \mathrm{Hol}(S) \cap W^{3,2}(S).$$

Remark: $A^{3,2}(S)$ is an algebra.

Semi-linear equations

Theorem (Ervedoza, Le Balc'h, Tucsnak (2021))

Let $T > 0$ *such that* $a_k(t,x) \in L^1([0,T], A^{3,2}(S))$ *and*

$$\exists \rho > 0, \ \sum_{k=2}^{+\infty} k \left\| a_k \right\|_{L^1([0,T],A^{3,2}(S))} \rho^k < +\infty.$$

Then $\exists \rho > 0$ *s.t.* $\forall \eta \in A^{3,2}(S)$, $\|\eta\|_{A^{3,2}(S)} \leq \delta$, *there exists* $u_0, u_\pi \in L^2([0,\tau])$ *s.t.*

$$\left\{ \begin{array}{ll} \partial_t y - \partial_x^2 y = \sum_{k=2}^{+\infty} a_k(t,x)(y(t,x))^k & \text{in } (0,T) \times (0,\pi), \\ \partial_x y(t,0) = u_0(t), \ \partial_x y(t,\pi) = u_\pi(t) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,\pi), \end{array} \right. \quad \Rightarrow y(T,\cdot) = \eta.$$

To be compared with [Laurent, Rosier (2021)]:

- Handles analytic function in y and $\partial_x y$, but no dependence in time.
- Requires stronger analyticity conditions on the coefficients in y.
- Small reachable states in $\mathrm{Hol}(B)$ for $S \subset\subset B$.

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Conclusion and perspectives

To sum up, for $\Sigma = (\mathbb{T}, \Phi)$ a STNCLS, with reachable space $\mathcal{R}$,

- $\mathbb{T}|_{\mathcal{R}}$ is a C^0 semi-group.
- $\tilde{\Sigma} = (\mathbb{T}|_{\mathcal{R}}, \Phi)$ is a small-time exact controllable system on $\mathcal{R}$.
- Handle small perturbations, compact perturbations and semi-linear equatitons.
- Find sharp results for the one-dimensional heat equation.

One can also obtain new results for

- Heat equations/Parabolic systems with internal controls in N-D case: Fernandez-Cara, Lu, Zuazua (2016), Lissy, Zuazua (2018).
- Heat equations with boundary controls in the multi-D case, based on Strohmaier Water (2021).

An interesting open question is:

Assume further that $\mathbb{T}$ is an analytic semi-group on H, its restriction to $\mathcal{R}$ is an analytic semi-group?