Un résultat de non-contrôlabilité pour l'équation de la demi-chaleur sur la droite réelle. Application à l'équation de Grushin

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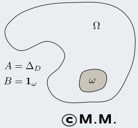
Summary

Introduction

A, B linear operators, y state, u parameter called the control, $T > 0$ final time.

Linear Control System

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \\ y(0) = y^0. \end{cases} \quad \text{(Syst-Cont)}$$



Goal: null controllability at time T

$\forall y^0$, find u such that the solution to (Syst-Cont) satisfies $y(T) = 0$.

Equivalent reformulation : observability in final time

$$\begin{cases} z' = A^* z, \\ z(0) = z_0. \end{cases} \quad \text{(Adj-Eq)}$$

Null-Controllability à 0 for (Syst-Cont)
$\Leftrightarrow$ Observability for (Adj-Eq) :
$\exists C, \forall z_0, ||z(T)||^2 \leqslant C \int_0^T ||B^* z||^2 dt.$

Interlude: an uncertainty principle in harmonic analysis (1)

What is an uncertainty principle?

A mathematical principle that formalizes the fact it is impossible for a non-zero function and its Fourier transform to be simultaneously "concentrated"'.

Here, "concentrated" will be understood in terms of support.

A trivial uncertainty principle

if $f \in L^2(\mathbb{R})$ is such that f and $\widehat{f}$ have compact support, then $f \equiv 0$.

Proof: if f has compact support included in $[-a, a]$, $\widehat{f}(\xi) = \int_{-a}^{a} f(x)e^{-ix\xi}dx$. By usual theorems, $\widehat{f}$ can be extended as an entire function. But $\widehat{f}$ is compactly supported, so $f \equiv 0$.

This can be quantified into an inequality (Amrein-Berthier'77 JFA).

Interlude: an uncertainty principle in harmonic analysis (2)

A spatial set $S \subset \mathbb{R}^d$ and a frequency set $\Sigma \subset \mathbb{R}^d$ is a strong a-pair if there exists $C > 0$ such that for any $f \in L^2(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} |f(x)|^2 dx \leqslant C \left(\int_{\mathbb{R}^n \setminus S} |f(x)|^2 dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(x)|^2 dx \right).$$

From now on, $\Sigma = B(0,r)$ for some $r > 0$. Then, we obtain

$$\int_{\mathbb{R}^d} |f(x)|^2 dx \leqslant C(r) \int_{\mathbb{R}^n \setminus S} |f(x)|^2 dx, \text{ if } \operatorname{supp}(\widehat{f}) \subset B(0,r).$$

Uncertainty principle for band-limited functions. We call this a Logvinenko-Sereda uncertainty principle (LSUP).

Characterization of the admissible sets S given in Paneah'61 DAN ($n = 1$), then Logvinenko-Sereda'74 TFFGAP.

Interlude: an uncertainty principle in harmonic analysis (3)

Characterization of the admissible sets S given in Paneah'61 DAN ($n = 1$), then Logvinenko-Sereda'74 TFFGAP.

Definition

A thick set of $\mathbb{R}^d$ is a mesurable set ω such that there exists $r > 0$ and $\gamma > 0$ such that, for any $x \in \mathbb{R}^d$, we have

$$\lambda(\omega \cap (x + r[-1,1]^d)) \geqslant \gamma\lambda(r[-1,1]^d).$$

We introduce $|\nabla|$ given by: for $h \in H^1(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$,

$$\widehat{|\nabla| h}(\xi) = ||\xi||\widehat{h}(\xi).$$

Definition

A Poisson set is a measurable set Ω such that there exists $t > 0$ and $\tilde{\gamma} > 0$ such that for $x \in \mathbb{R}^d$,

$$\left(e^{-t|\nabla|}\mathbb{1}_\Omega\right)(x) \geqslant \tilde{\gamma}.$$

Interlude: an uncertainty principle in harmonic analysis (4)

The crucial point is

Proposition (Havin'Joricke'92)

A set ω is thick iff it is a Poisson set.

We then have:

Theorem (Logvininko-Sereda'74 TFFGAP)

The set S satisfies a Logvinenko-Sereda uncertainty principle if and only if $\mathbb{R}^d \setminus S$ is a thick set.

We propose a proof here, with an important drawback does not give the optimal behaviour of the constant in the LSUP, even in the simples cases.

Proof of the LSUP for $d = 1$ (1)

Fix any $t > 0$ and consider f such that $\mathrm{supp}(\widehat{f}) \subset (-c, c)$.

Step 1: if $g = e^{icx} f$, then $\mathrm{supp}(\widehat{g}) \subset (0, 2c) \subset \mathbb{R}^+$. So $g \in \mathcal{H}^2(\mathbb{C}^+)$.

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Step 2: One (amongst many) Jensen inequality, for functions in the Hardy class:

$$\log \left| e^{-t|\nabla|} g(x) \right| \leqslant \left(e^{-t|\Delta|} \log |g| \right)(x), \; x + it \in \mathbb{C}^+.$$

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Step 3:

$$\int_{\mathbb{R}} |f|^2 = \int_{\mathbb{R}} |g|^2 = C \int_0^{2c} |\widehat{g}|^2.$$

So

$$\int_{\mathbb{R}} |f|^2 \leqslant C e^{4tc} \int_0^{2c} |e^{-t|\xi|} \widehat{g}(\xi)|^2 d\xi.$$

Proof of the LSUP for $d = 1$ (2)

Step 4: Integrate $\exp(|\text{step } 2|^2)$, use Plancherel and step 3:

$$\int_{\mathbb{R}} |f|^2 \leqslant Ce^{4tc} \int_{\mathbb{R}} \exp\left(e^{-t|\Delta|} \log\left(|g|^2\right)\right).$$

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Step 5: The theorem on two constants of Gorin'85: If μ is a probability measure on $\mathbb{R}$ and Ω is a measurable set such that $\mu(x+\Omega) \geqslant \tilde{\gamma}$, $x \in \mathbb{R}$, then, if $h \in L^1(\mathbb{R})$ with $h \geqslant 0$,

$$\int_{\mathbb{R}} \exp\left(\mu * \log(h)\right) \leqslant 2 \left(\int_{\Omega} h\right)^{\tilde{\gamma}} \left(\int_{\mathbb{R}} h\right)^{1-\tilde{\gamma}}.$$

Proof of the LSUP for $d = 1$ (2)

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Step 6: Apply step 5 with $\mu = \frac{t}{\pi(x^2+t^2)}$, $\Omega = \mathbb{R} \setminus S$, $h = |g|^2$, and remark that $\mu * \log(h) = e^{-t|\nabla|} \log(h)$. So by step 4,

$$\int_{\mathbb{R}} |f|^2 \leqslant 2Ce^{4tc} \left(\int_{\Omega} |f|^2\right)^{\tilde{\gamma}} \left(\int_{\mathbb{R}} |f|^2\right)^{1-\tilde{\gamma}}.$$

Heuristic of the next part

- Link between the half-heat equation and the LSUP.
- Let us go back to the observability inequality

$$\exists C, \forall z_0, ||z(T)||^2 \leqslant C \int_0^T ||B^* z||^2 dt.$$

Non-observability related to "concentration" of solutions outside of the control region ω, if $B = B^* = 1_\omega$.

This it suggests that a way to find counter-example to the controllability for the half-heat equation: take initial conditions that "saturate" the uncertainty principle outside the control domain to disprove observability, hoping that they stay quite "well-concentrated" over time. Here we restrict to $n = 1$.

Here, we will rediscover a result by A. Koenig (PhD thesis) on the half-heat equation, giving a constructive (and easier) proof and also being applicable to the Grushin equation.

Summary

Definition

Let $c > 0$. We introduce

$$F_c : \varphi \in L^2\left(-1,1\right) \mapsto \int_{-1}^{1} \varphi\left(\xi\right) e^{icx\xi} d\xi \in L^2\left(-1,1\right).$$

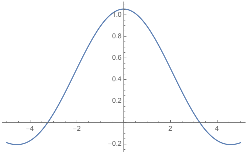
F_c is a compact on $L^2(-1,1)$. λ_c is its largest eigenvalue, and ψ_c "the" first eigenvector of F_c. ψ_c is the first PSWF with parameter c. It verifies

$$\lambda_c \psi_c\left(x\right) = \int_{-1}^{1} e^{icx\xi} \psi_c\left(\xi\right) d\xi.$$

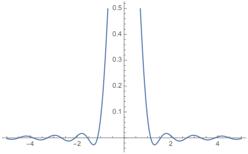
$\psi_c \in L^2(\mathbb{R})$ and is c-band-limited, so it is an entire function of exponential type c. Moreover, for any $x + it \in \mathbb{C}$,

$$\psi_c\left(x+it\right) = \frac{1}{\lambda_c} \int_{-1}^{1} e^{ic(x+it)\xi} \psi_c\left(\xi\right) d\xi. \qquad \text{(PSWF-C)}$$

The shape of the PSWF



(a) The first PSWF ($c = 1$)



(b) The first PSWF ($c = 5$)

Additional properties (1)

- ψ_c, real, even on $\mathbb{R}$, no roots on $(-1,1)$.
- ψ_c, even, hermitian on $\mathbb{C}$.
- Normalization of ψ_c in $L^2(\mathbb{R})-$norm and assume $\psi_c > 0$ in $(-1,1)$.

We can prove

$$\int_{\mathbb{R}\setminus[-1,1]} \psi_c(x)^2\, dx = 1 - \mu_c, \;\; \mu_c = \frac{c}{2\pi}\lambda_c^2$$

with a nice asymptotic behaviour (Fuchs'64, JMAA):

$$1 - \mu_c \sim 4\sqrt{\pi}c^{\frac{1}{2}}e^{-2c} \text{ as } c \to \infty. \qquad \text{(Asy-Mu)}$$

Additional properties (2)

- Any other function f which is c-band-limited and has $L^2(\mathbb{R})$-norm equal to 1 is such that

$$\int_{\mathbb{R}\setminus[-1,1]} f(x)^2 dx \geqslant 1-\mu_c.$$

- ψ_c: the c-band-limited function which concentrates the most on $[-1,1]$.
- $1-\mu_c$: best constant for the LSUP on c- band-limited functions, with $S=\mathbb{R}\setminus[-1,1]$.
- Another estimation (Fuchs'64 JMAA):

$$|\psi_c(1)| \sim 2\pi^{\frac{1}{4}} c^{\frac{3}{4}} e^{-c} \text{ as } c\to\infty. \qquad \text{(Val-1)}$$

The dual nature of the PSWF

The "lucky accident" of Slepian

Let

$$L_c : \varphi \mapsto \left(x \mapsto -\left(1 - x^2\right)\varphi''(x) + 2x\varphi(x) + c^2x^2\varphi(x)\right).$$

F_c and L_c commute. Hence, ψ_c eigenvector of L_c, with associated eigenvalue $\chi_c > 0$ (that is $\sim c$ as $c \to \infty$).

By analyticity, $\forall z \in \mathbb{C}$,

$$-\left(1 - z\right)^2 \psi_c''(z) + 2z\psi_c'(z) + c^2z^2\psi_c(z) = \chi_c\psi_c(z). \qquad \text{(EDO-C)}$$

Absolutely crucial for our proof.
PSWF: introduced in the pionner works of Slepian-Landau-Pollak in the 60'ies. Then many developments, extensions, applications. See also the book by Osipov-Rokhlin-Xiao'13.

Summary

The half-heat equation

Let $T > 0$, ω open subset of $\mathbb{R}$ whose exterior contains an interval.

The half-heat equation

$$\begin{cases} \partial_t y(t,x) + |\nabla| y(t,x) = 1_\omega v(t,x) \text{ in } (0,T) \times \mathbb{R}, \\ y(0,x) = y^0(x). \end{cases} \qquad \text{(Half-H)}$$

We have well-posedness in L^2.

Theorem (Lissy'20)

System (Half-H) *is not null-controllable: for any $T > 0$, there exists at least one initial condition $y^0 \in L^2(\mathbb{R})$ such that there exists no $u \in L^2((0,T) \times \mathbb{R})$ such that the solution y of* (Half-H) *verifies $y(T) = 0$.*

Reduction to observability

By duality, we study

$$\begin{cases} \partial_t u\,(t,x) + |\nabla| u\,(t,x) = 0 \text{ in } (0,T)\times\mathbb{R}, \\ u\,(0,x) = u^0\,(x) \in L^2\,(\mathbb{R})\,. \end{cases} \qquad \text{(Half-H-Adj)}$$

In order to prove our Theorem, it is sufficient to exhibit a family of initial conditions $u^0(c)$ depending on some parameter $c > 0$, look at the solution u_c to (Half-H-Adj) and make to quotient

$$Q(c) = \frac{\int_0^T ||u_c\,(t,\cdot)\,||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\,dt}{||u_c\,(T,\cdot)\,||^2_{L^2(\mathbb{R})}}.$$

Goal

$Q(c) \to 0$ as $c \to \infty$.

Some basic facts and reductions

- Let $t \geqslant 0$. If $u^0(x) = \psi_c(x) e^{icx}$, then the solution u is

$$u(t,x) = e^{ic(x+it)}\psi_c(x+it), \; t \in \mathbb{R}^+, \; x \in \mathbb{R}. \qquad \text{(u-Half)}$$

  Proof : direct computation by Fourier transform, or remark that on the Hardy class $H^2(\mathbb{C}^+)$, (Half-H-Adj) is just the Cauchy-Riemann relation.

- Non-controllability on $\tilde{\omega} \supset \omega \Rightarrow$ non-controllability on ω.
- Here, $B(x_0, \varepsilon) \subset \mathbb{R} \setminus \overline{\omega}$ for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}$.

After translation and dilatation arguments, we reduce to the following
Goal: non-observability on $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$.

A very general estimate on complex second order ODEs

Proposition

Let $a < b$. Consider $w, u, \beta, \gamma : [a, b] \to \mathbb{C}$ of class C^1, verifying: $\forall t \in [a, b]$,

$$\begin{pmatrix} w'(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(t) \\ \gamma(t) & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}.$$

Assume β and γ do not vanish. Introduce

$$R(t) = \frac{|\beta(t)|}{|\gamma(t)|} \text{ and } Q(t) = |w(t)|^2 + R(t)|w'(t)|^2.$$

Then, for any $a \leqslant t_0 \leqslant t_1 \leqslant b$, we have

$$\frac{\sqrt{Q(t_0)}}{R(t_0)^{\frac{1}{4}}} \leqslant \frac{\sqrt{Q(t_1)}}{R(t_1)^{\frac{1}{4}}} \exp\left(\int_{t_0}^{t_1} \sqrt{\left(\frac{R'(s)}{4R(s)}\right)^2 + \frac{|\beta(s)\gamma(s)| + Re(\beta(s)\gamma(s))}{2}} ds \right).$$

An important corollary for the PSWF in the upper half plane

Proposition

There exists $C \geqslant 1$ such that for any $x \geqslant 2$, $t \geqslant 0$ and $c > 0$ large enough (independently of x or t), we have

$$|\psi_c(x+it)| \leqslant Ce^{ct} \frac{\psi_c(1)}{c\lambda_c\sqrt{x^2+t^2}} \exp\left(\frac{Cct}{x^2+t^2}\right).$$

Proof : For $x+it \in \mathbb{C}$, with $x \geqslant 2$ and $t \geqslant 0$, we introduce

$$\phi_c(x+it) = \psi_c(x+it)\sqrt{(x+it)^2-1}.$$

Then by (EDO-C), ϕ_c verifies

$$\phi_c''(x+it) + \left(\frac{c^2(x+it)^2-\chi_c}{(x+it)^2-1} + \frac{1}{((x+it)^2-1)^2}\right)\phi_c(x+it) = 0.$$

Then apply the previous proposition to $w(x) = \phi_c(x+it)$ and $u(x) = \phi_c'(x+it)$ and make direct computations.

A L^2 estimate and choice of the initial condition

Corollary

There exists $C \geqslant 1$ such that for any $c > 0$ large enough and any $t \geqslant 0$, we have

$$\left(\int_{\mathbb{R}\setminus[-\sqrt{c},\sqrt{c}]} |\psi_c(x+it)|^2 dx\right) e^{-2ct} \leqslant C\frac{(1-\mu_c)}{\sqrt{c}+t}.$$

- If u is a solution of (Half-H-Adj), $u(\alpha\cdot,\alpha\cdot)$ is still a solution.
- Hence, we choose as an initial condition

$$u^0(x) = e^{i\frac{c^{3/2}}{\varepsilon}x}\psi_c\left(\frac{\sqrt{c}}{\varepsilon}x\right)$$

.

- The corresponding solution of (Half-H-Adj) is

$$u(t,x) = e^{i\frac{c^{3/2}}{\varepsilon}(x+it)}\psi_c\left(\frac{\sqrt{c}}{\varepsilon}(x+it)\right).$$

Estimations on u

Proposition

There exists $C(\varepsilon) > 0$, depending only on ε, such that for any $t \geqslant 0$ and any c large enough (independently on t, but possibly depending on ε),

$$||u_{c,\varepsilon}(t,\cdot)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])} \leqslant C(\varepsilon)\frac{(1-\mu_c)}{c(1+t)}. \qquad \text{(Maj-Num)}$$

This is just an immediate consequence of the previous result.

Proposition

There exist $K(T,\varepsilon) > 0$ and $C(T,\varepsilon) > 0$ (depending only on T and ε) such that for any $c \geqslant K(T,\varepsilon)$, we have

$$\int_{\mathbb{R}} |u(T,x)|^2 dx \geqslant C(T,\varepsilon)\frac{|\psi_c(1)|^2}{c^{\frac{3}{2}}}. \qquad \text{(Min-Den)}$$

Proof : Plancherel, localize the integral around an adequate point, and direct computations involving the ODE.

End of the proof

By the previous estimations (Maj-Num) and (Min-Den),

$$Q(c,\varepsilon) \leqslant C'(T,\varepsilon)\,\frac{\sqrt{c}\,(1-\mu_c)}{|\psi_c(1)|^2},$$

for $C'(T,\varepsilon)$ depending only on T and ε. Hence, using (Asy-Mu) together with (Val-1), we deduce that for $c \geqslant K(T,\varepsilon)$ and some constant $C''(T,\varepsilon)$ depending only on T and ε,

$$Q(c,\varepsilon) \leqslant 2\sqrt{c}\,C'(T,\varepsilon)\,\frac{4\sqrt{\pi}\,c^{\frac{1}{2}}\,e^{-2c\,T}}{4\pi^{\frac{1}{2}}\,c^{\frac{3}{2}}\,e^{-2c}} \leqslant \frac{C''(T,\varepsilon)}{\sqrt{c}}.$$

This ends the proof of the non-observability result by making c go to ∞.

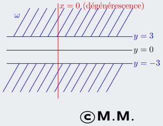
Summary

The Grushin equation

Control domain $\omega = \mathbb{R} \times (\mathbb{R} \setminus [-\varepsilon, \varepsilon])$. f^0 and g in L^2.

The Grushin equation

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - x^2 \partial_{yy}^2 f = g 1_\omega, \\ f(0, \cdot) = f^0(\cdot). \end{cases} \qquad \text{(Grushin)}$$



Theorem

System (Grushin) *is not null-controllable for no time* $T > 0$.

First studied Beauchard-Cannarsa-Guglielmi'13 JEMS. Then studied by many authors (Allonsius, Boyer, Dardé, Duprez, Ervedoza, Koenig, Miller, Morancey,...). Koenig'17 CRAS: (on $(-1, 1) \times (0, 1)$, non-controllability from outside of horizontal strips. Novelty here : $y \in \mathbb{R}$.

Duality

We consider the adjoint equation

$$\begin{cases} \partial_t v\,(t,x,y) - \partial_{xx}^2 v(t,x,y) - x^2\partial_{yy}^2 v(t,x,y) = 0 \text{ in } (0,T)\times\mathbb{R}^2, \\ v\,(0,x,y) = v^0\,(x,y) \in L^2\left(\mathbb{R}^2\right). \end{cases} \tag{Gru-Adj}$$

If we consider the Fourier transform in the second variable,

$$\begin{cases} \partial_t \widehat{v}\,(t,x,\xi) - \partial_{xx}^2 \widehat{v}(t,x,\xi) + x^2|\xi|^2 v(t,x,y) = 0 \text{ in } (0,T)\times\Omega, \\ \widehat{v}\,(0,x,\xi) = \widehat{v^0}\,(x,\xi) \in L^2\,(\Omega)\,. \end{cases}$$

The elliptic operator $-\partial_{xx}^2 + x^2|\xi|^2$ is exactly the harmonic oscillator.
The first eigenvector is $x \mapsto e^{-\frac{|\xi|x^2}{2}}$.

The initial condition

We consider

$$\widehat{v^0}(x,\xi) = \frac{2\pi}{c\lambda_c} e^{-\frac{x^2|\xi|}{2}} \psi_c\left(\frac{\xi - c}{c}\right) 1_{[-c,c]}(\xi - c).$$

Then, the solution of (Gru-Adj) can be expressed thanks to the solution of (Half-H-Adj):

$$v(t,x,y) = u\left(t + \frac{x^2}{2}, y\right).$$

We introduce the quotient

$$Q' = \frac{\int_0^T \int_{\mathbb{R}} \int_{(\mathbb{R}\setminus[-\varepsilon,\varepsilon])} |v(t,x,y)|^2 dydxdt}{\int_{\mathbb{R}} \int_{\mathbb{R}} |v(T,\cdot)|^2 dydx}$$

and show totally similarly that it goes to 0 as $c \to +\infty$, at rate $1/\sqrt{c}$.

Conclusion

Short-term extensions

- Multi-D for the half-heat equation: outside of a ball of $\mathbb{R}^n$, use of prolate with radial symmetry? Work "in progress".
- For Grushin, study the case $(-1, 1) \times \mathbb{R}$ ($\mathbb{T} \times \mathbb{R}$ should be OK).
- Characterization of the initial condition that cannot be brought to 0?

Long-term goals

Hope that it gives a general method to prove non-controllability in weak diffusivity case: consider functions that saturate some uncertainty principle. Generalizations in other geometries : how to get rid of the Fourier transform ?

Reference

Reference

Pierre Lissy, *A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation*, en révision, disponible sur HAL.

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Merci pour votre attention!