Controllability of the 1D bilinear Schrödinger equation by a power series expansion

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ANR TRECOS

E-STLC in finite dimension

$$\begin{cases} \frac{dx}{dt} = f(x, u), \quad t \in (0, T) \\ x(0) = x_0, \end{cases}$$

where, at time t,

- $x(t) \in \mathbb{R}^n$: **state** of this system,
- $u(t) \in \mathbb{R}$: **control**.

Definition (STLC)

The system is **STLC** if

$$\forall T > 0, \quad \forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall\ |x_0| + |x_f| < \delta,$$
$$\exists u \in L^\infty(0, T) \text{ with } \|u\|_{L^\infty(0,T)} < \varepsilon \quad \text{s. t.} \quad x(T;\ u,\ x_0) = x_f.$$

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Definition (E-STLC)

Let $(E_T, \| \cdot \|_{E_T})$ be a family of normed vector spaces of real functions defined on $[0, T]$ (ex: $E_T = L^\infty(0, T), H^1(0, T)$). The system is **E-STLC** if

$$\forall T > 0, \quad \forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall\ |x_0| + |x_f| < \delta,$$
$$\exists u \in E_T \text{ with } \|u\|_{E_T} < \varepsilon \quad \text{s. t.} \quad x(T;\ u,\ x_0) = x_f.$$

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Consider a control-affine system $\dot{x} = f_0(x) + uf_1(x)$.
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- If $\mathcal{S}_{2k} \subset S_{2k-1} := \{W(0);\ W$ bracket with $2k-1$ times $f_1\}$ for all $k \in \mathbb{N}^*$; [Hermes, Sussmann]

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- If $\mathcal{S}_{2k} \subset S_{2k-1} := \{W(0);\ W \text{ bracket with } 2k-1 \text{ times } f_1\}$ for all $k \in \mathbb{N}^*$; [Hermes, Sussmann]
- If there exists $\theta \in [0, 1]$ such that every bracket involving f_0 an **odd** number l of times and f_1 an **even** number k of times is a linear combination of brackets involving k_i times f_1 and l_i times f_0 with $\mathbf{k_i} + \theta \mathbf{l_i} < \mathbf{2} + \theta \mathbf{l}$. [Sussmann $\mathcal{S}(\theta)$ condition]

Susmann's example

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases}$$

This is a **control-affine** system of the form

$$\dot{x} = f_0(x) + uf_1(x)$$

with the vector fields

$$f_0(x) = (0, x_1, x_1^3 + x_2^2)^{\mathrm{tr}} \quad \text{and} \quad f_1(x) = (1, 0, 0).$$

Question

E-STLC for which E ? (At least $E = L^\infty$ by Sussmann's $\mathcal{S}(\theta)$ condition.)

Sussmann's example: Quadratic wins over cubic

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases} \qquad \begin{cases} x_1 = u_1, \\ x_2 = u_2, \\ x_3(T) = \int_0^T u_1(t)^3 dt + \int_0^T u_2(t)^2 dt. \end{cases}$$

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- The **quadratic** term **wins** when $\|u'\|_{L^\infty(0,T)} \to 0$:

$$\int_0^T u_1(t)^3 dt = -\int_0^T u_2(t) 2u(t) u_1(t) dt = \int_0^T u_2(t)^2 \mathbf{u}'(\mathbf{t}) dt$$

Then, when $(T, \|u'\|_{L^\infty(0,T)}) \to 0$,

$$x_3(T) \geqslant \left(1 - \|u'\|_{L^\infty(0,T)}\right) \int_0^T u_2(t)^2 dt > 0$$

$\rightsquigarrow$ No $W^{1,\infty}$-STLC because the **quadratic** term entails a **drift**

Sussmann's example: Cubic wins over quadratic

- The **cubic** term **wins** for controls of the form:

$$u_\lambda(t) = \lambda^{\frac{3}{4}} \phi''\left(\frac{t}{\lambda}\right), \quad \lambda \to 0.$$

Size of the controls:

$$\|u_\lambda\|_{H^1(0,T)} \approx \lambda^{\frac{1}{4}} \ll 1 \quad \text{but} \quad \|u'_\lambda\|_{L^\infty(0,T)} \approx \lambda^{-\frac{1}{4}} \gg 1.$$

Computation of the solution

$$\begin{aligned} x_3(T) &= \int_0^T u_1(t)^3 dt + \int_0^T u_2(t)^2 dt \\ &= \lambda^{\frac{11}{2}} \int_0^1 \phi'(\theta)^3 d\theta + \lambda^6 \int_0^1 \phi(\theta)^2 d\theta \\ &= a + o(a). \end{aligned}$$

$\rightsquigarrow$ H^1-STLC because the **cubic** term **absorbs the drift** for controls small in less regular spaces

Sussmann's example

Controllability of the following control-affine system,

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 + x_2^2. \end{cases}$$

Theorem

- *The system is **not** $W^{1,\infty}$-**STLC** (≈ Hermes condition) [Beauchard, Marbach - 2018].*
- *But the system is H^1-**STLC** (≈ Sussmann's $\mathcal{S}(\theta)$ condition).*

Question

Same phenomenon for a control-affine system in **infinite dimension** ?

Schrödinger equation

$$\begin{cases} i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T). \end{cases}$$

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Bilinear control system

- the **state**: ψ, such that $\|\psi(t)\|_{L^2(0,1)} = 1$ for all time,
- $\mu : (0,1) \to \mathbb{R}$ **dipolar moment** of the quantum particle,
- and $u : (0,T) \to \mathbb{R}$ denotes a scalar **control**.

Equation under study

To do as in finite dimension:

$$f_0(\varphi) = -\varphi'' \quad \text{with} \quad \mathrm{Dom}(f_0) = H^2 \cap H^1_0(0,1).$$

Orthonormal basis of $L^2(0,1)$ of **eigenvectors**:

$$\forall j \in \mathbb{N}^*, \quad \varphi_j := \sqrt{2}\sin(j\pi\cdot) \quad \text{associated with} \quad \lambda_j := (j\pi)^2.$$

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Definition

Let $(E_T, \|\cdot\|_{E_T})$ be a family of normed vector spaces of real functions defined on $[0,T]$ and X a vector space of functions defined on $[0,1]$. The Schrödinger equation is said to be **E-STLC around the ground state with targets in** X if:

$$\forall T > 0, \quad \forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall (\psi_*, \psi_f) \in X \text{ with}$$

$$\|\psi_* - \varphi_1\|_X < \delta \text{ and } \|\psi_f - \varphi_1 e^{-i\lambda_1 T}\|_X < \delta,$$

$$\exists u \in L^2(0,T) \cap E_T \text{ with } \|u\|_{E_T} < \varepsilon \quad \text{s. t.} \quad \psi(T;\ u,\ \psi_*) = \psi_f.$$

State of the art

Theorem (Ball, Marsden, Slemrod - 1982 & Turinici - 2000)

When μ is in $W^{2,\infty}$, the Schrödinger equation is ***not controllable*** *in $H^2 \cap H^1_0(0,1)$ with controls in $L^r_{\mathrm{loc}}((0,+\infty),\mathbb{R})$ for $r > 1$.*

$\rightsquigarrow$ *Bad choice of functional settings*

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Theorem (Beauchard, Laurent - 2010)

When μ is in $H^3((0,1),\mathbb{R})$ such that there exists a constant $c > 0$ such that

$$\forall j \in \mathbb{N}^*, \quad |\langle \mu\varphi_1, \varphi_j\rangle| \geq \frac{c}{j^3},$$

*the Schrödinger equation is L^2-**STLC** with targets in $H^3_{(0)}(0,1)$.*

$\rightsquigarrow$ *Choice of μ such that the linearized system is controllable*

Bibliography

Theorem (Beauchard, Morancey - 2014)

When μ is in $H^3((0,1),\mathbb{R})$ such that

$$\langle \mu\varphi_1, \varphi_K\rangle = 0 \quad \text{and} \quad A_K^1(\mu) := \langle {\mu'}^2 \varphi_1, \varphi_K\rangle \neq 0,$$

*the Schrödinger equation is **not** L^2-**STLC** due to a drift quantified by the H^{-1}-norm of the control.*

$\rightsquigarrow$ *Choice of μ s. t.* $\langle \psi(T) - \psi_1(T), \varphi_K\rangle = A_K^1(\mu)\int_0^T u_1(t)^2 dt + \dots$

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Question

If $A_K^1(\mu) = 0$, what happens ?

Main result

1. **Choice** of μ such that there exists $K \in \mathbb{N}^*$,

$$\langle \psi(T) - \psi_1(T), \varphi_K \rangle \approx A_K^3(\mu) \int_0^T u_3(t)^2 dt \; + \; C_K(\mu) \int_0^T u_1(t)^2 u_2(t) dt \; + \; \text{error terms}$$

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2. Study of the **quadratic/cubic competition**
   - In an asymptotic $(T, \|u\|_{H^3}) \to 0$, one has $(\text{Cub}) = o(\text{Quad})$.
   - "In an asymptotic $(T, \|u\|_{H^2}) \to 0$", one has $(\text{Quad}) = o(\text{Cub})$.

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   - "In an asymptotic $(T, \|u\|_{H^2}) \to 0$", one has (Quad) $= o$(Cub).

Theorem (B., 2022)

There exists a choice of μ such that the Schrödinger equation

- *is **not** H^3-**STLC** because of a **drift** quantified by the H^{-3}-norm of the control,*
- *but is H^2-**STLC** thanks to the **cubic term**.*

Computation of the expansion: The linear term

$$i\partial_t \psi = -\partial_x^2 \psi - u\mu\psi \qquad \text{(Lin)} \;\; i\partial_t \Psi = -\partial_x^2 \Psi - u\mu\varphi_1 e^{-i\lambda_1 t}$$

Explicit resolution:

$$\Psi(T) = i \sum_{j=1}^{+\infty} \left(\langle \mu\varphi_1, \varphi_j \rangle \int_0^T u(t) e^{i(\lambda_j - \lambda_1)t} dt \right) \varphi_j e^{-i\lambda_j T}.$$

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- If $\langle \mu\varphi_1, \varphi_K \rangle = 0$ then

$$\langle \Psi(t), \varphi_K \rangle \equiv 0.$$

$\rightsquigarrow$ *Go further into the expansion*

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- If $\langle \mu\varphi_1, \varphi_K \rangle = 0$ then

$$\langle \Psi(t), \varphi_K \rangle \equiv 0.$$

$\rightsquigarrow$ *Go further into the expansion*

- If for all $j \in \mathbb{N}^*$, $\langle \mu\varphi_1, \varphi_j \rangle \neq 0$, then $\Psi(T) = \psi_f$ is equivalent to

$$\forall j \in \mathbb{N}^*, \quad \int_0^T u(t) e^{i(\lambda_j - \lambda_1)t} dt = -i \frac{\langle \psi_f, \varphi_j \rangle}{\langle \mu\varphi_1, \varphi_j \rangle} e^{i\lambda_j T}.$$

$\rightsquigarrow$ *Solvability of a moment problem*

Computation of the expansion: The quadratic term

$$i\partial_t \psi = -\partial_x^2 \psi - u\mu\psi \qquad \text{(Quad) } i\partial_t \xi = -\partial_x^2 \xi - u\mu\Psi$$

Explicit computations:

$$\langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle = \int_0^T u(t) \int_0^t u(\tau) h(t,\tau) d\tau dt.$$

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Lemma (Coercivity of the quadratic term)

One can ***choose*** μ *such that there exists* $T^* > 0$ *such that for all* $T \in (0, T^*)$ *and for all* $u \in L^2(0,T)$,

$$-\operatorname{sign}(A_K^3) \langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle \geq \frac{|A_K^3|}{4} \int_0^T u_3(t)^2 dt.$$

$\rightsquigarrow$ *Integrations by parts*

Computation of the expansion: The quadratic term

Sketch of the proof.

$$\langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle = \int_0^T u(t) \int_0^t u(\tau) h(t,\tau) d\tau dt$$

$$= -iA_K^1(\mu) \int_0^T u_1(t)^2 e^{i(\lambda_K - \lambda_1)(t-T)} dt + \int_0^T u_1(t) \int_0^t u_1(\tau) \partial_1 \partial_2 h(t,\tau) d\tau dt$$

$+$(boundary terms)

$$= -i \sum_{p=1}^{3} A_K^p(\mu) \int_0^T u_p(t)^2 e^{i(\lambda_K - \lambda_1)(t-T)} dt$$

$$+ \int_0^T u_3(t) \int_0^t u_3(\tau) \partial_1^3 \partial_2^3 h(t,\tau) d\tau dt$$ $+$(boundary terms)

And so on.... $\square$

Computation of the expansion: The cubic term

$$i\partial_t \psi = -\partial_x^2 \psi - u\mu\psi \qquad \text{(Cub) } i\partial_t \zeta = -\partial_x^2 \zeta - u\mu\xi$$

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Goal (Behavior of the cubic term)

One can **choose** μ such that

$$\langle \zeta(T), \varphi_K e^{-i\lambda_1 T} \rangle \approx C_K(\mu) \int_0^T u_1(t)^2 u_2(t) dt.$$

$\rightsquigarrow$ *Okay for oscillating controls*

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Lemma

When $(T, \|u_1\|_{L^\infty}) \to 0$, *the cubic remainder is estimated by*

$$\langle (\psi - \psi_1 - \Psi - \xi)(T), \varphi_K e^{-i\lambda_1 T} \rangle = \mathcal{O}\left(\|u_1\|^3_{L^2(0,T)} \right).$$

Quadratic obstruction: No H^3-STLC

1. The **quadratic** term has a **coercivity**:

$$\langle \xi(T), \varphi_K e^{-i\lambda_1 T} \rangle \geq A_K^3 \int_0^T u_3(t)^2 dt.$$

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2. Estimate of the cubic remainder: When $(T, \|u_1\|_{L^\infty}) \to 0$,

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$$\langle (\psi - \psi_1 - \Psi - \xi)(T), \varphi_K e^{-i\lambda_1 T} \rangle = \mathcal{O}\left(\|u_1\|_{L^2(0,T)}^3 \right).$$

3. The **quadratic** term **prevails**: for $u \in H^3(0, T)$,

$$\|u_1\|_{L^2(0,T)}^3 \leq C \left(\|u^{(3)}\|_{L^2(0,T)} + T^3 \|u\|_{L^2(0,T)} \right) \|u_3\|_{L^2(0,T)}^2.$$

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4. The **nonlinear** solution has a **coercivity**: for $u \in H^3(0,T)$ sufficiently small,

$$\langle \psi(T), \varphi_K e^{-i\lambda_1 T} \rangle \geq A \int_0^T u_3(t)^2 dt.$$

STLC result despite the drift: H^2-STLC

We assume that the space of the **targets** is sliced in **two**:

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- The subspace spanned by the **lost direction** $\varphi_{\mathbf{K}}$ at the linear level.

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- The subspace spanned by the **lost direction** $\varphi_{\mathbf{K}}$ at the linear level.
- The subspace spanned by every other components $(\varphi_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^*-\{\mathbf{K}\}}$, **controllable** at the linear level.

Theorem

For every $\psi_0, \psi_f \in H^{11}_{(0)}(0,1)$ 'small', there exists $v \in H^2_0(0,T)$ such that

$$\forall j \in \mathbb{N}^* - \{K\}, \quad \langle \psi(T;\ v,\ \psi_0), \varphi_j \rangle = \langle \psi_f, \varphi_j \rangle,$$

with the following estimate

$$\|v\|_{H^2_0(0,T)} \leq C \left(\|\psi_0 - \varphi_1\|_{H^{11}_{(0)}(0,1)} + \|\psi_f - \varphi_1 e^{-i\lambda_1 T}\|_{H^{11}_{(0)}(0,1)} \right).$$

STLC result despite the drift: H^2-STLC

1. There exists a family of controls $(u_b)_{b\in\mathbb{R}}$ small in $H^2_0(0,T)$ such that
$$\langle \psi(T;\ u_b,\ \varphi_1), \varphi_K\rangle = b + o(b).$$
$\rightsquigarrow$ *Use the cubic term*

2. ⚠ There exists a family of controls $(v_b)_{b\in\mathbb{R}}$ small in $H^2_0(T, 2T)$ such that
$$\psi(2T;\ v_b,\ \varphi_1) = b\varphi_K + o(b),$$

$$|\langle \psi(2T;\ v_b) - \psi(T;\ u_b), \varphi_K\rangle| \leq C\|\mathbf{v}\|^2_{\mathbf{L^2(0,T)}},$$
$$\|\mathbf{v}\|_{\mathbf{H^2_0(0,T)}} \leq C\|\psi(T;\ u_b,\ \varphi_1) - \varphi_1 e^{-i\lambda_1 T}\|_{H^{11}_{(0)}(0,1)}.$$

$\rightsquigarrow$ Not working...

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$$\psi(2T;\ v_b,\ \varphi_1) = b\varphi_K + o(b),$$

$$|\langle \psi(2T;\ v_b) - \psi(T;\ u_b), \varphi_K\rangle| \leq C\|\mathbf{v_3}\|^{\mathbf{2}}_{\mathbf{L^2(0,T)}},$$
$$\|\mathbf{v}\|_{\mathbf{H^2_0(0,T)}} \leq C\|\psi(T;\ u_b,\ \varphi_1) - \varphi_1 e^{-i\lambda_1 T}\|_{H^{11}_{(0)}(0,1)}.$$

$\rightsquigarrow$ Not working...

STLC result despite the drift: H^2-STLC

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$\rightsquigarrow$ *Use the cubic term*

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$\rightsquigarrow$ Working!

STLC result despite the drift: H^2-STLC

3. ⚠ There exists two families of controls $(v_b^{\Re})_{b\in\mathbb{R}}$ and $(v_b^{\Im})_{b\in\mathbb{R}}$ small in $H_0^2(T,2T)$ such that

$$\psi(2T;\ v_b^{\Re},\ \varphi_1) = b\varphi_K + o(b),$$
$$\psi(2T;\ v_b^{\Im},\ \varphi_1) = ib\varphi_K + o(b).$$

4. For all target $\psi_f \in H^{11}_{(0)}(0,1)$ 'small', there exists $w \in H_0^2((0,T),\mathbb{R})$ arbitrary small in $H^2(0,2T)$ such that

$$\psi(2T;\ w,\ \varphi_1) = \psi_f.$$

$\rightsquigarrow$ *Brouwer fixed point theorem*

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Let $\dot{x} = f(x, u)$ a finite or infinite dimensional control system.

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$$x(T;\ u_b^i,\ 0) = b\xi_i + \mathcal{O}(|b|^{1+s_i}) \quad \text{with} \quad \|u_b^i\|_{E_T} = \mathcal{O}(|b|^{s_i}).$$

Then, the system is E-**STLC**.

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Then, the system is E-**STLC**.

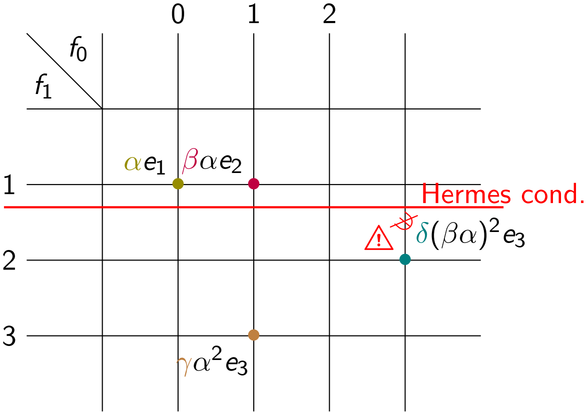
Open questions

Application to other equations (KdV)? Recovering an infinite number of lost directions?

Thanks for your attention!

Sussmann's example: The Lie Brackets

$$\begin{cases} \dot{x}_1 = \alpha u, \\ \dot{x}_2 = \beta x_1, \\ \dot{x}_3 = \gamma x_1^3 + \delta x_2^2. \end{cases}$$



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