Cumulative structure for Hawkes processes

Laetitia Colombani,
under the direction of Manon Costa and Patrick Cattiaux

IMT

2nd June 2022,
Workshop New trends on Hawkes processes, Toulouse

INSTITUT de MATHEMATIQUES de TOULOUSE

Definition

Let $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function.
A Hawkes process N^h is a self-influencing point process whose intensity is given at each time $t \geq 0$ by:

$$\Lambda^h(t) = \Phi\left(\int_{(-\infty,t)} h(t-u)N^h(du)\right) = \Phi\left(\sum_{i\geq 1} h(t-U_i)\right)$$

where $\Phi : \mathbb{R} \to \mathbb{R}^+$, and U_i are the jumps of N^h.

Definition

Linear process

If Φ is linear or affine ($\lambda \geq 0$), and h is positive:

$$\Lambda^h(t) = \lambda + \int_{(-\infty,t)} h(t-u)N^h(du).$$

Definition

Linear process

If Φ is linear or affine ($\lambda \geq 0$), and h is positive:

$$\Lambda^h(t) = \lambda + \int_{(-\infty,t)} h(t-u) N^h(du).$$

Framework on this work

$\Phi(x) = \max(0, \lambda + x)$ and signed h:

$$\Lambda^h(t) = \left(\lambda + \int_{(-\infty,t)} h(t-u) N^h(du)\right)^+.$$

Construction from an EDS

Proposition

Let Q a $(\mathcal{F}_t)_{t\geq 0}$-Poisson point process on $(0,+\infty)^2$ with unit intensity. Let $\lambda > 0$ and $h:(0,\infty)\to\mathbb{R}$ a signed reproduction function with $\|h^+\|_1 < 1$. Then, there exists a unique strong solution of:

$$\begin{cases} N^h = \int \delta_u \mathbb{1}_{\theta \leq \Lambda^h(u)} Q(du, d\theta) \\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s) N^h(du)\right)^+, \ u > 0, \end{cases}$$

and this solution is a Hawkes process.

Remarks

Remark

This proposition provides a way of coupling Hawkes processes.

Remarks

Remark

This proposition provides a way of coupling Hawkes processes.

Proposition

In the sense of measure, $N^h \leq N^{h^+}$.

Remarks

Remark

This proposition provides a way of coupling Hawkes processes.

Proposition

In the sense of measure, $N^h \leq N^{h^+}$.

Example

If $h \leq 0$, then $h^+ = 0$.
Eventually: N^h is lower than a Poisson process of parameter λ.

Assumptions

Object of interest: number of jumps on an interval $[0, t]$: $N^h([0, t]) = N^h_t$, when t tends to ∞.

First assumptions

- $\| h^+ \|_1 < 1$
- Empty initial condition: $N^h(]-\infty, 0]) = 0$
- h has a compact support, included in $[0, L(h)]$

Literature

Linear case: h non-negative, we assume $\| h \|_{L^1(du)} < 1$.
LLN[1]

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\lambda}{1 - \| h \|_{L^1(du)}} := \mu \quad ,$$

and CLT

$$\frac{N_t^h - \mu t}{\sqrt{t}} \xrightarrow[t\to\infty]{law} \mathcal{N}^h\left(0, \frac{\lambda}{(1 - \| h \|_{L^1(du)})^3}\right) .$$

.

[1] Daryl J. Daley and David Vere-Jones. *An Introduction to the Theory of Point Processes*. 2nd ed. Probability and Its Applications. New York: Springer-Verlag, 2003. 471 pp.

Literature

Linear case: h non-negative, we assume $\| h \|_{L^1(du)} < 1$.
Bordenave and Torrisi[1] obtain a Large Deviation Principle for N_t^h/t, when $h \geq 0$ with rate function

$$I(x) = x \ln\left(\frac{x}{\lambda + x \| h \|_{L^1(du)}}\right) - x(1 - \| h \|_{L^1(du)}) + \lambda\,.$$

(The explicit expression is obtained by Zhu[2] p.761.)

[1] Charles Bordenave and Giovanni Luca Torrisi. "Large Deviations of Poisson Cluster Processes". In: *Stochastic Models* 23.4 (2007), pp. 593–625.

[2] Lingjiong Zhu. "Central Limit Theorem for NonLinear Hawkes Processes". In: *Journal of Applied Probability* 50.3 (2013), pp. 760–771. JSTOR: 43283499.

1. Hawkes processes
   - Definition
   - Construction of Hawkes
2. Cumulative processes
   - Framework
   - Definition
   - Results
3. Link between Hawkes and Cumulative processes
   - Link
   - Results for Hawkes processes
4. Conclusion

Cumulative process (from Asmussen[3])

Definition from Asmussen

Let $(\tau_i, W_i)_i$ i.i.d. couples of random variable.
Let M_t the counting process associated with $(\tau_i)_i$:

$$M_t = \sup_{n\in\mathbb{N}} \left\{ \sum_{i=1}^{n} \tau_i \leq t \right\}.$$

The *cumulative process* associated with $(\tau_i, W_i)_i$ is

$$Z_t = \sum_{i=1}^{M_t} W_i.$$

[3] Soeren Asmussen. *Applied Probability and Queues*. 2nd ed. Stochastic Modelling and Applied Probability. New York: Springer-Verlag, 2003.

Cumulative process

Example 1

$\tau_i \sim \mathcal{E}(a)$ and $W_i = 1$ *a.s.* The cumulative process associated is a Poisson process.

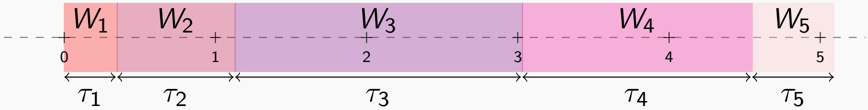
Cumulative process

Example 1

$\tau_i \sim \mathcal{E}(a)$ and $W_i = 1$ $a.s.$ The cumulative process associated is a Poisson process.

Example 2

$(\tau_i)_i$ a random sequence which represents intervals on $\mathbb{R}$.
W_i a random quantity associated to i-th interval.



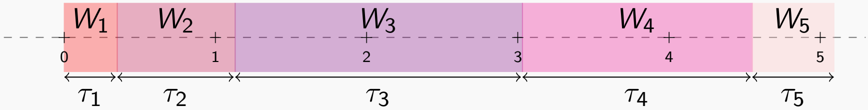
Cumulative process

Example 1

$\tau_i \sim \mathcal{E}(a)$ and $W_i = 1$ *a.s.* The cumulative process associated is a Poisson process.

Example 2

$(\tau_i)_i$ a random sequence which represents intervals on $\mathbb{R}$.
W_i a random quantity associated to i-th interval.



Warning : $(\tau_i, W_i)_i$ are i.i.d. couples but τ_i and W_i aren't supposed independent.

Literature for cumulative process

Object of interest: process $Z_t = \sum_{i=1}^{M_t} W_i$ at time t when t tends to ∞.

- Law of large numbers (can be found in Asmussen)
- Central limit theorem (can be found in Asmussen)
- Large deviations principle (Borovkov, Mogul'skii[4] ; Cattiaux, C., Costa[5]; Zamparo[6])

[4] Alexander A. Borovkov and Anatolii A. Mogul'skii. "Large Deviation Principles for Trajectories of Compound Renewal Processes. I - II". In: *Theory of Probability & Its Applications* 60.2 (2016), pp. 207–224.

[5] Patrick Cattiaux, Laetitia Colombani, and Manon Costa. "Large Deviation Principles for Cumulative Processes and Applications". 2021.

[6] Marco Zamparo. "Large Deviation Principles for Renewal-Reward Processes". 2021.

Results

Results for process $Z_t = \sum_{i=1}^{M_t} W_i$

Law of large numbers:

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} := m.$$

Results

Results for process $Z_t = \sum_{i=1}^{M_t} W_i$

Law of large numbers:

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} := m.$$

Central limit theorem:

$$\sqrt{t}\left(\frac{Z_t}{t} - m\right) \xrightarrow[t\to\infty]{law} \mathcal{N}\left(0, \frac{Var\,(W - m\tau)}{\mathbb{E}(\tau)}\right).$$

Results

Results for process $Z_t = \sum_{i=1}^{M_t} W_i$

Law of large numbers:

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]} := m.$$

Central limit theorem:

$$\sqrt{t}\left(\frac{Z_t}{t} - m\right) \xrightarrow[t\to\infty]{law} \mathcal{N}\left(0, \frac{Var\,(W - m\tau)}{\mathbb{E}(\tau)}\right).$$

Large deviations equation:

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \geq m + a\right) \leq -\min\left[\inf_{z\geq m+(a/2)} J(z)\,,\ \frac{\theta_0 a}{4}\right],$$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i < m - a\right) \leq -\min\left[\inf_{z\leq m-(a/2)} J(z)\,,\ \frac{\theta_0 a}{4}\right].$$

Link between Hawkes process and cumulative process

Assumption: h has a compact support.
Hawkes process is *almost* a cumulative process.

Intensity

$\Lambda^h(t) = \left(\lambda + \sum_{i\geq 1} h(t - U_i)\right)^+$, where U_i are the jumps of N^h.
If $t > U_i + L(h)$ for each $U_i < t$, then $\Lambda^h(t) = \lambda$.

Link between Hawkes process and cumulative process

Assumption: h has a compact support.
Hawkes process is *almost* a cumulative process.

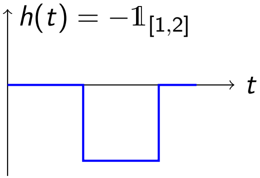
Intensity

$\Lambda^h(t) = \left(\lambda + \sum_{i\geq 1} h(t - U_i)\right)^+$, where U_i are the jumps of N^h.
If $t > U_i + L(h)$ for each $U_i < t$, then $\Lambda^h(t) = \lambda$.

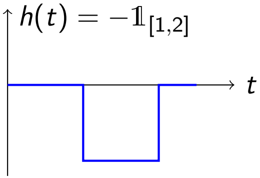
We can define

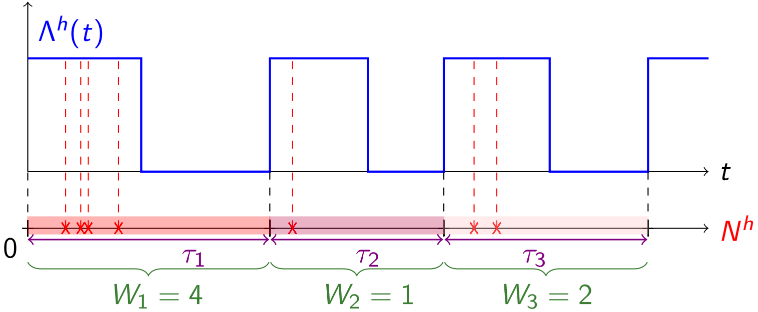
$\tau_1 = \inf\{t > U_1^1, N^h((t - L(h), t]) = 0\}$,
$W_1 = N^h([0, \tau_1])$,
...
Then : $N_t^h = \sum_{i=1}^{M_t} W_i + R_t$ with $0 \leq R_t \leq W_{M_t+1}$.

Definition of τ and W

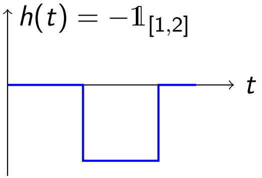


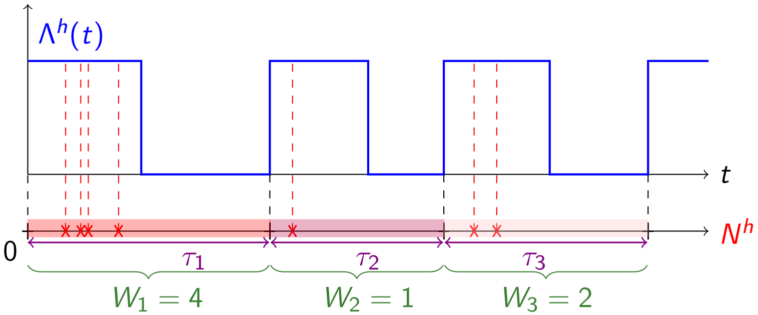
Definition of τ and W





Definition of τ and W





$\tau_1 = \inf\{t > U_1, N^h((t - L(h), t]) = 0\}$

$W_1 = N^h([0, \tau_1])$.

$N^h(t) = \sum_{i=1}^{M_t} W_i + R_t$

where $0 \leq R_t \leq W_{M_t+1}$

Results

First assumptions

- $\| h^+ \|_1 < 1$
- Empty initial condition: $N^h(]-\infty, 0]) = 0$
- h has a compact support, included in $[0, L(h)]$

Results

Law of large numbers and Limit Central Theorem for Hawkes process [Cattiaux, Costa, C.]

Let h be a signed function, with a support includes in $[0, L(h)]$. Then we have:

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$$

and

$$\frac{1}{\sqrt{t}}\left(N_t^h - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = Var\left(W_1 - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right)$.

Large Deviations Principle

Important assumptions for LDP for cumulative processes

- $\exists \beta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\beta\tau}] < \infty$ for $\beta < \beta_0$,
- $\exists \theta_0 \in (0, +\infty]$ such that $\mathbb{E}[e^{\theta|W|}] < \infty$, for $\theta < \theta_0$,
- for all interval $\mathcal{I}$ such that $\mathbb{P}(W \in \mathcal{I}) > 0$, it holds : for all $t \geq 0$, $\mathbb{P}(\tau > t\,,\, W \in \mathcal{I}) > 0$

Proposition for Hawkes process

Let θ_0 such that $\forall \theta < \theta_0, \mathbb{E}(e^{\theta|W|}) < \infty$.

Rate functions

For W^n a well-chosen reduction of W, we introduce the Cramer transform for $(a,b) \in \mathbb{R}^2$, and the rate function J^n associated for $z \in \mathbb{R}^+$

$$\Lambda_n^*(a,b) = \sup_{x,y} \left\{ ax + by - \ln \mathbb{E}\left(e^{x\tau + yW^n} \right) \right\} \quad \text{and} \quad J^n(z) = \inf_{\beta > 0} \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{z}{\beta} \right).$$

We also define

$$\tilde{J}(z) = \sup_{\delta > 0} \liminf_{n \to \infty} \inf_{|y - z| < \delta} J^n(y).$$

For W,

$$\Lambda^*(a,b) = \sup_{x,y} \left\{ ax + by - \ln \left(\mathbb{E}\left[e^{x\tau + yW} \right] \right) \right\} \quad \text{and} \quad J(z) = \inf_{\beta > 0} \beta \Lambda^* \left(\frac{1}{\beta}, \frac{z}{\beta} \right).$$

Proposition for Hawkes process

Let θ_0 such that $\forall\theta < \theta_0, \mathbb{E}(e^{\theta|W|}) < \infty$.

Theorem

- If $\theta_0 = \infty$, then N_t^h/t satisfies a LDP with rate function $\tilde{J}$.
- If $\theta_0 < +\infty$, we have for all $a > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} > m + a\right) \leq -\min\left[\inf_{z\geq m+a/2} J(z)\,,\ \theta_0 a/4\right].$$

Similarly

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} < m - a\right) \leq -\min\left[\inf_{z\leq m-a/2} J(z)\,,\ \theta_0 a/4\right].$$

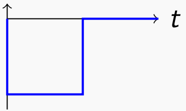
Examples

Can we apply this theorem?

Examples

Can we apply this theorem?

Let $\lambda = 1$ and let:



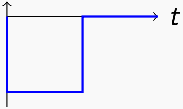
$h_3(t) = -\mathbb{1}_{[0,1]}$

$W = 1$, $\tau \sim 1 + \mathcal{E}(\lambda)$.
$\theta_0 = \infty$
(Renewal process)

Examples

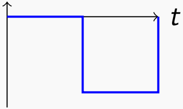
Can we apply this theorem?

Let $\lambda = 1$ and let:



$h_3(t) = -\mathbb{1}_{[0,1]}$

$W = 1$, $\tau \sim 1 + \mathcal{E}(\lambda)$.
$\theta_0 = \infty$
(Renewal process)



$h_4(t) = -\mathbb{1}_{[1,2]}$

$W \sim 1 + Poi(\lambda)$, the law of τ can be described. The joint law is known.
$\theta_0 = \infty$

Idea of proofs

$N^h(t) = \sum_{i=1}^{M_t} W_i + R_t$ where $0 \leq R_t \leq W_{M_t+1}$.

- Check the necessary assumptions for Theorems on Cumulative Process
- Study the behavior of R_t

Conclusion

We have:

- Law of large numbers : signed h and compact support
- Central limit theorem : signed h and compact support
- Large deviation principle : signed h, compact support and stronger assumptions
  Deviations inequalities: signed h and compact support.

Conclusion

We have:

- Law of large numbers : signed h and compact support
- Central limit theorem : signed h and compact support
- Large deviation principle : signed h, compact support and stronger assumptions
  Deviations inequalities: signed h and compact support.

Leads

- Understand these assumptions for Hawkes process
- Obtain more properties on cumulative process

Thank you