On two spatially structured Hawkes processes as spike train models

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Based on joint works with A. Duarte, E. Löcherbach, A. Melnykova, G. Ost and I. Tubikanec

Université Grenoble Alpes (LJK)

New trends on Hawkes processes – Toulouse
2022/06/01

Outline

Notation

- Point process (temporal): N, $(N_t)_{t\geq 0}$ or $N(dt) = \sum_{T\in N} \delta_T(dt)$.
- Stochastic intensity: $(\lambda_t)_{t\geq 0}$ and $\Lambda_t = \int_0^t \lambda_s ds$.
- Standard Poisson process (temporal or spatial): Π (rate =1).
- Number of interacting processes (neurons): n.

Thinning - theory

Theorem (Thinning)

If $(\lambda_t)_{t\geq 0}$ is predictable and L^1_{loc} a.s., then

$$N(dt) = \int_0^{+\infty} \mathbf{1}_{z\leq\lambda_t}\Pi(dt,dz)$$

defines a point process with intensity given by $(\lambda_t)_{t\geq 0}$.

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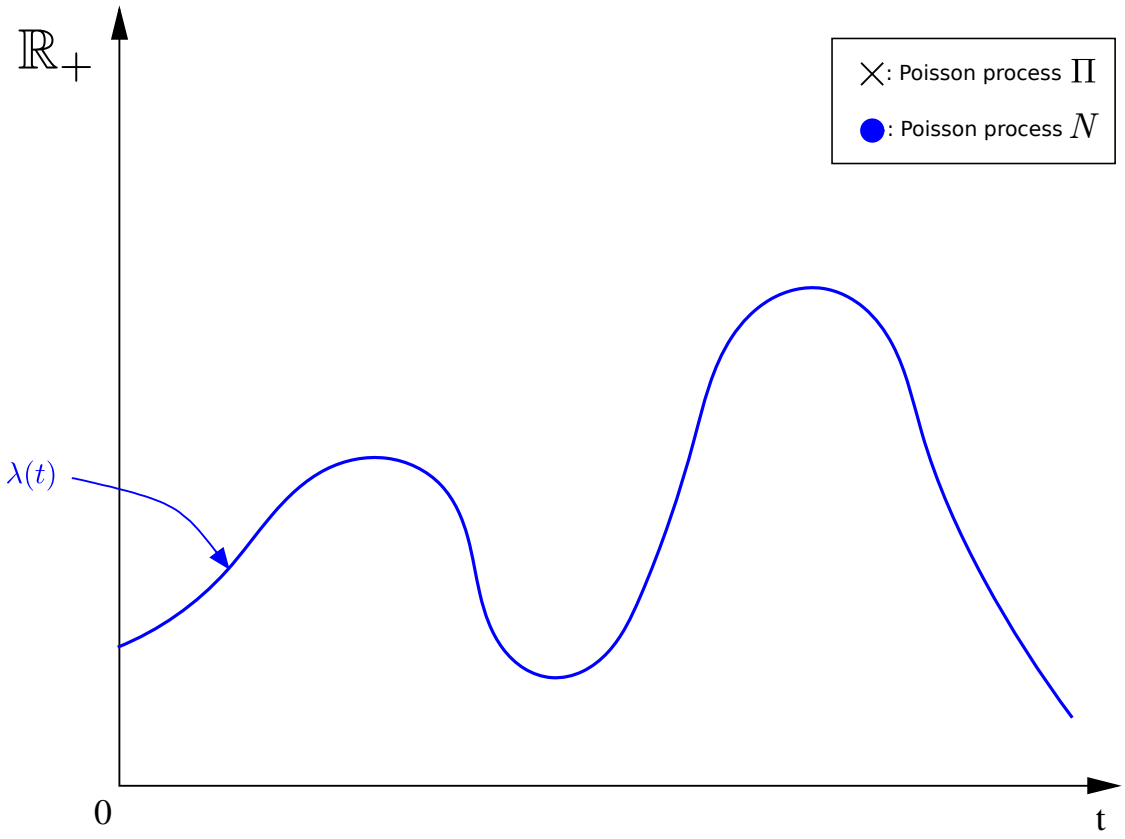

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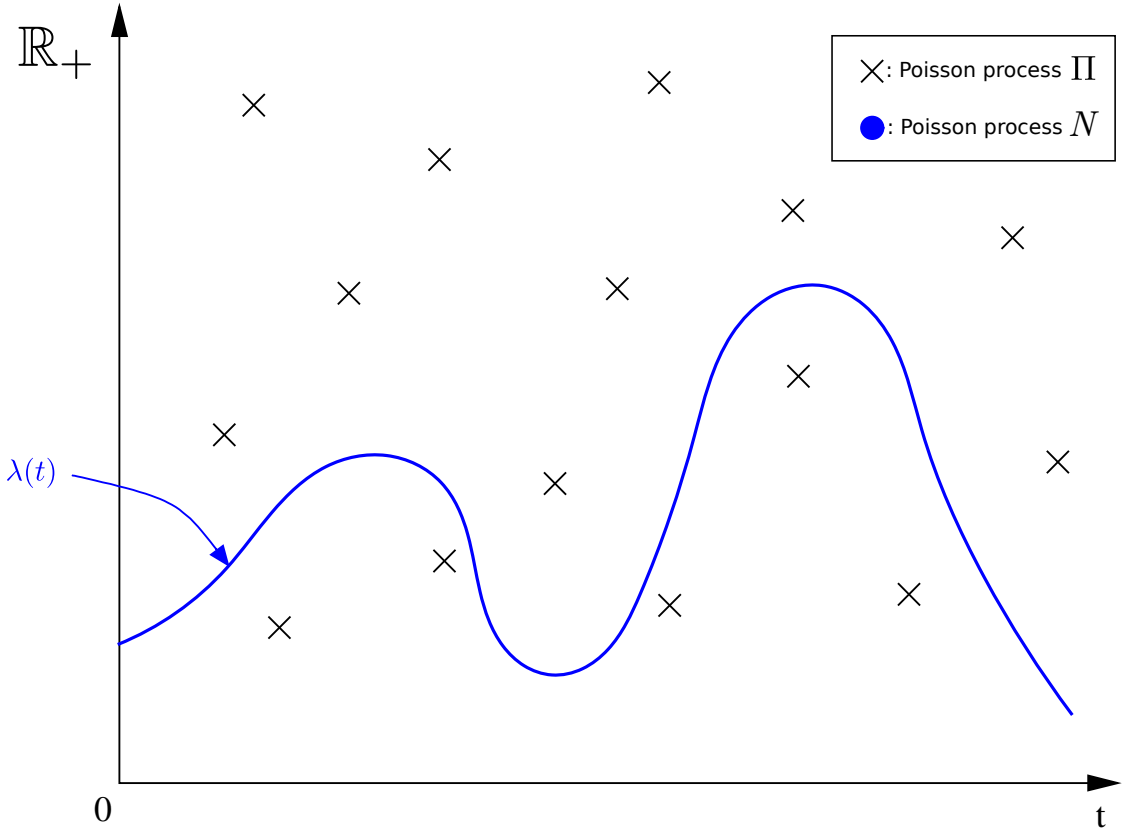

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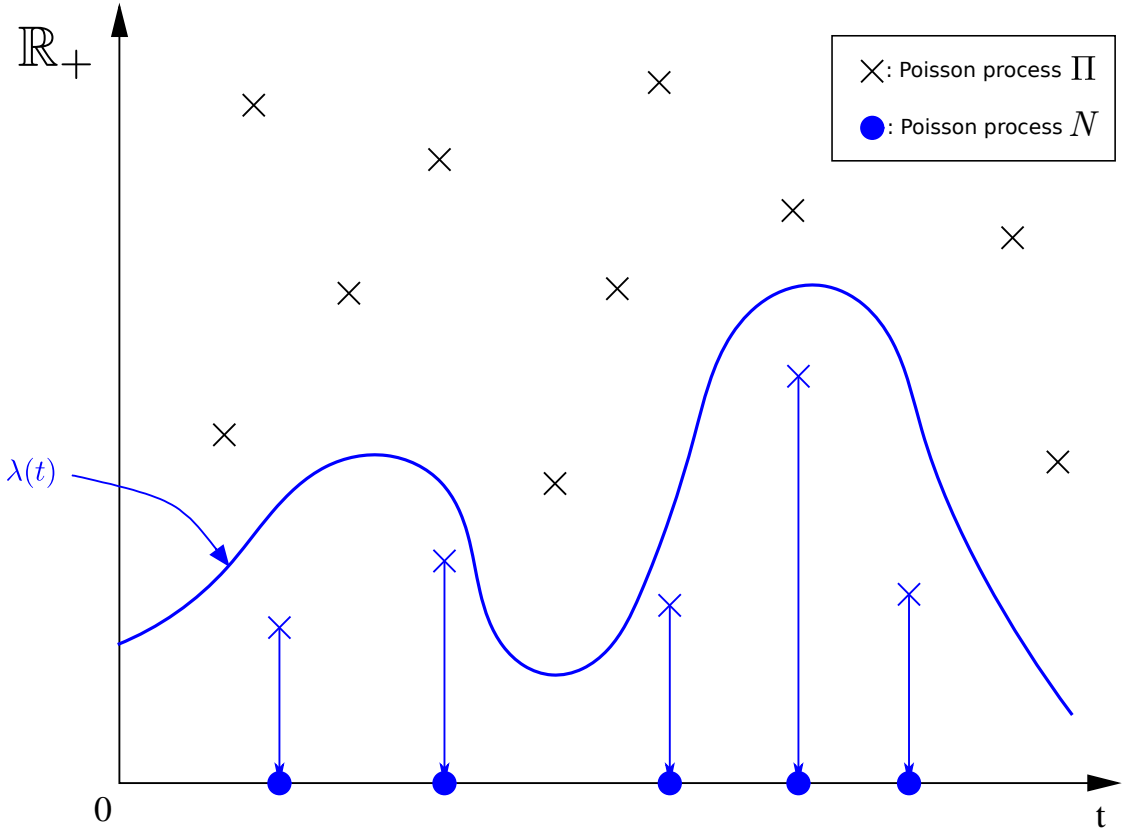

Thinning - simulation

- Exact simulation algorithm.
- Π is simulated piecewisely over finite rectangles.
- Need to upperbound the intensity λ_t. Not trivial in general.
  - If the bound is bad, the method is slow (cf. rejection sampling).

Change time - theory

Theorem (Change time)

If $(\lambda_t)_{t\geq 0}$ is predictable and L^1_{loc} a.s., then

$$N_t = \Pi_{\Lambda_t}$$

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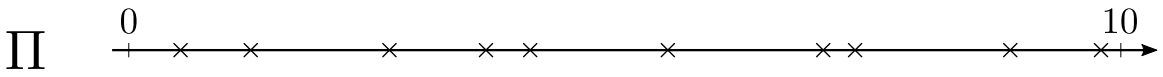

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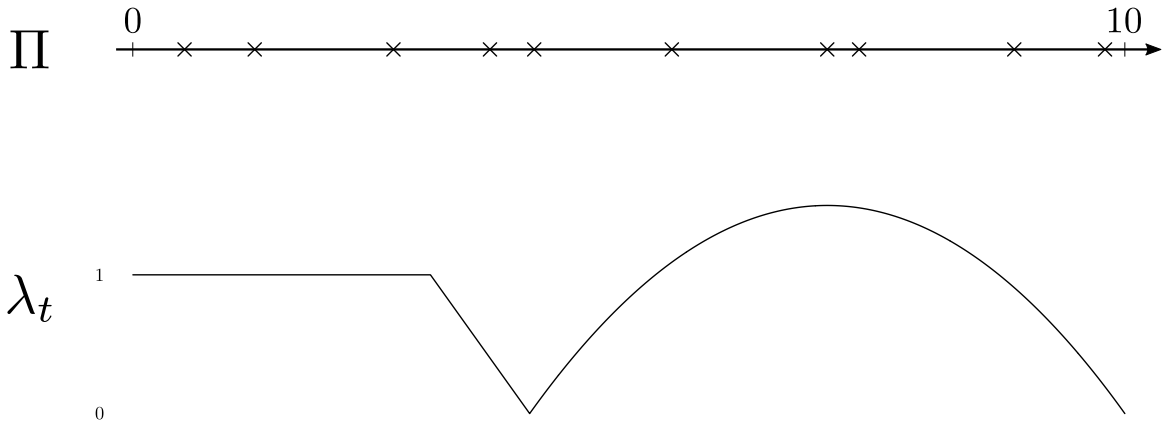

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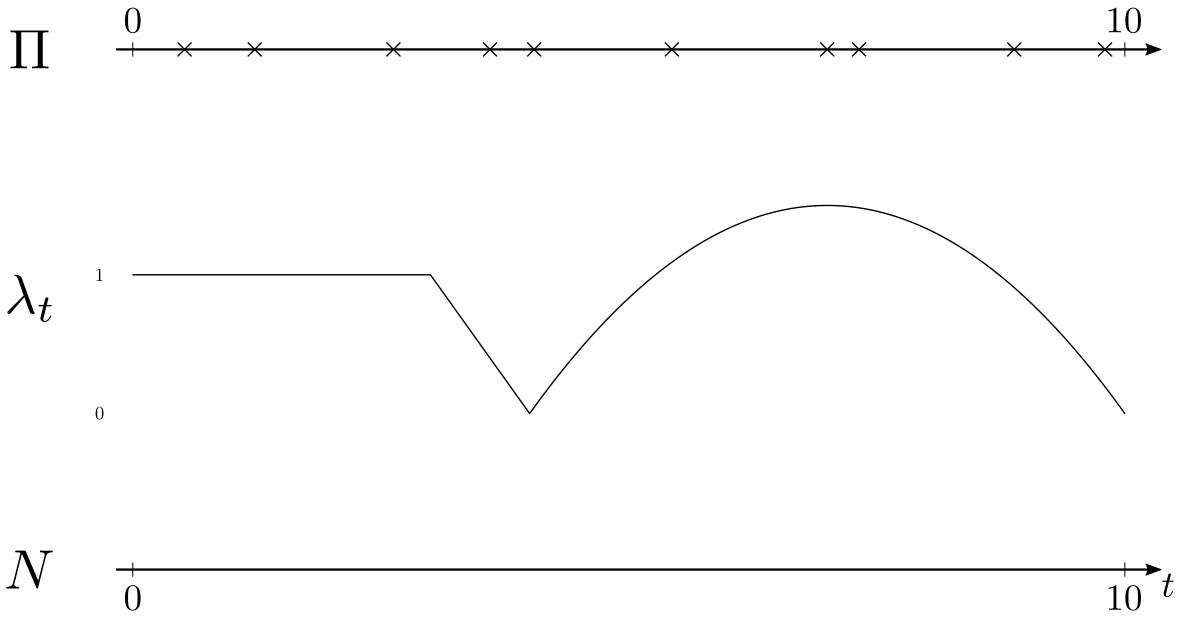

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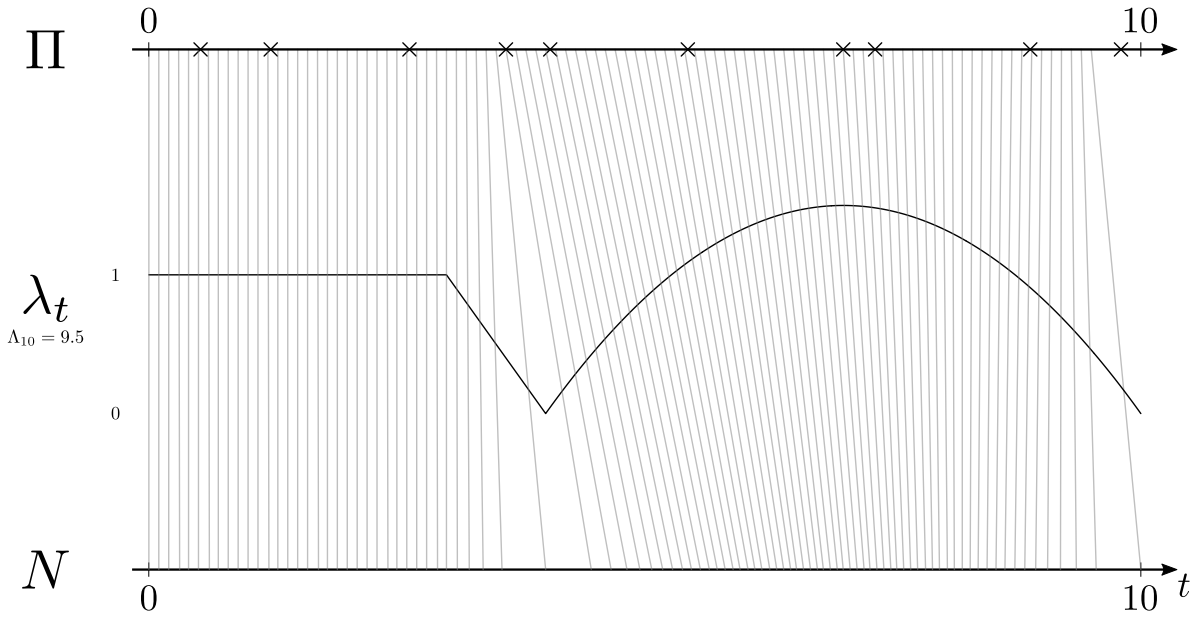

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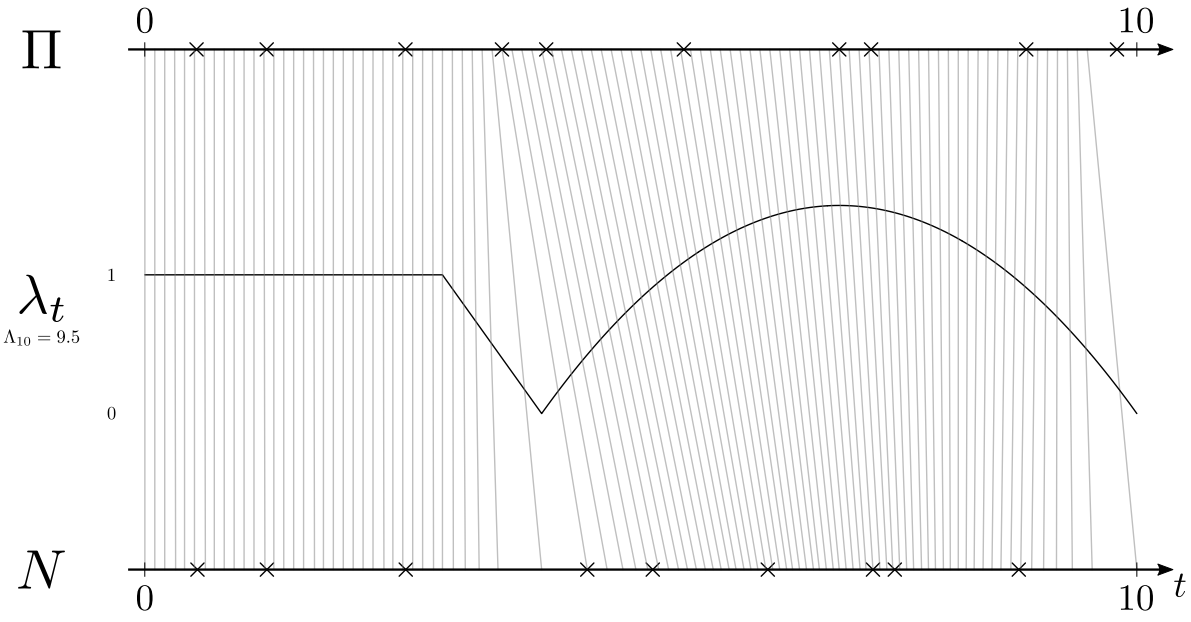


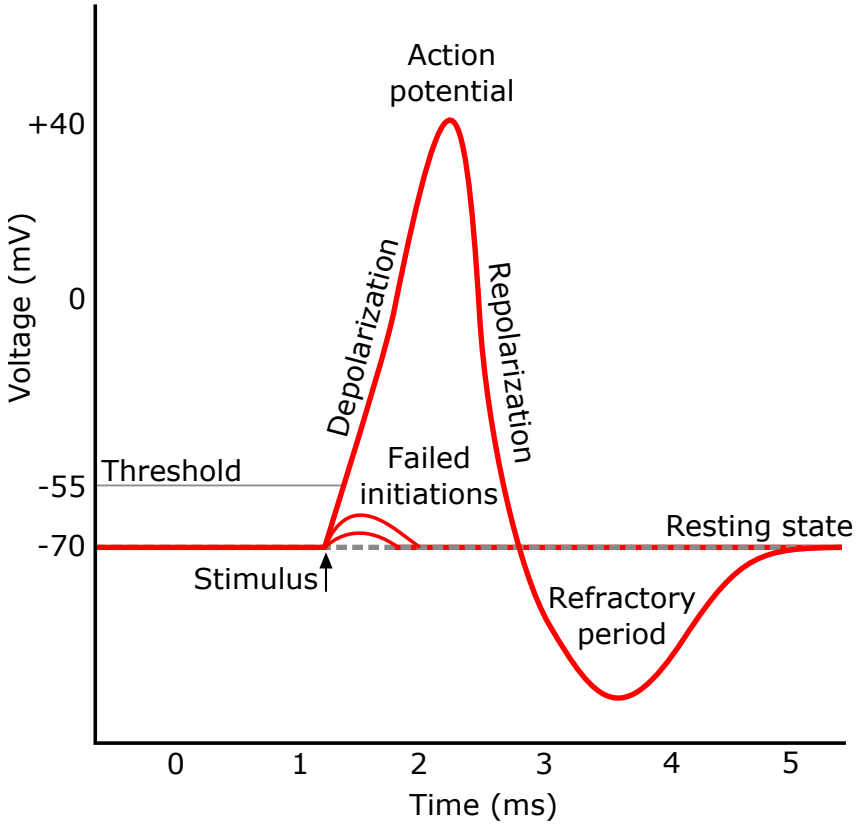
$$T \in N \Leftrightarrow \Lambda_T \in \Pi$$

Change time - simulation

- If $\Lambda^{-1}(t)$ is explicit then the method is fast and exact.
- If not, integral approximation is needed :
  - $\Rightarrow$ approximation method.
  - $\Rightarrow$ computational cost depends on the approximation step.

Outline

Biological context - Several scales



- Neurons = electrically excitable cells.
- Action potential = spike of the membrane potential.
- Interactions: synaptic integration.

Biological context - Several scales

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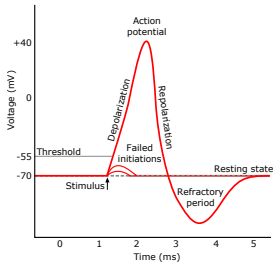

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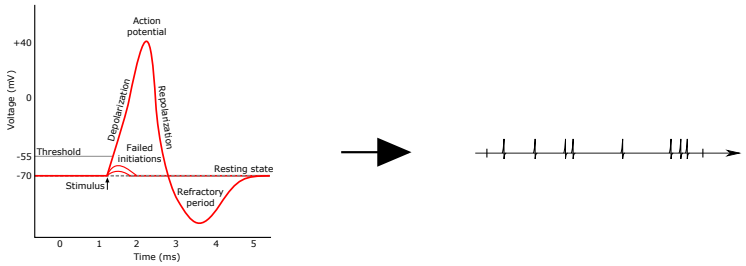

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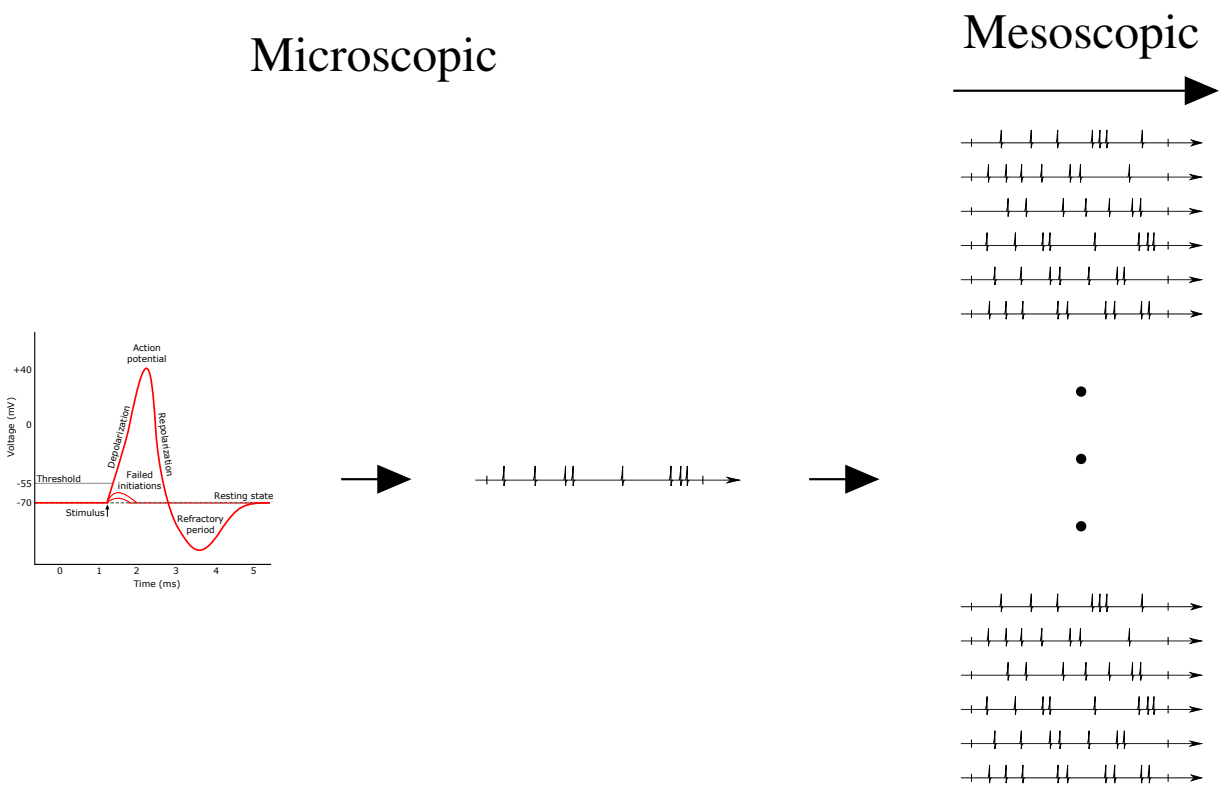

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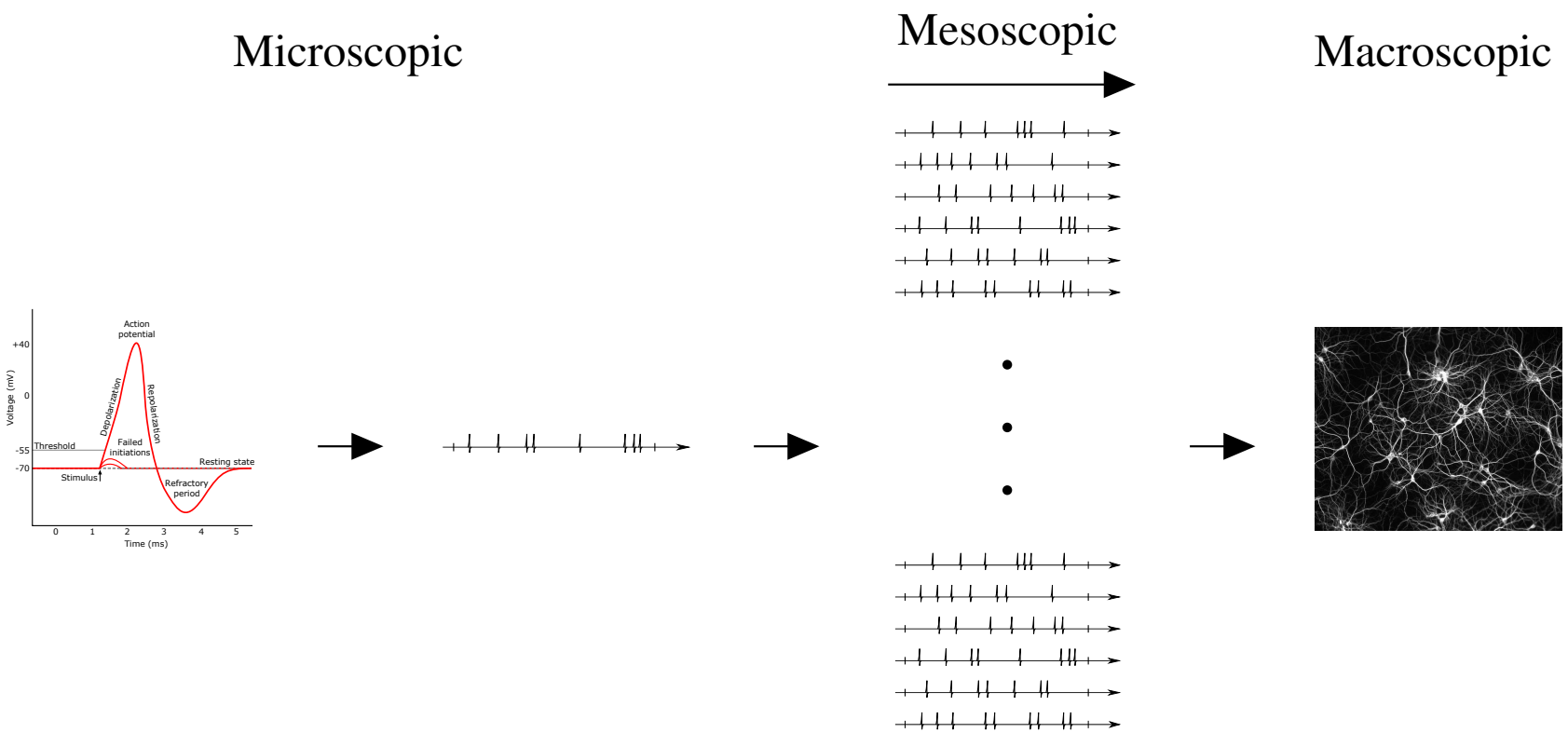

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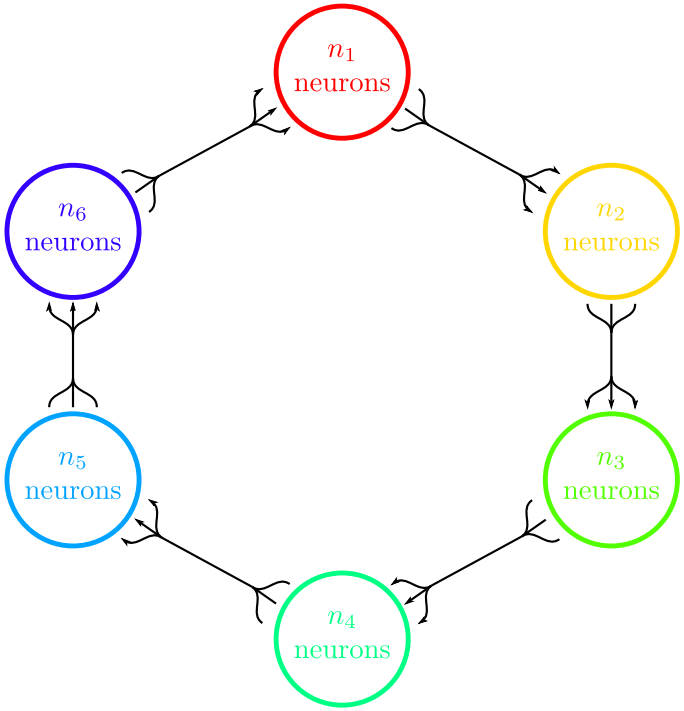

Multi-class

(Ditlevsen, Löcherbach, '17)
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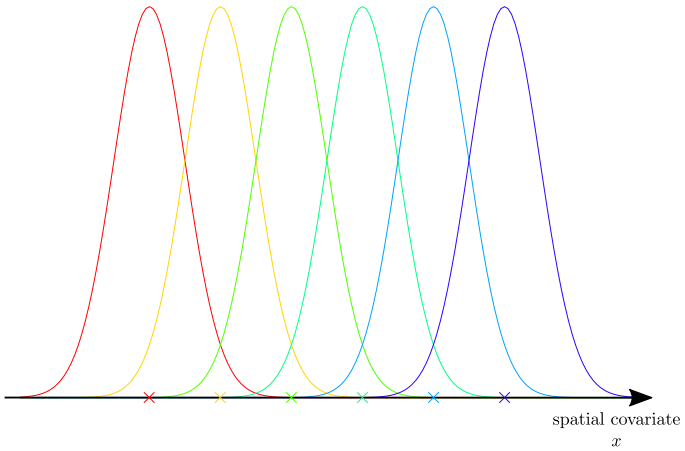

Continuous spatial covariates

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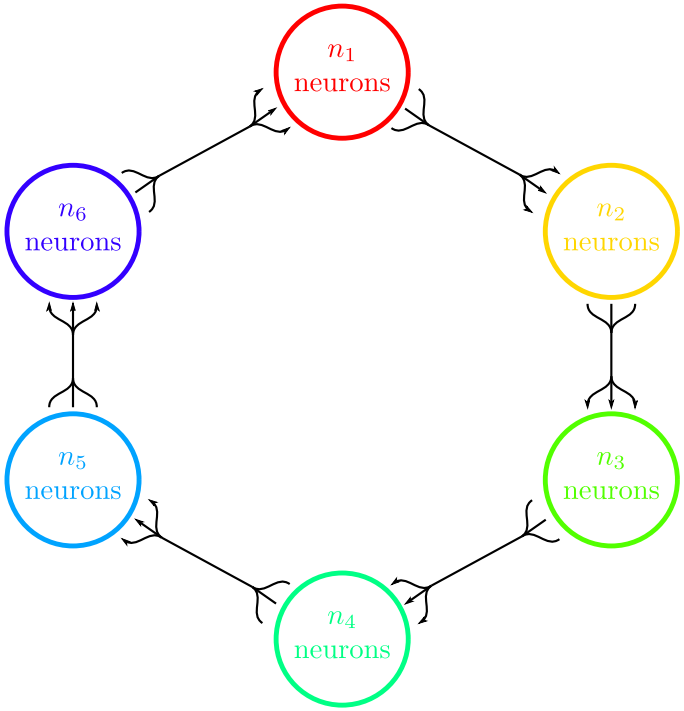

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Outline

The model - simplified with 2 populations

- Two populations of neurons of size n_1 and n_2.
- Two non-null interacting functions $h_{1\to 2}$ and $h_{2\to 1}$.
- Two intensity functions $f_k > 0$, non-decreasing, bounded and Lipschitz.

For each $k = 1, 2$, and $m = 1, \dots, n_i$, the spikes of the m-th neuron of population k are described by $N^{k,m}$ with intensity

$$\lambda_t^k = f_k \left(\frac{1}{n_{\hat{k}}} \sum_{\ell=1}^{n_{\hat{k}}} \int_0^{t-} h_{\hat{k}\to k}(t-t') N^{\hat{k},\ell}(dt') \right).$$

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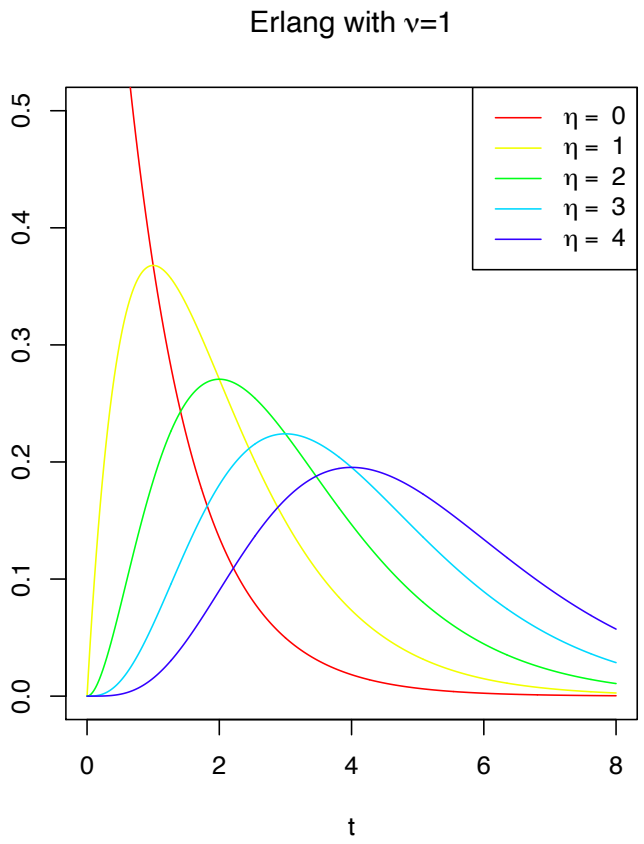
Main assumption (Erlang kernels)

$$h_{\hat{k}\to k}(t) = c_k e^{-\nu_k t} \frac{t^{\eta_k}}{\eta_k!},$$

where $c_k = \pm 1$ (excitatory/inhibitory), $\nu_k > 0$ real, $\eta_k \geq 0$ integer.

Erlang kernels



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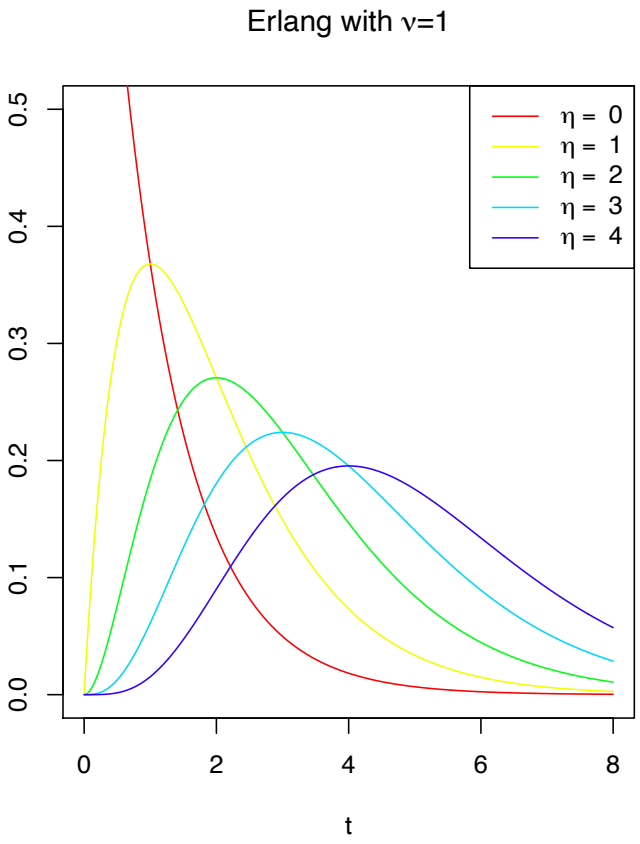

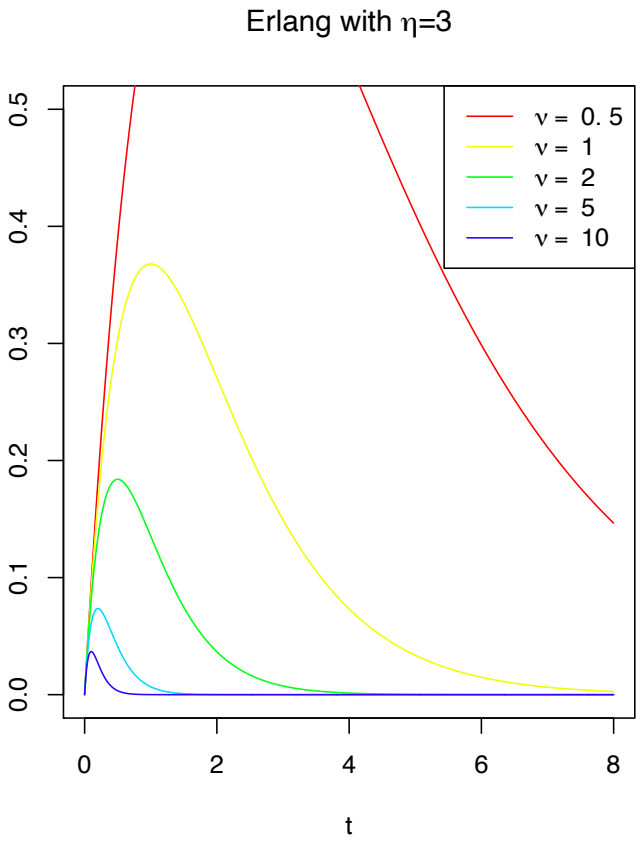

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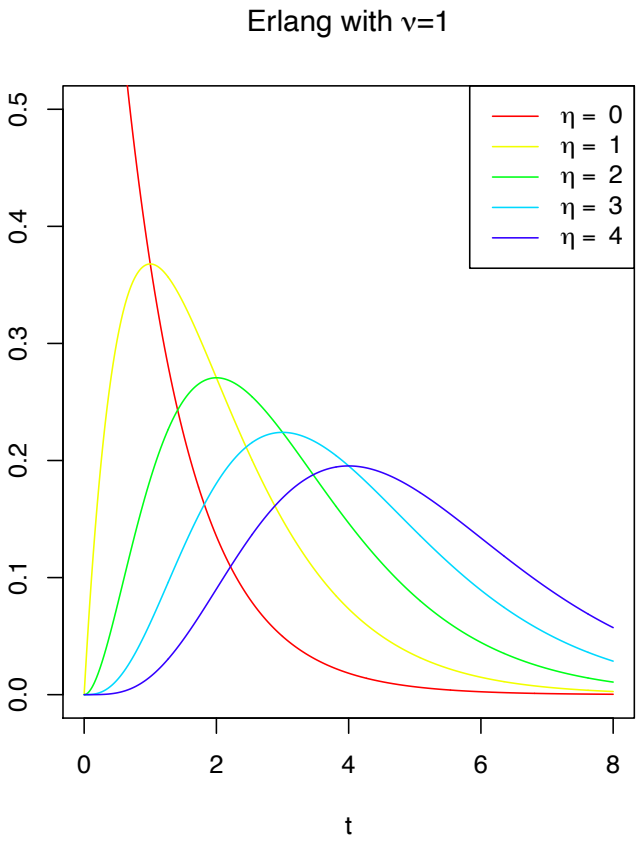

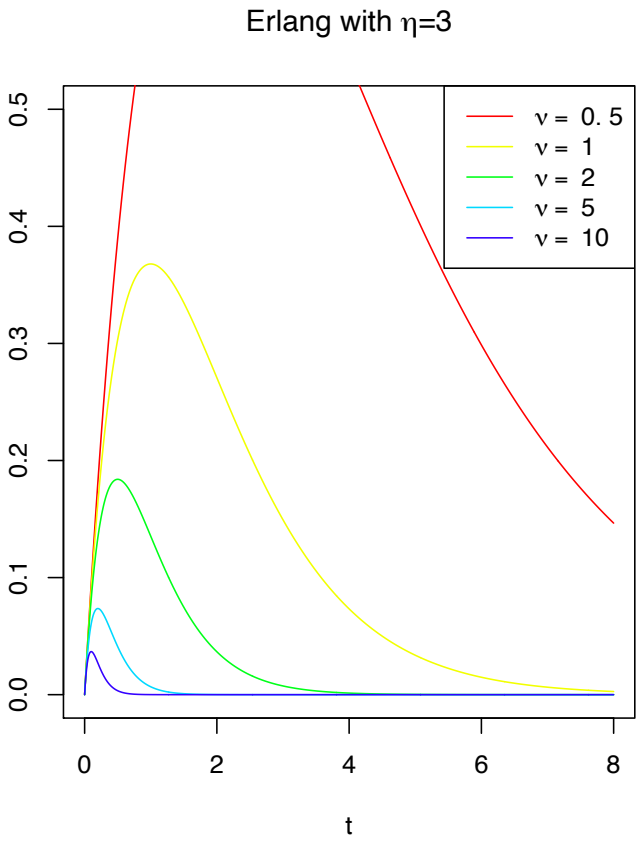

Main property (Derivative cascade)

$$\frac{d}{dt}\left\{c_k e^{-\nu_k t}\frac{t^{\eta_k}}{\eta_k!}\right\} = -\nu_k\left\{c_k e^{-\nu_k t}\frac{t^{\eta_k}}{\eta_k!}\right\} + c_k e^{-\nu_k t}\frac{t^{\eta_k-1}}{(\eta_k-1)!}$$

PDMP structure

The intensity of each population is given by $\lambda_t^k = f_k(X_{t-}^{k,1})$ where

$$X_t^{k,1} = \int_0^t h_{\hat{k}\to k}(t-t') \underbrace{\left\{ \frac{1}{n_{\hat{k}}} \sum_{\ell=1}^{n_{\hat{k}}} N^{\hat{k},\ell}(dt') \right\}}_{\tilde{N}^{\hat{k}}(dt')}.$$

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The derivative cascade gives the Markovian system (X is a PDMP) :

$$\begin{cases} dX_t^{k,j} = \left[-\nu_k X_t^{k,j} + X_t^{k,j+1} \right] dt, \quad \text{for } j = 1, \dots, \eta_k, \\ dX_t^{k,\eta_k+1} = -\nu_k X_t^{k,\eta_k+1} dt + c_k \tilde{N}^{\hat{k}}(dt), \\ X_0 = x_0 \in \mathbb{R}^{\eta_1+\eta_2+2}, \end{cases}$$

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and $\tilde{N}^{\hat{k}}$ is a pure jump process with jumps of size $\frac{1}{n_{\hat{k}}}$ and intensity $n_{\hat{k}} f_{\hat{k}}(X_{t-}^{k,1})$.

Mean-field framework

Let $n = n_1 + n_2$ and assume that $n_k/n \xrightarrow{n\to\infty} p_k > 0$ for $k = 1,2$.
Assume that $f_1, f_2 \in C^1$ with bounded derivatives and that $h_{1\to 2}, h_{2\to 1} \in L^2_{loc}$.

- Hawkes processes $N^{k,m}$ with intensity

$$\lambda_t^k = f_k\left(\int_0^{t-} h_{\hat{k}\to k}(t-t')\tilde{N}^{\hat{k}}(dt')\right)$$

should be well approximated by independent Poisson processes $\bar{N}^{k,m}$ with intensity

$$\bar{\lambda}^k(t) = f_k\left(\int_0^{t-} h_{\hat{k}\to k}(t-t')\mathbb{E}\left[\bar{N}^{\hat{k},1}(dt')\right]\right).$$

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- Note that $\bar{\Lambda}^k(t) = \int_0^t \bar{\lambda}^k(t')dt' = \mathbb{E}\left[\bar{N}_t^{k,1}\right]$ satisfies the following integral equation

$$\bar{\Lambda}^k(t) = \int_0^t f_k\left(h_{\hat{k}\to k}(t-t')\bar{\lambda}^{\hat{k}}(t')dt'\right). \tag{1}$$

- $\bar{\Lambda}^k$ is uniquely characterized by (1).

Law of Large Numbers - Proof via Thinning

- Let $(\Pi^{k,m})$ be i.i.d. standard Poisson processes and

$$N^{k,m}(dt) = \int_0^{+\infty} \mathbf{1}_{z \leq \lambda_t^k} \Pi^{k,m}(dt, dz), \quad \bar{N}^{k,m}(dt) = \int_0^{+\infty} \mathbf{1}_{z \leq \bar{\lambda}^k(t)} \Pi^{k,m}(dt, dz).$$

- This coupling is such that

$$\mathbb{P}\left((N^{k,m})_{[0,T]} \neq (\bar{N}^{k,m})_{[0,T]} \right) \leq \mathbb{E}\left[\int_0^T \left| \lambda_t^k - \bar{\lambda}^k(t) \right| dt \right].$$

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Theorem (Fournier, Delattre '16; Ditlevsen, Löcherbach '17)

For any $m \geq 1$,

$$\left((N^{1,m'})_{m' \leq m}, (N^{2,m'})_{m' \leq m} \right) \xrightarrow[n \to +\infty]{\mathscr{L}} \left((\bar{N}^{1,m'})_{m' \leq m}, (\bar{N}^{2,m'})_{m' \leq m} \right),$$

where D (the space of cadlàg functions) is endowed with the Skorokhod topology.

Central Limit Theorem

Theorem (Fournier, Delattre '16; Ditlevsen, Löcherbach '17)

As n and t go to ∞ with :

1. $t/n \to 0$ in the subcritical case,
2. $e^{\alpha_0 t} t^{-1/2} n^{-1/2} \to 0$ in the supercritical case (α_0 depends on the parameters),

we have

$$\left(\left(\frac{N_t^{1,m'} - \bar{\Lambda}^1(t)}{\sqrt{\bar{\Lambda}^1(t)}} \right)_{m' \leq m}, \left(\frac{N_t^{2,m'} - \bar{\Lambda}^2(t)}{\sqrt{\bar{\Lambda}^2(t)}} \right)_{m' \leq m} \right) \xrightarrow{\mathscr{L}} \mathscr{N}(0, I_{2m}).$$

Oscillations

- Remind the cascade structure :

$$\begin{cases} dX_t^{k,j} = \left[-\nu_k X_t^{k,j} + X_t^{k,j+1}\right] dt, \quad \text{for } j = 1, \dots, \eta_k, \\ dX_t^{k,\eta_k+1} = -\nu_k X_t^{k,\eta_k+1} dt + c_k \tilde{N}^{\hat{k}}(dt), \\ X_0 = x_0 \in \mathbb{R}^{\eta_1+\eta_2+2}, \end{cases}$$

where $X_t^{k,j} = \int_0^t c_k e^{-\nu_k(t-t')} \frac{(t-t')^{\eta_k+1-j}}{(\eta_k+1-j)!} \tilde{N}^{\hat{k}}(dt')$.

- Similarly, we have the *monotone cyclic feedback* system :

$$\begin{cases} dx^{k,j} = \left[-\nu_k x^{k,j}(t) + x^{k,j+1}(t)\right] dt, \quad \text{for } j = 1, \dots, \eta_k, \\ dx^{k,\eta_k+1} = -\nu_k x^{k,\eta_k+1}(t) dt + c_k f_{\hat{k}}(x^{\hat{k},1}(t)) dt, \\ x(0) = x_0 \in \mathbb{R}^{\eta_1+\eta_2+2}, \end{cases}$$

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- Monotone cyclic feedback system :

$$\begin{cases} \frac{dx^{k,j}}{dt} = -\nu_k x^{k,j}(t) + x^{k,j+1}(t), \quad \text{for } j = 1, \dots, \eta_k, \\ \frac{dx^{k,\eta_k+1}}{dt} = -\nu_k x^{k,\eta_k+1}(t) + c_k f_{\hat{k}}(x^{\hat{k},1}(t)), \\ x(0) = x_0 \in \mathbb{R}^{\eta_1+\eta_2+2}, \end{cases}$$

where $x^{k,j}(t) = \int_0^t c_k e^{-\nu_k(t-t')} \frac{(t-t')^{\eta_k+1-j}}{(\eta_k+1-j)!} \bar{\lambda}^{\hat{k}}(t')dt'$.

Oscillations

- Monotone cyclic feedback system :

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Theorem (Ditlevsen, Löcherbach '17)

Assume that f_1, f_2 are non-decreasing bounded analytic functions and that $c_1 c_2 < 0$. Then, the system admits a unique equilibrium point x^*.
There are some parameters for which x^* is unstable and the system is attracted to a periodic orbit.

Diffusion approximation

Remind the cascade structure :

$$\begin{cases} dX_t^{k,j} = \left[-\nu_k X_t^{k,j} + X_t^{k,j+1}\right] dt, \quad \text{for } j = 1, \dots, \eta_k, \\ dX_t^{k,\eta_k+1} = -\nu_k X_t^{k,\eta_k+1} dt + c_k \tilde{N}^{\hat{k}}(dt), \\ X_0 = x_0 \in \mathbb{R}^{\eta_1+\eta_2+2}, \end{cases}$$

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where W^1, W^2 are independent standard Brownian motions.

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where W^1, W^2 are independent standard Brownian motions.

Weak approximation (Ditlevsen, Löcherbach '17)

Assume $f_1, f_2 \in C_b^5$. There exists C such that for all $\phi \in C_b^4$,

$$||P_t^X \phi - P_t^Y \phi||_\infty \leq Ct \frac{||\phi||_{4,\infty}}{n^2}.$$

Diffusion approximation

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where W^1, W^2 are independent standard Brownian motions.

Strong approximation (C., Melnykova, Tubikanec '21)

Assume f_1, f_2 are upper and lower bounded. Then, X and Y can be coupled such that, for all $T > 0$,

$$\sup_{t \le T} \|X_t - Y_t\|_\infty \le \Theta e^{CT} \frac{\log(n)}{n},$$

where Θ is a random variable with exponential moments.

Proof of the strong approximation - three ingredients (Kurtz, '77)

Since the drift part are the same for the PDMP and the diffusion, it suffices to control the difference between $\tilde{N}_t^k$ and

$$\tilde{W}_t^k = \int_0^t \left\{ f_k(Y_{t'}^{k,1})dt' + \frac{\sqrt{f_k(Y_{t'}^{k,1})}}{\sqrt{n_k}} dW_{t'}^k \right\}$$

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1. Change-time representation : $\tilde{N}_t^k = \frac{1}{n_k}\Pi_{n_k\Lambda_t^k}$ where $\Lambda_t^k = \int_0^t f_k(X_{t'}^{k,1})dt'$,

$$\tilde{W}_t^k = \frac{1}{n_k} B_{n_k\tilde{\Lambda}_t^k}, \quad \text{where} \quad \tilde{\Lambda}_t^k = \int_0^t f_k(Y_{t'}^{k,1})dt',$$

   and $(B_t)_t$ is a Brownian motion with unit drift.
2. KMT coupling : Π and B can be coupled such that the random variable

$$\sup_{t\geq 0} \frac{|\Pi_t - B_t|}{1 \vee \log t} \quad \text{is sub-exponential.}$$

3. Brownian motion modulus of continuity : the random variable

$$\sup_{s,t\leq T} \frac{|B_t - B_s|}{\sqrt{|t-s|\{1+\log(T/|t-s|)\}}} \quad \text{is sub-gaussian.}$$

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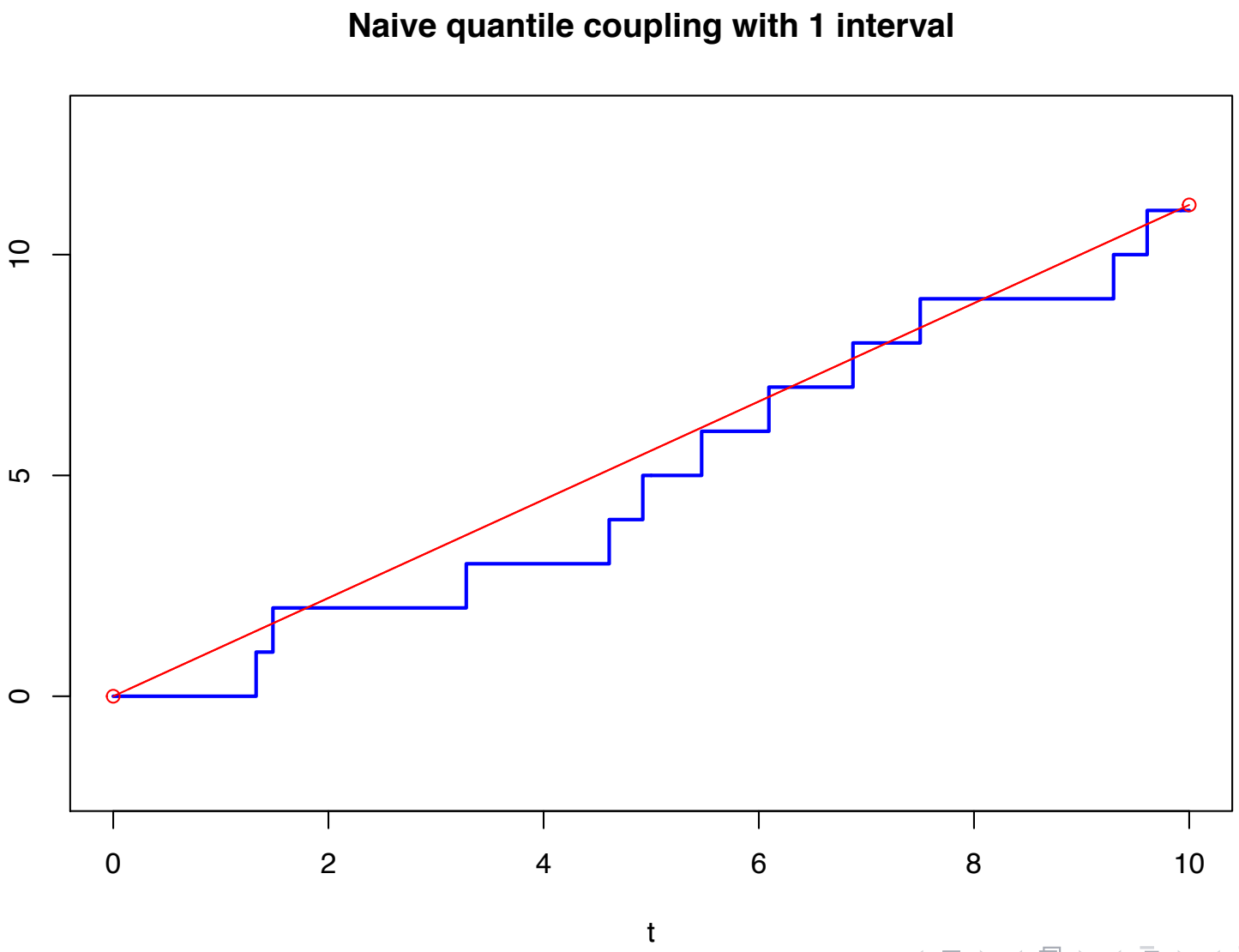
Naive quantile coupling

Poisson increment $\Pi_t - \Pi_s \rightsquigarrow$ uniform r.v. in $[0,1] \rightsquigarrow$ Brownian increment $B_t - B_s$



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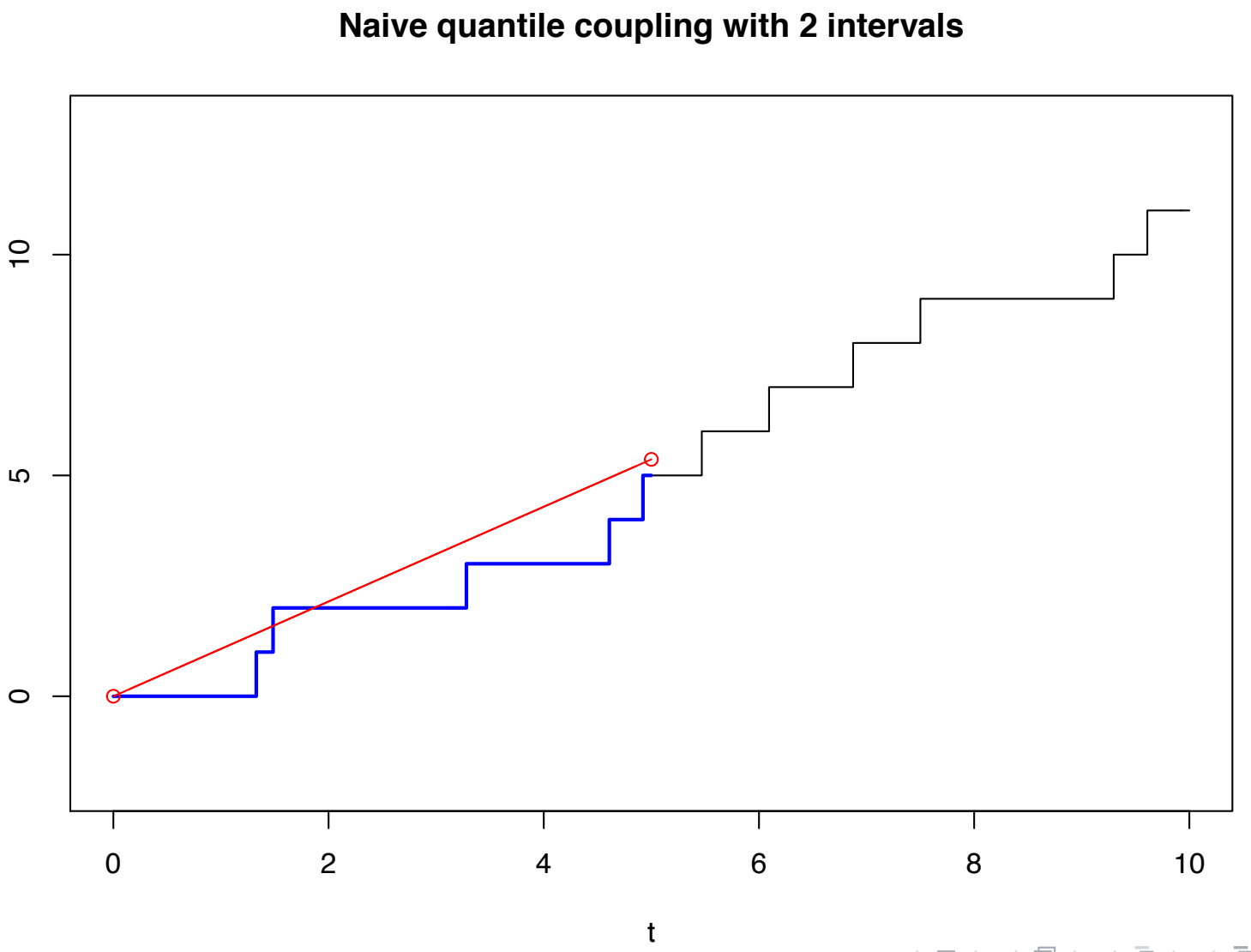

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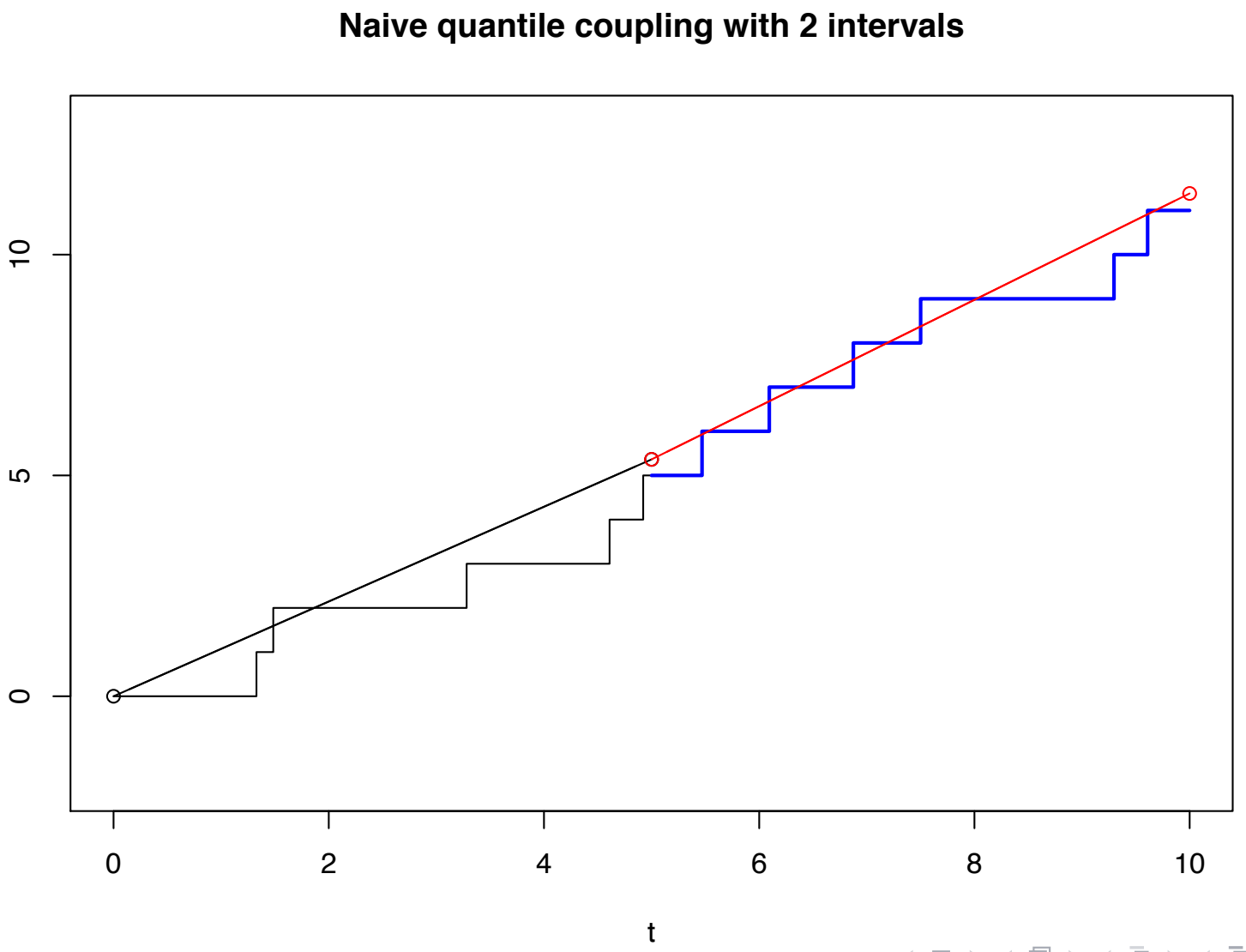

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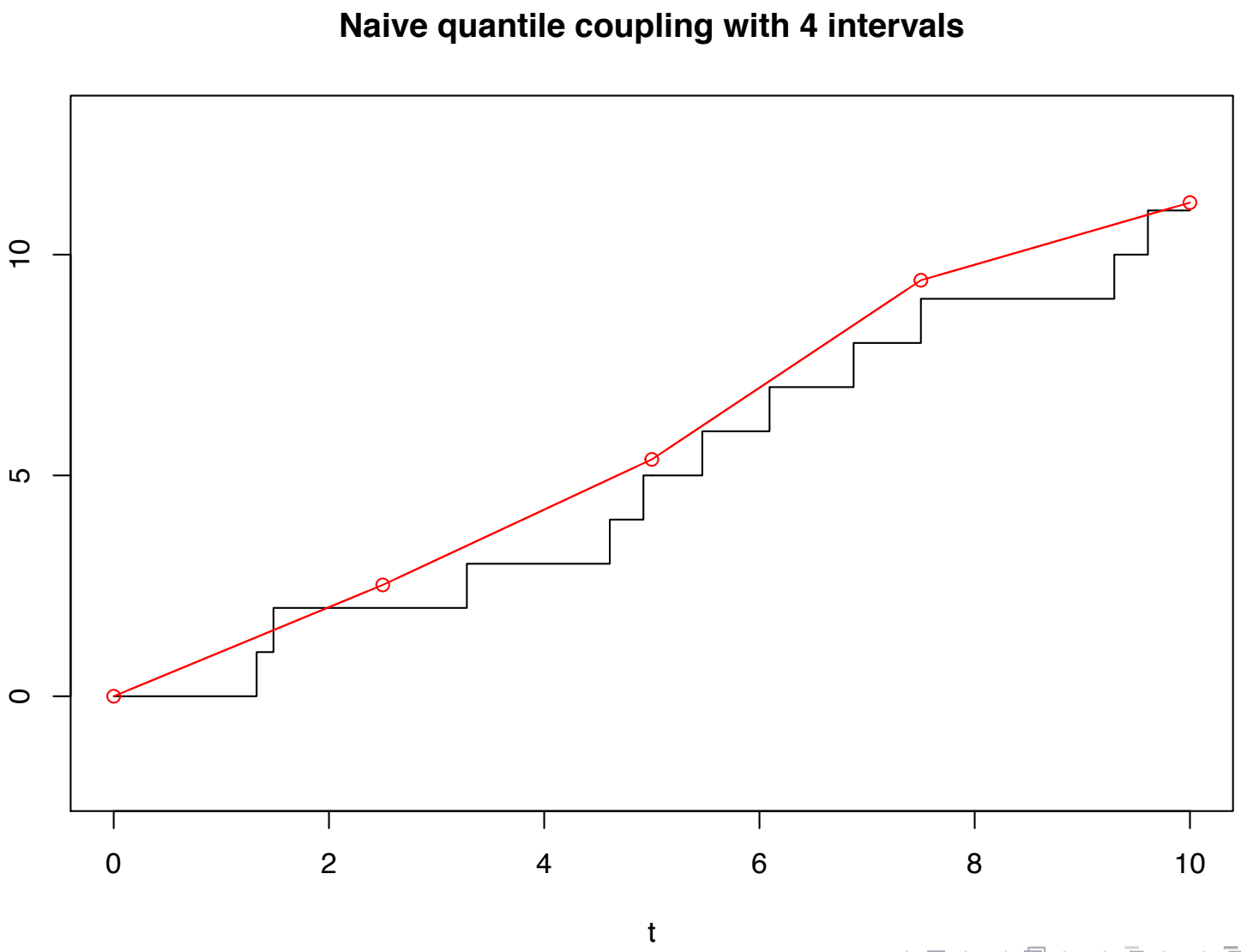

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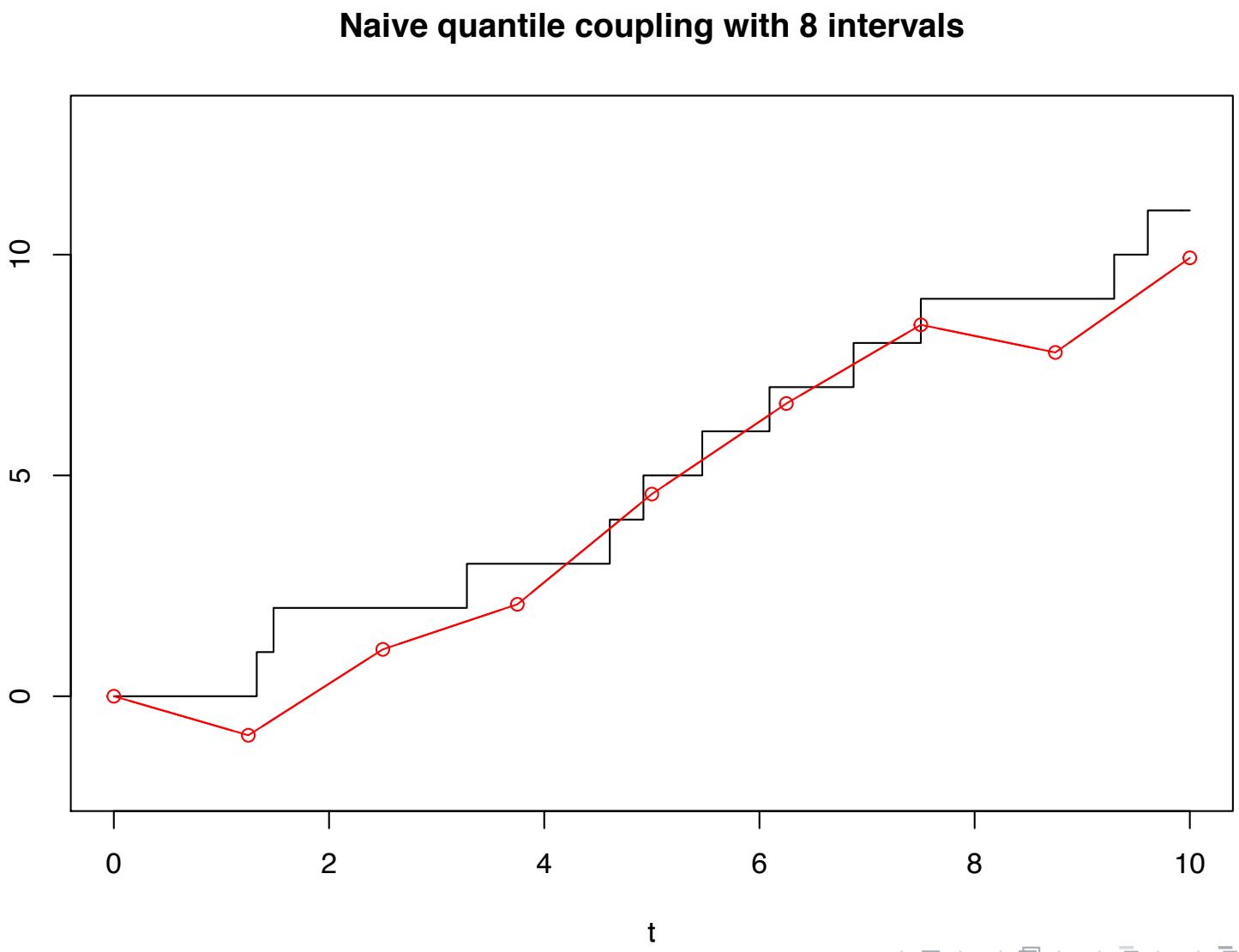

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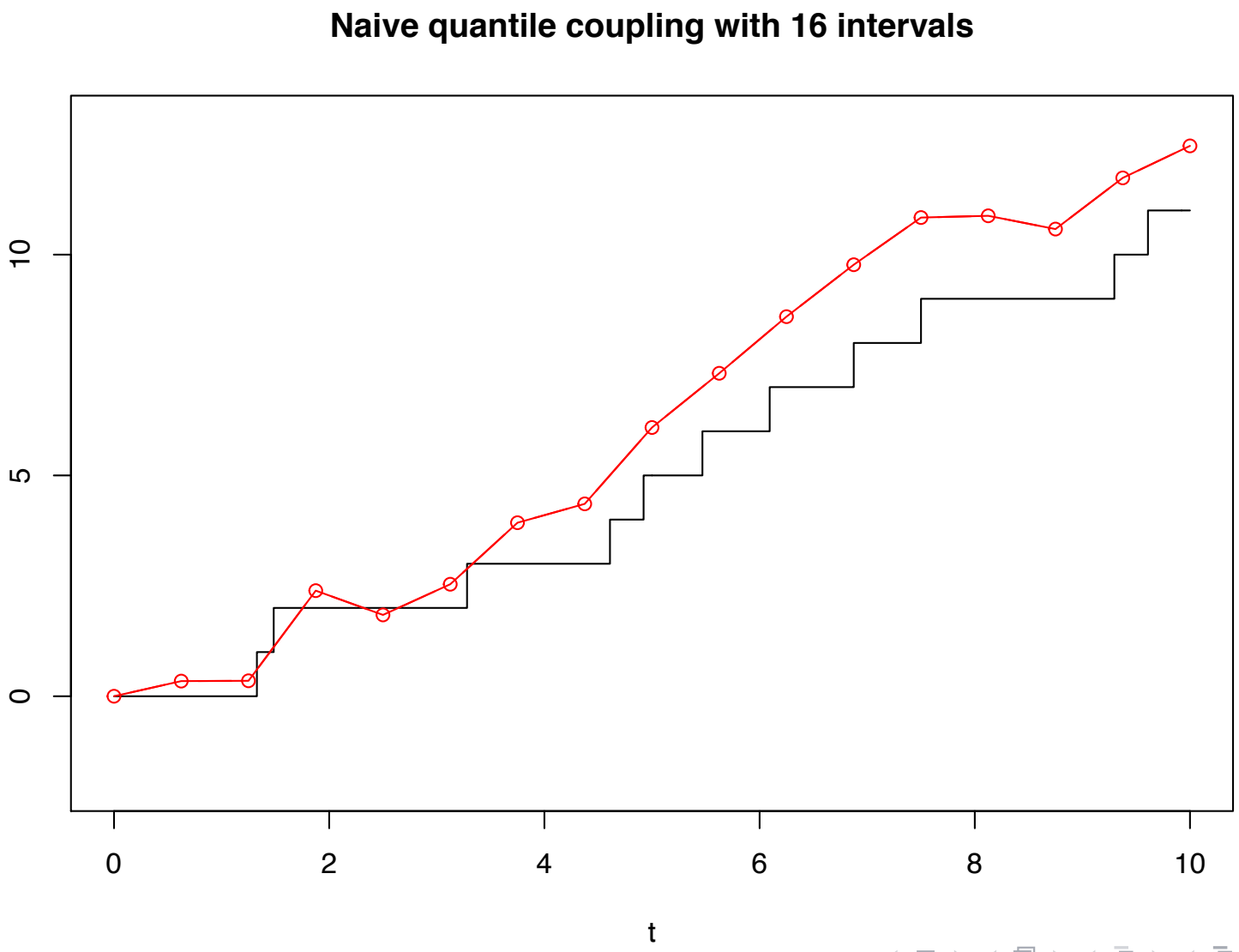

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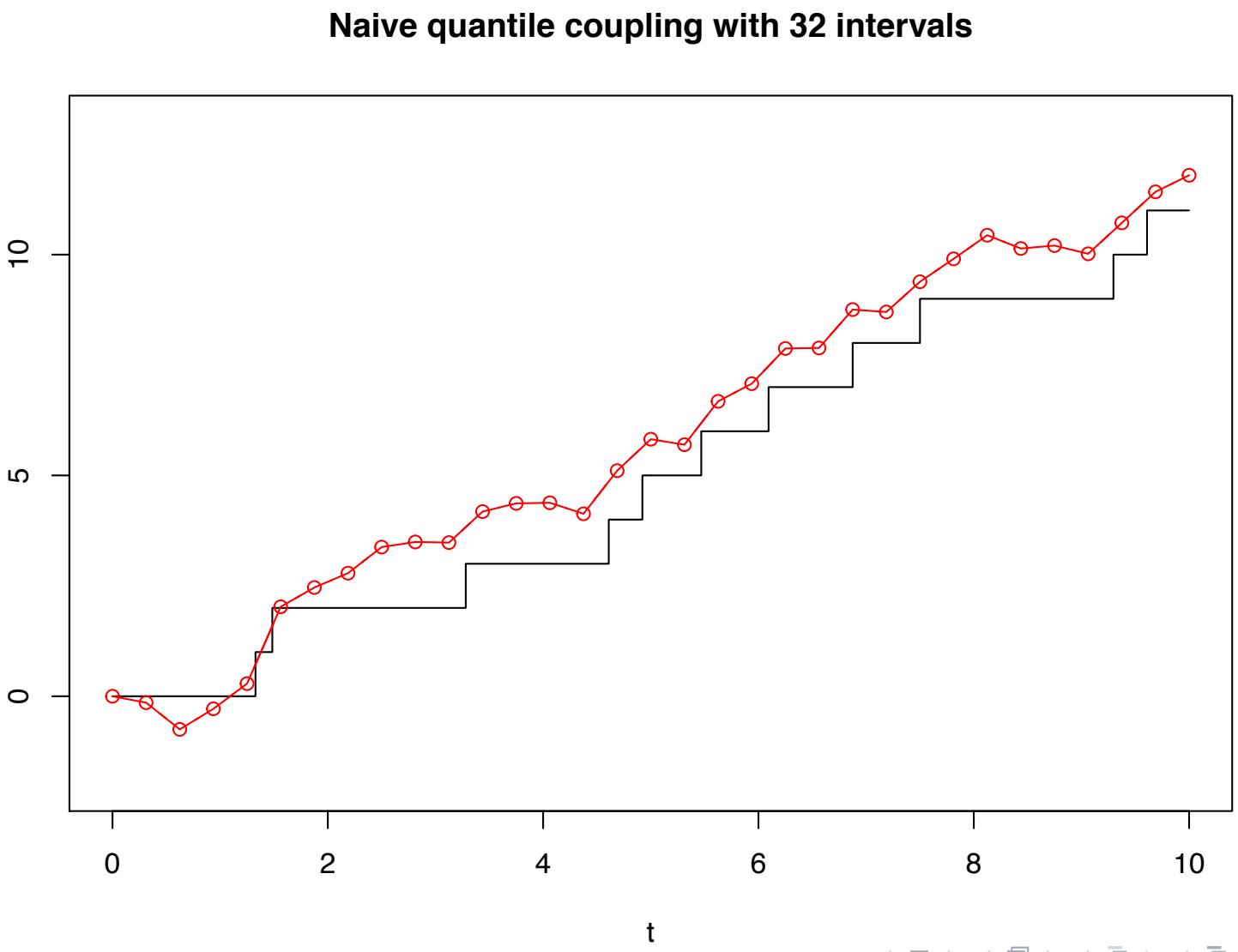

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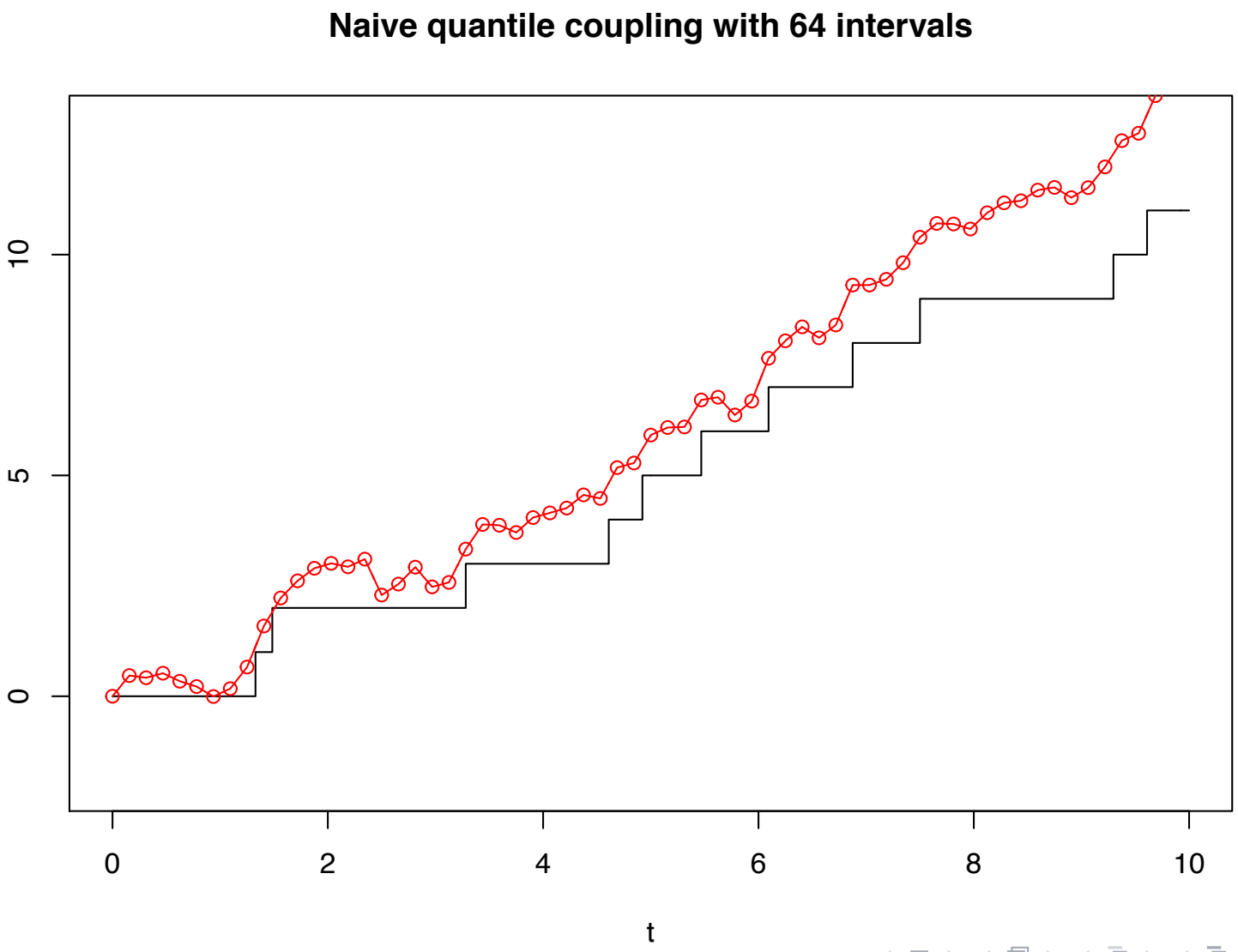
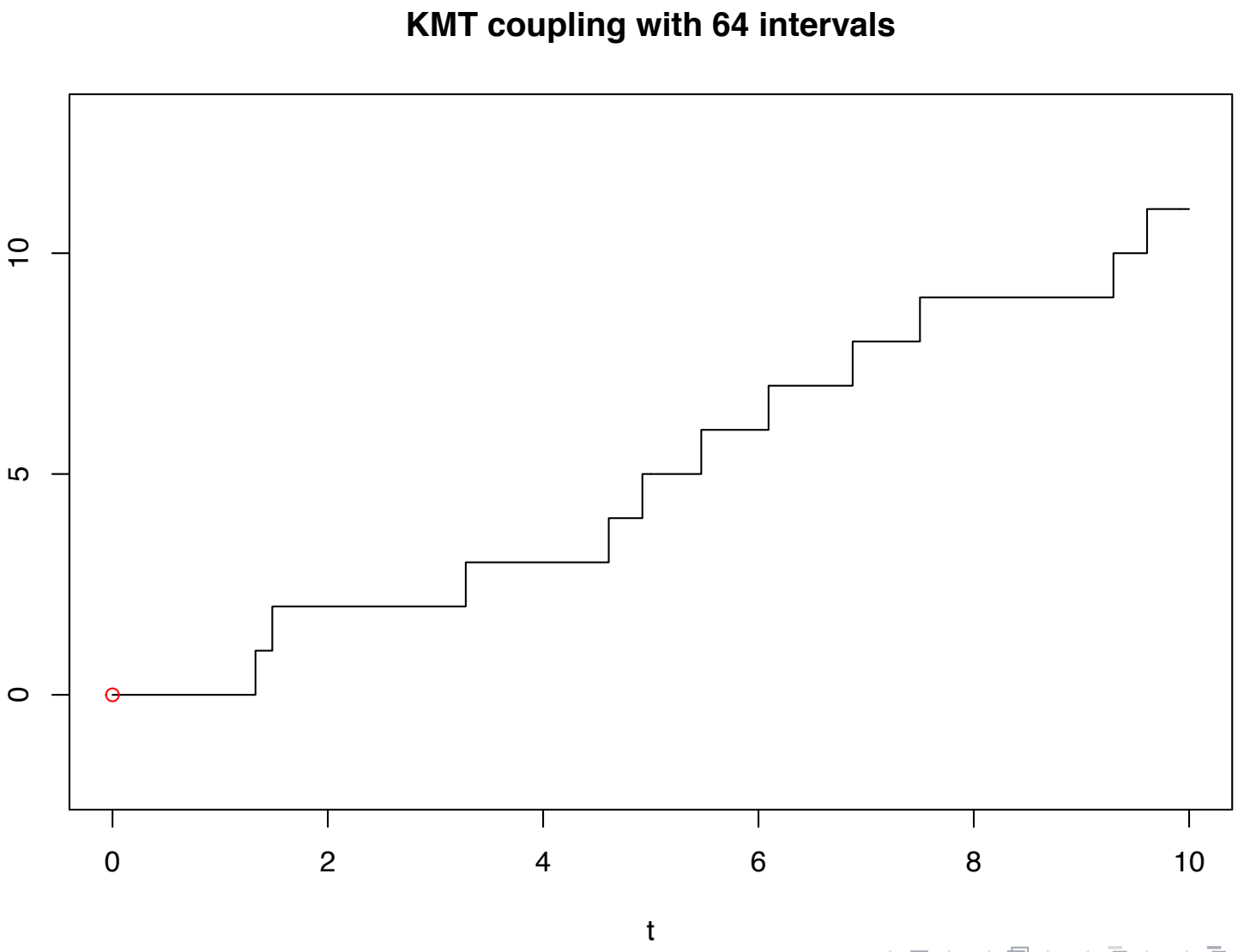


KMT coupling

Apply quantile coupling to larger intervals and refine by dichotomy.



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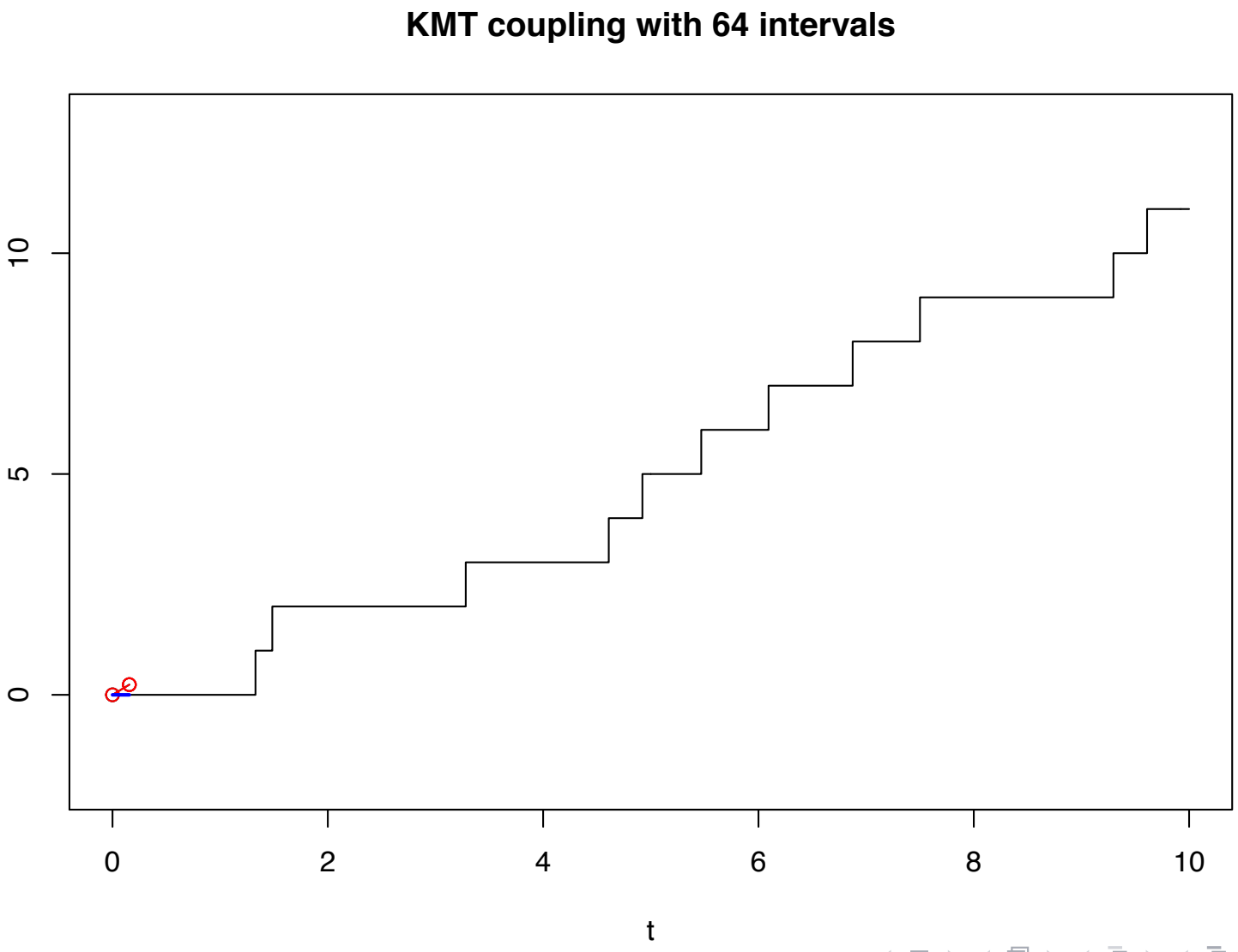

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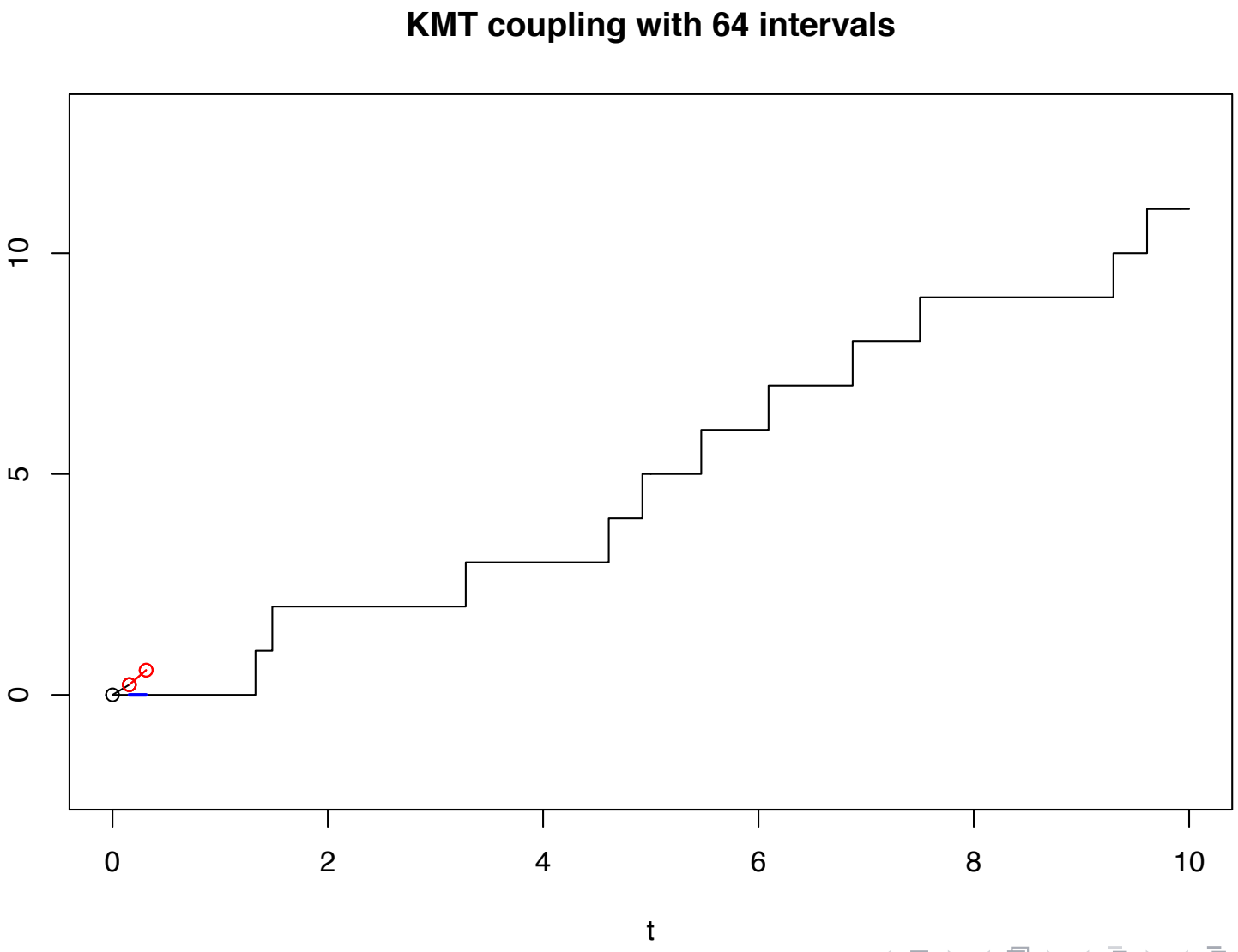

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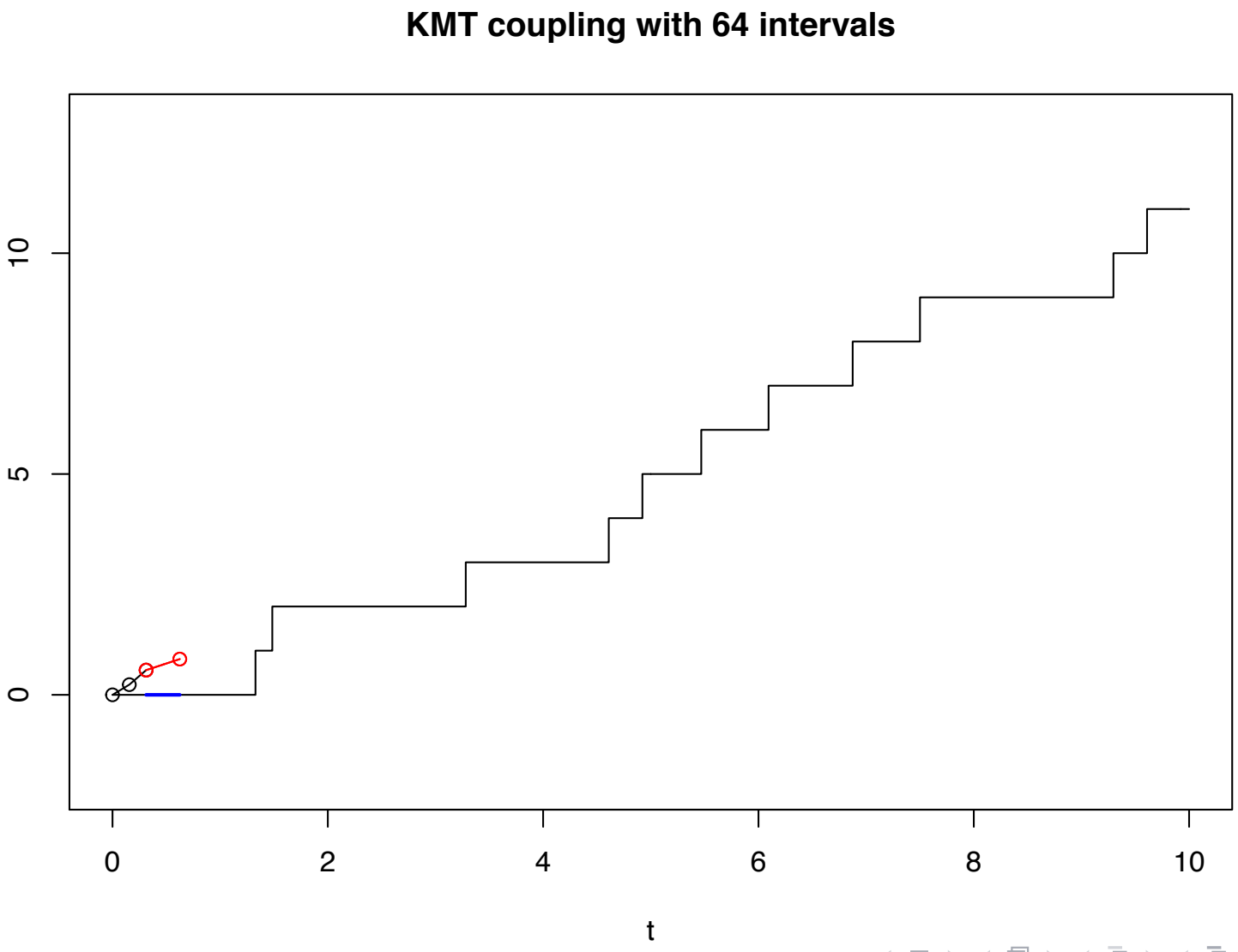

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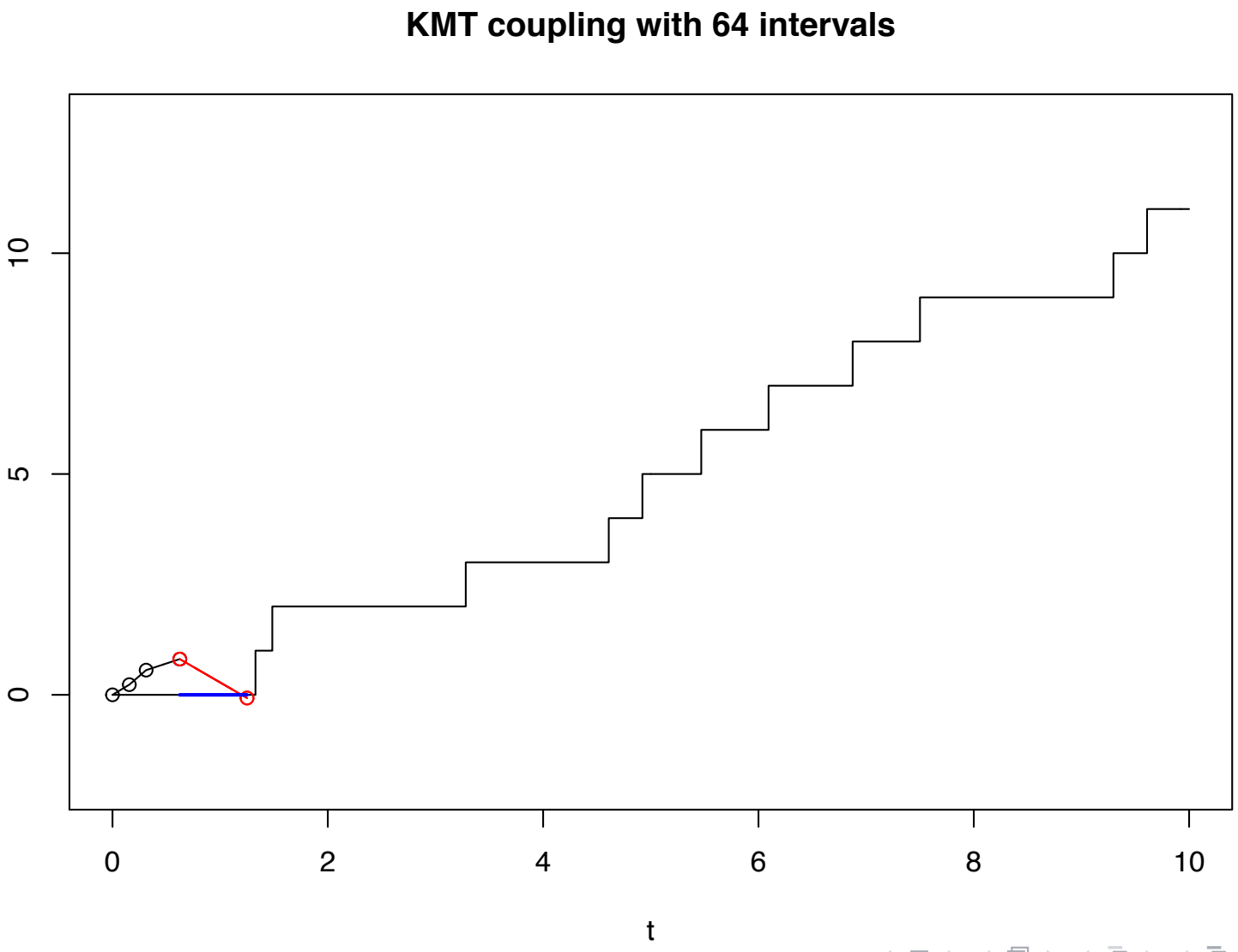

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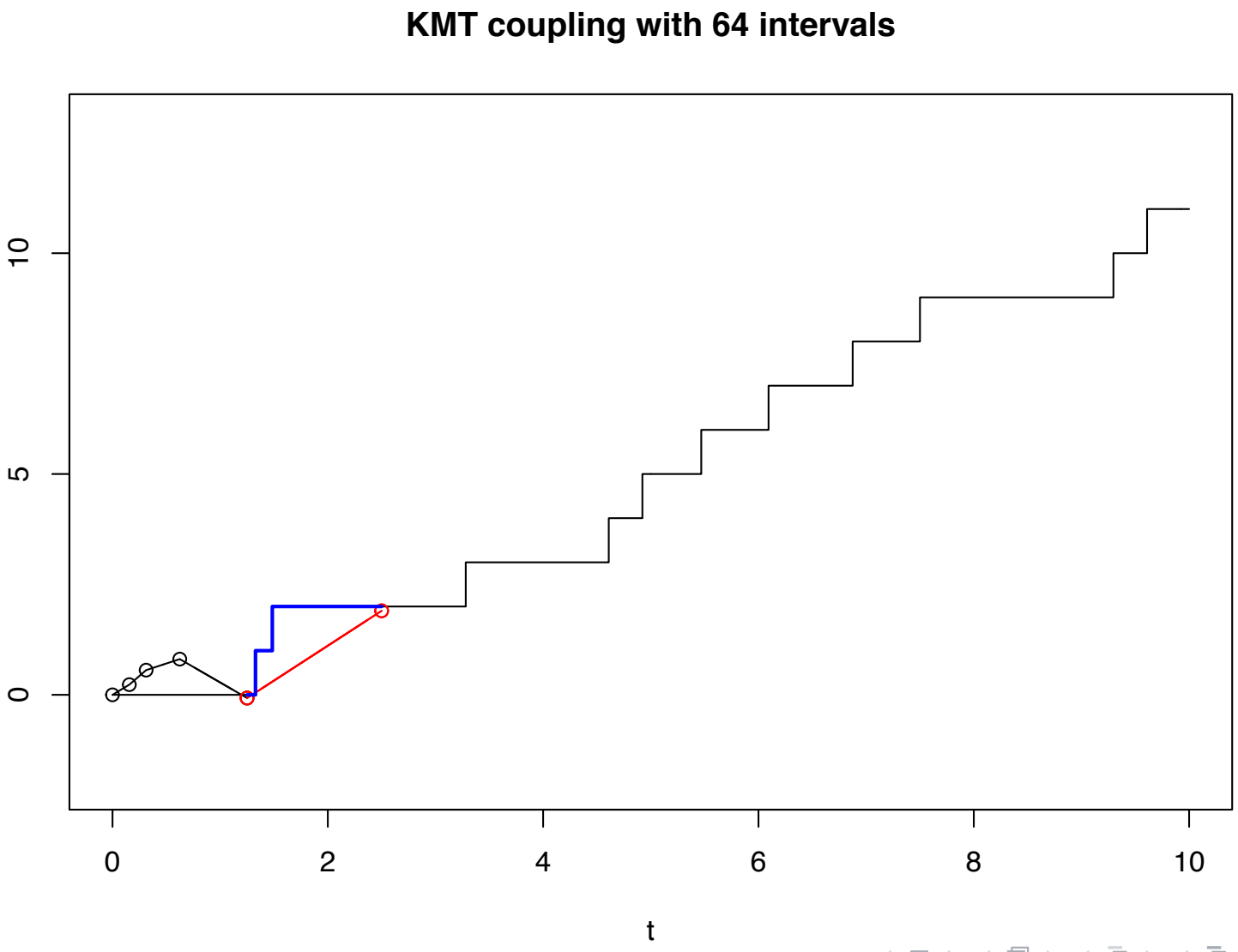

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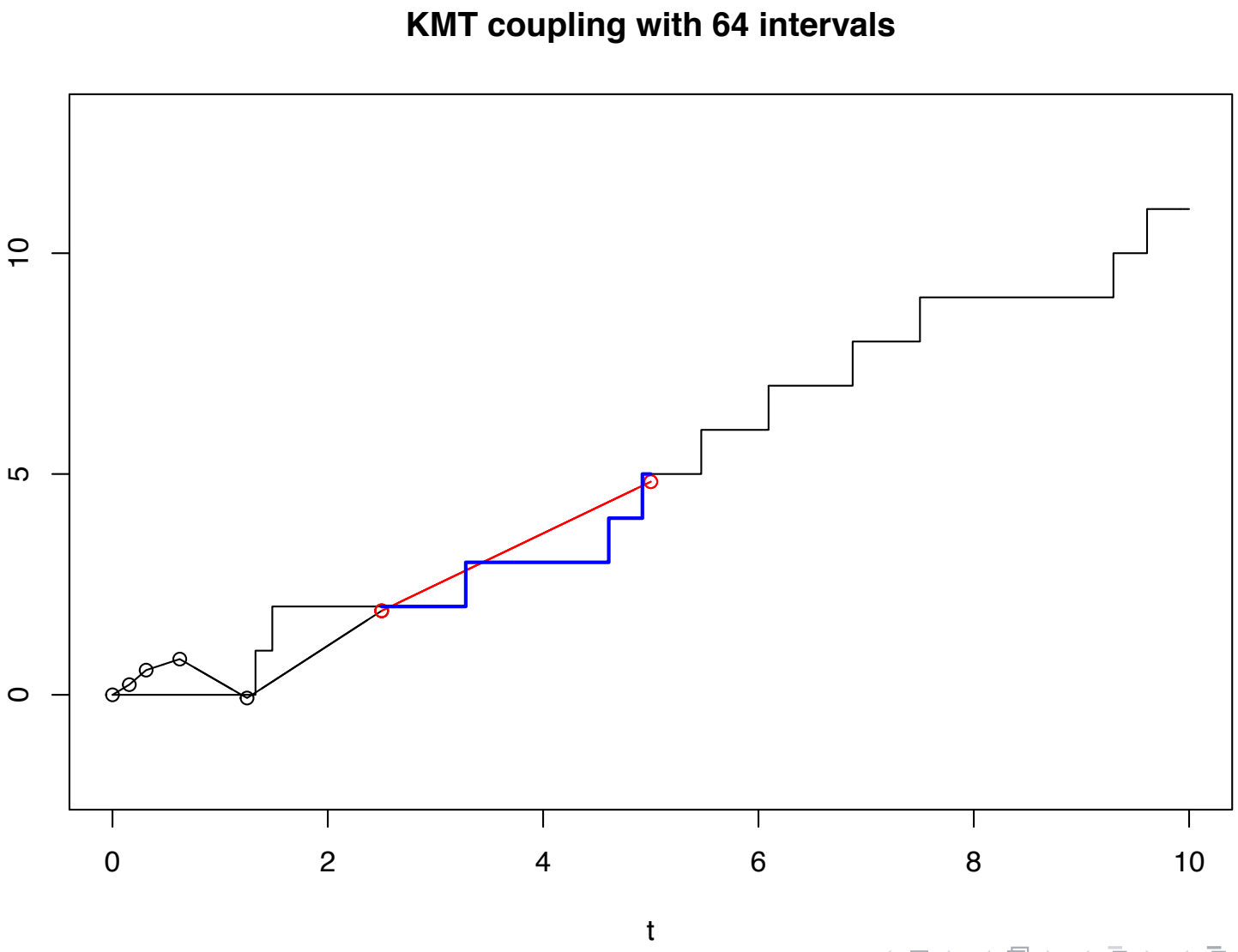

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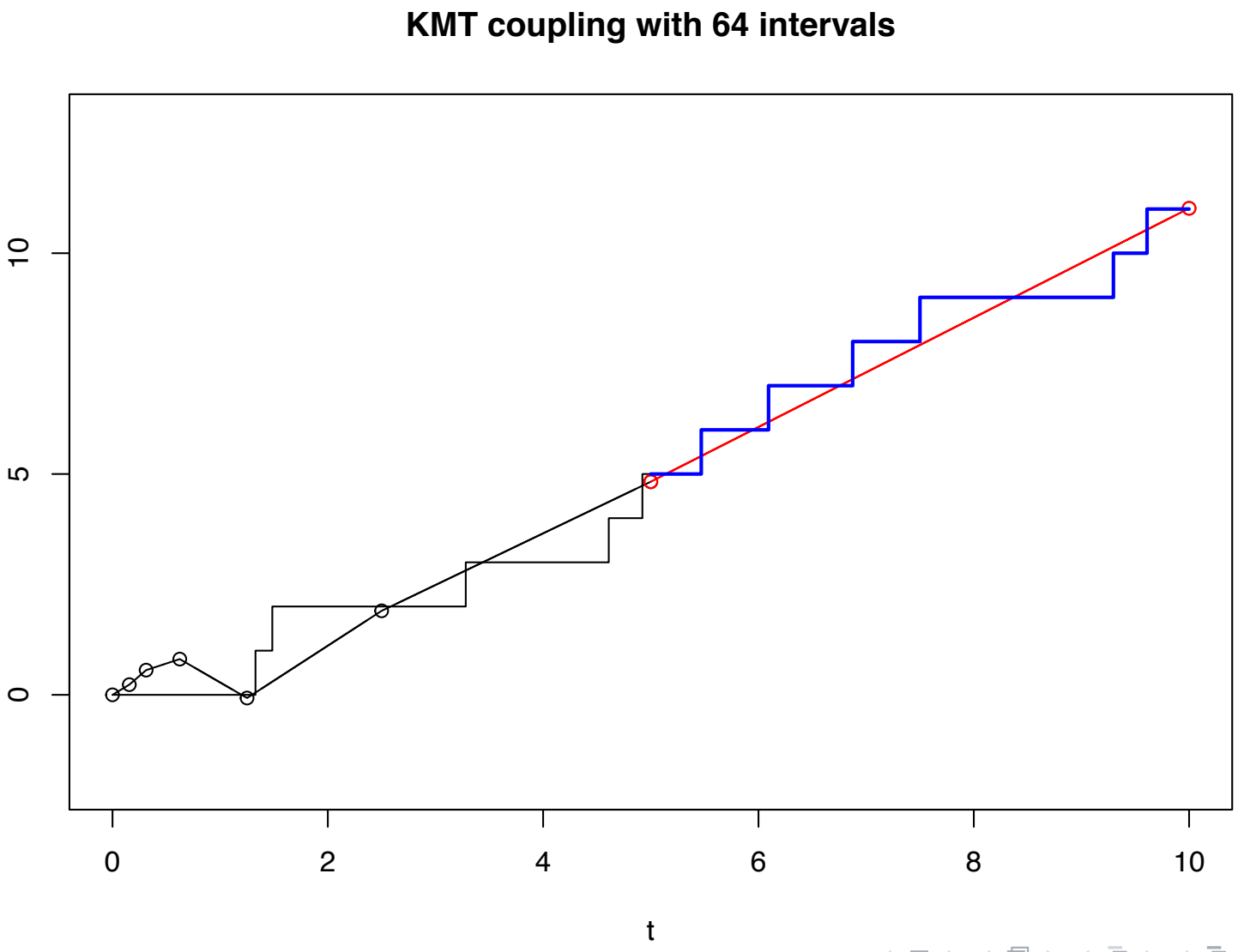

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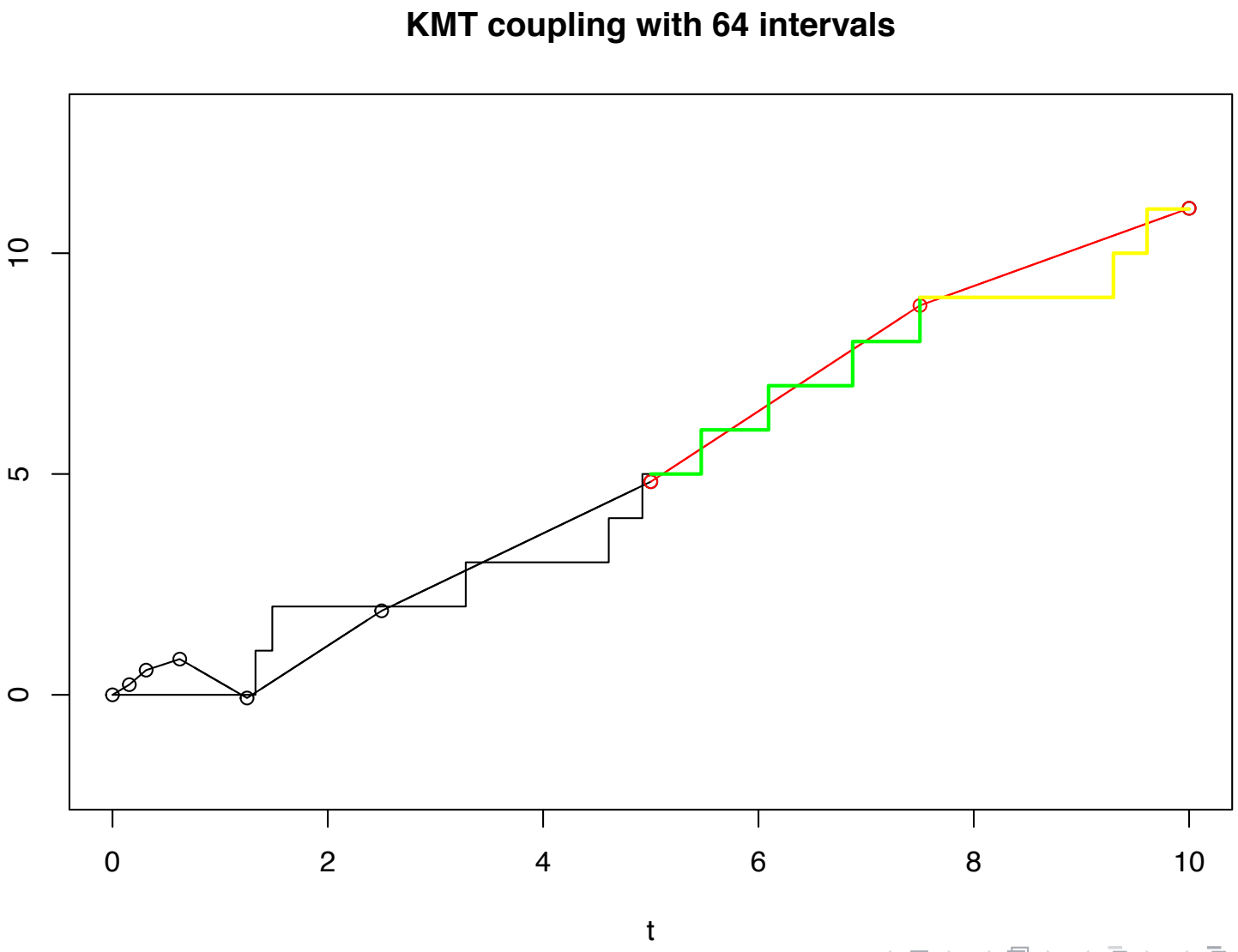

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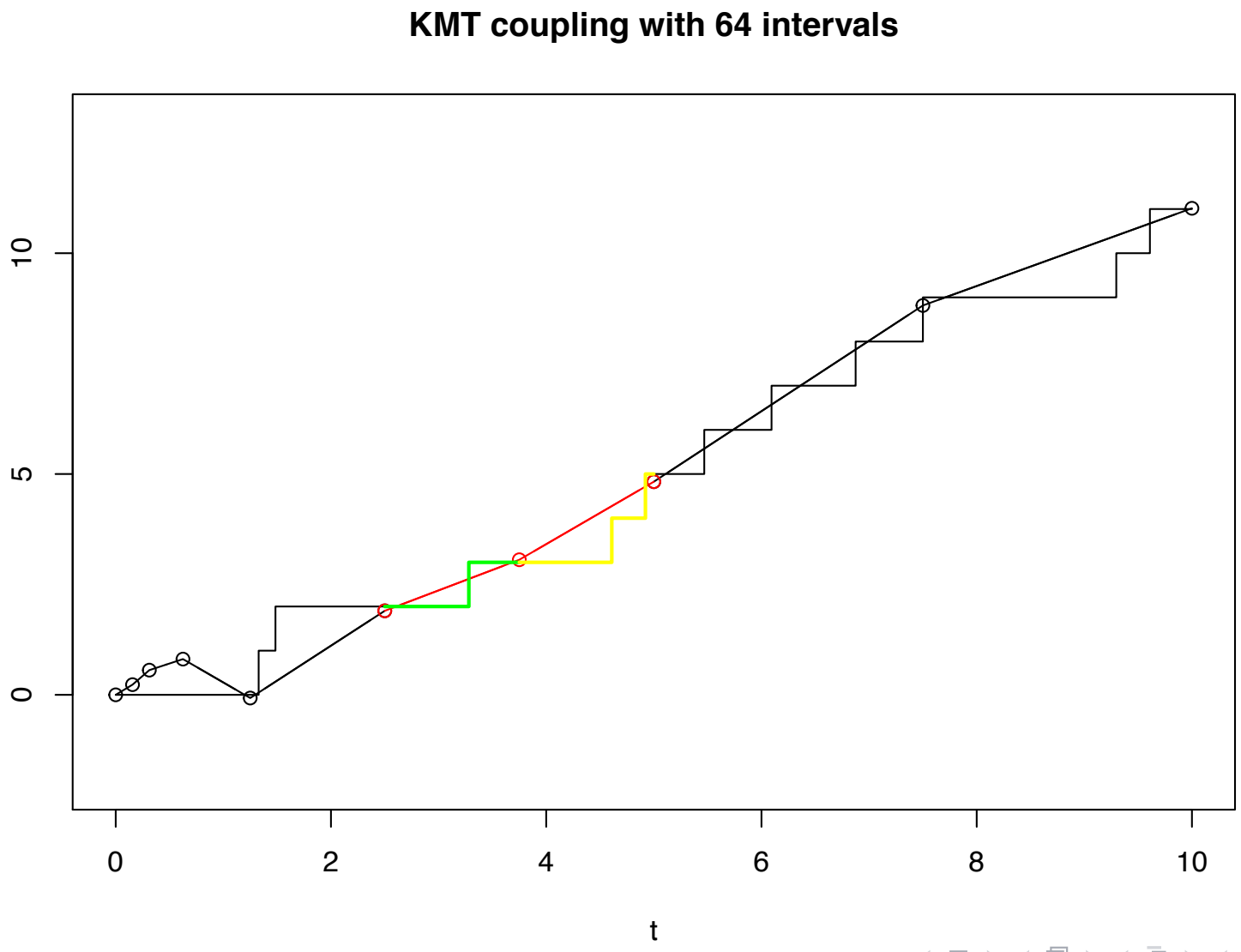

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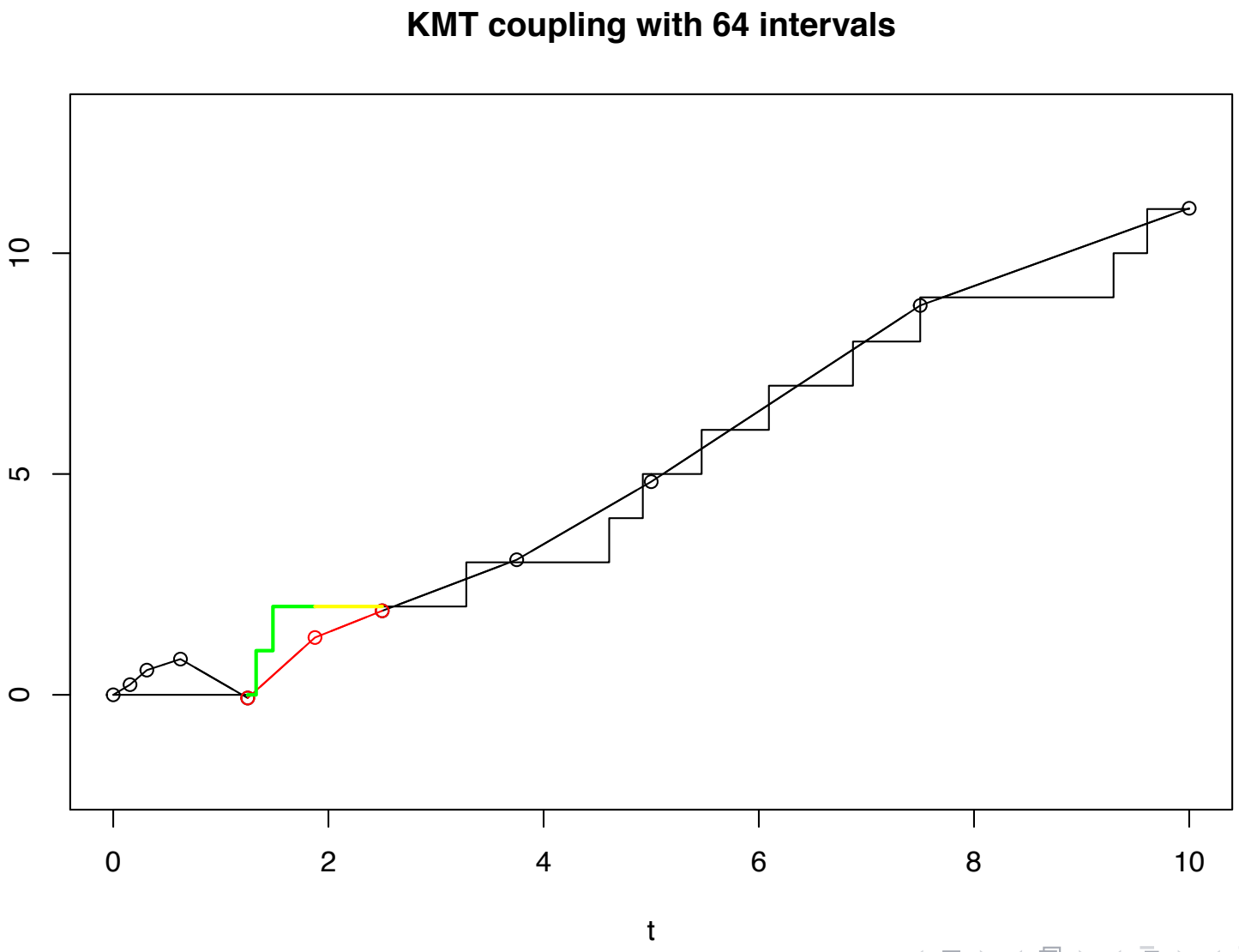

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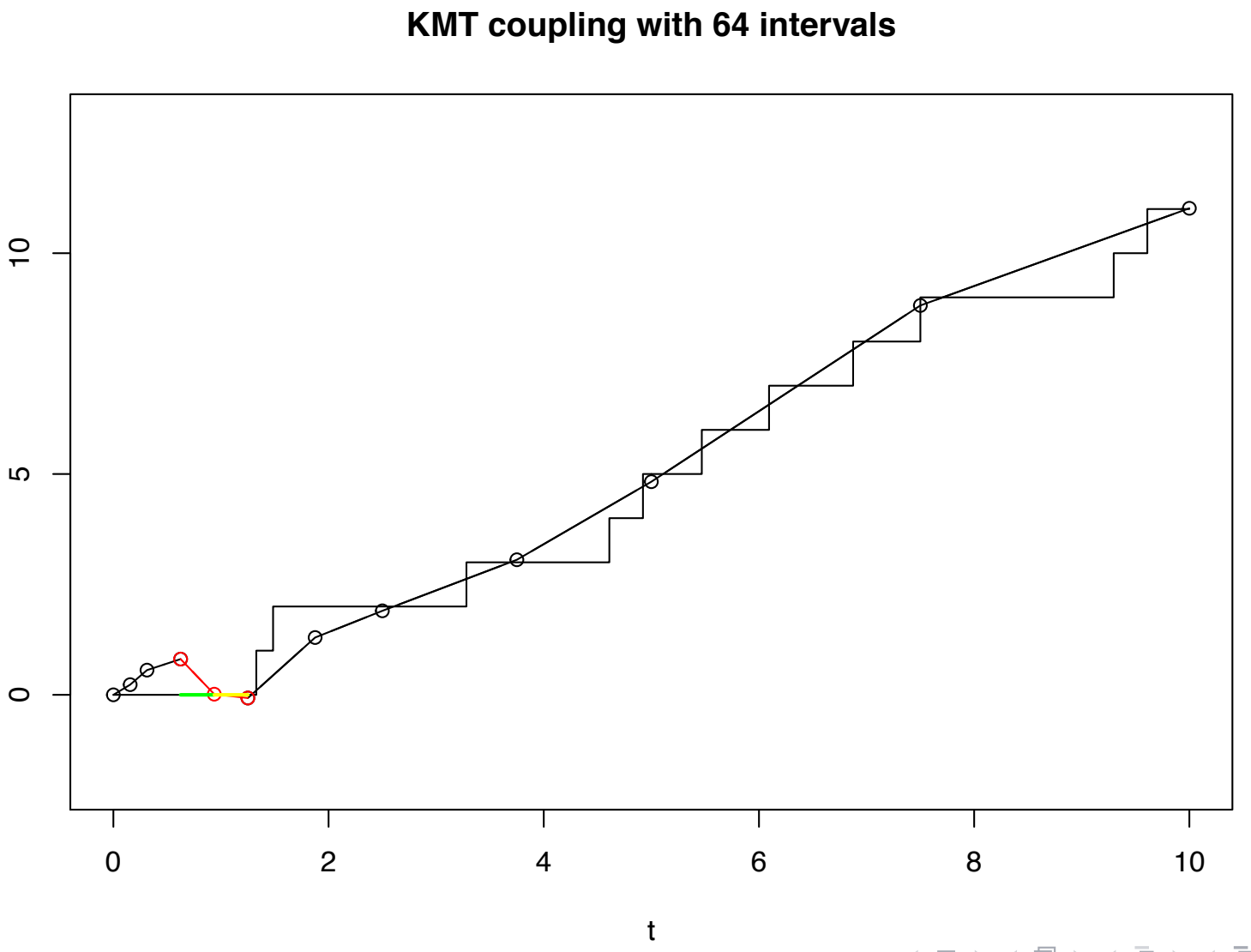

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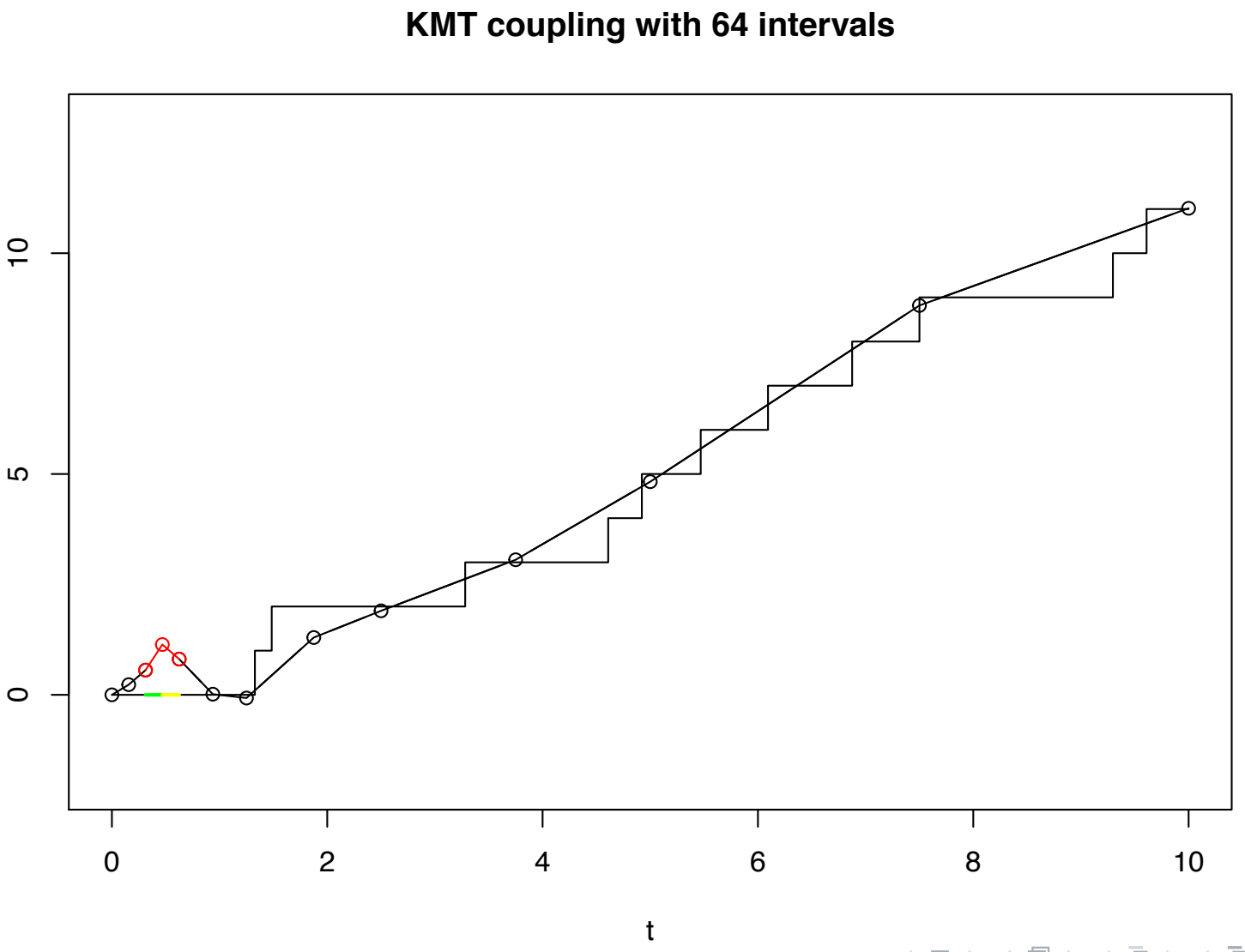

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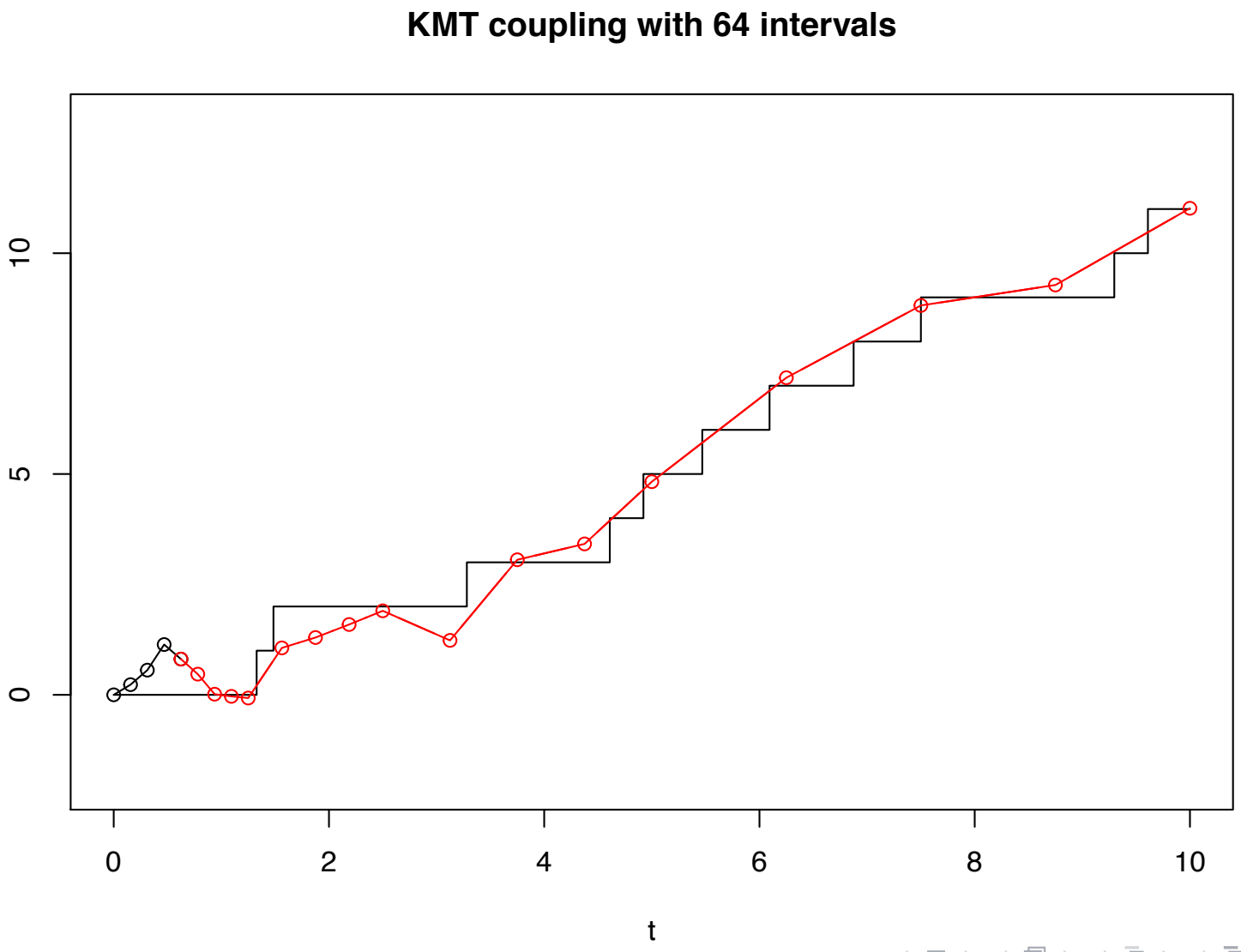

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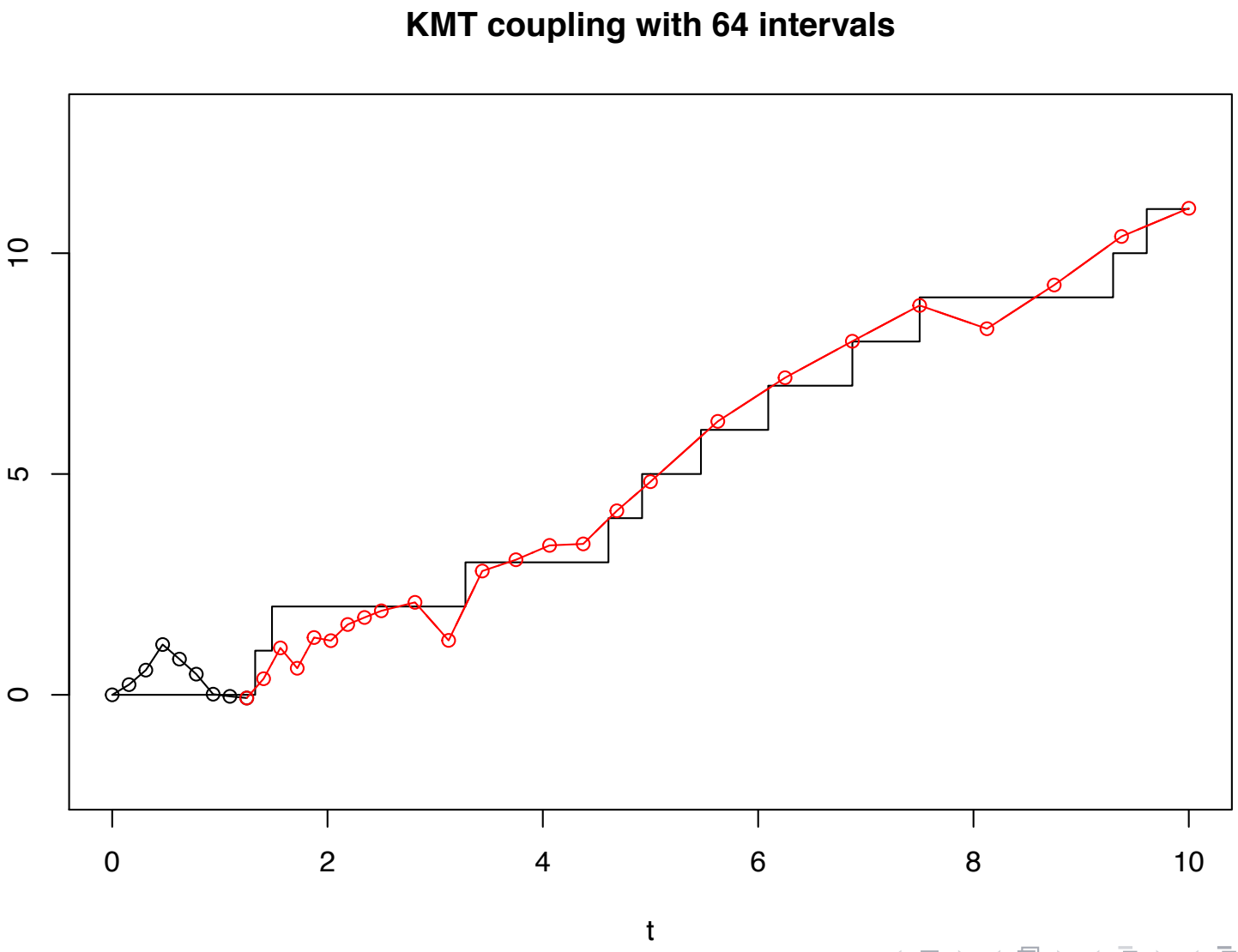

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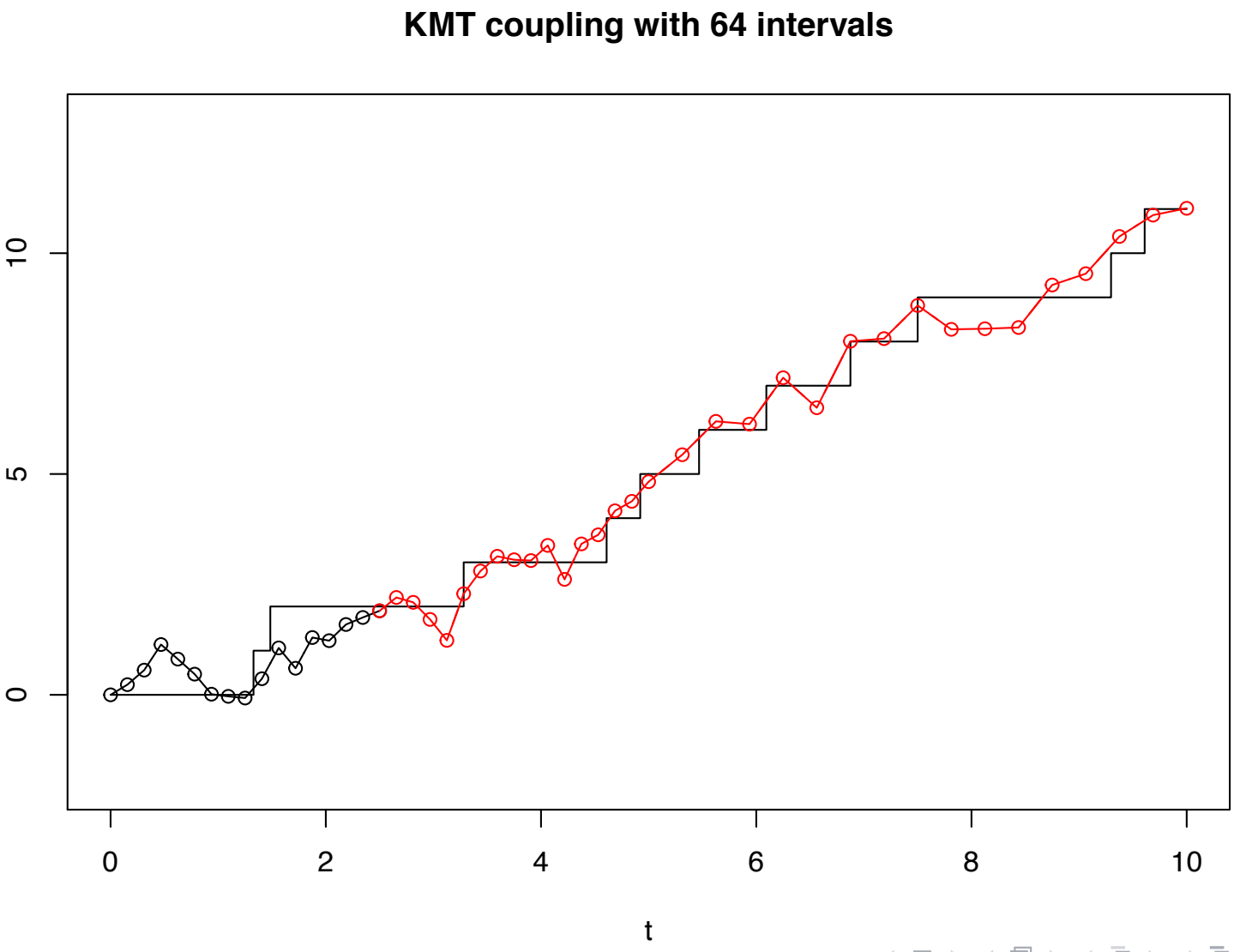

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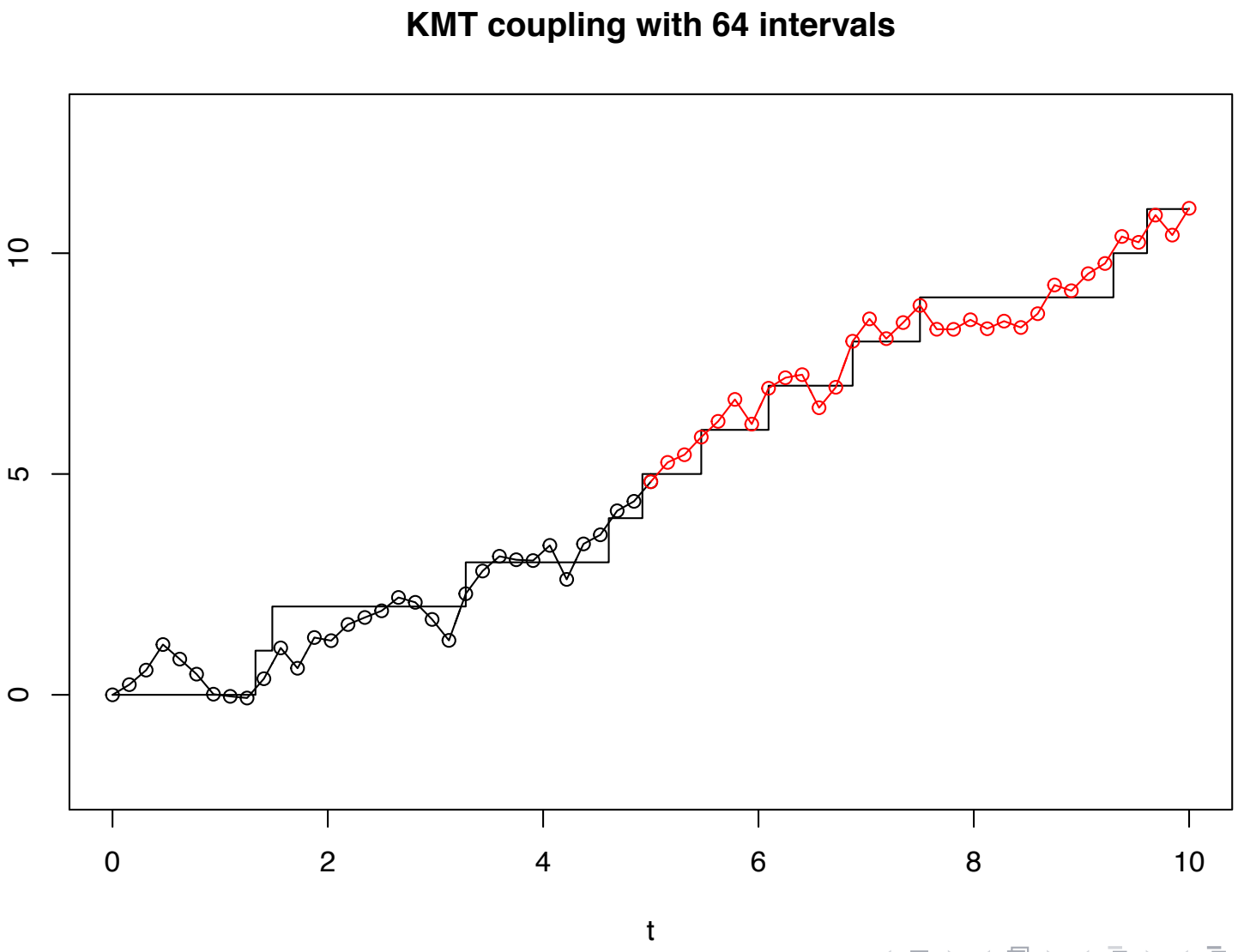

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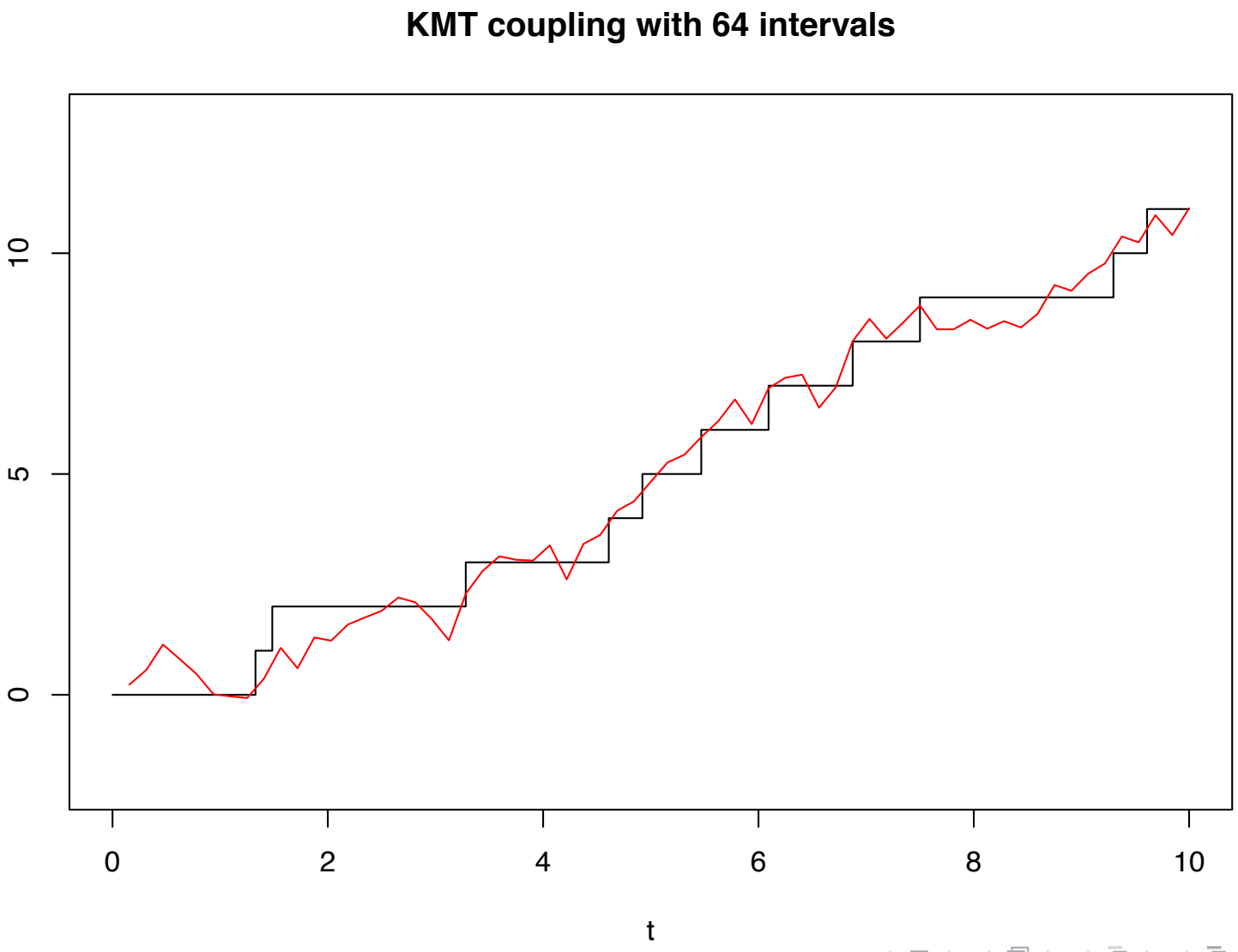
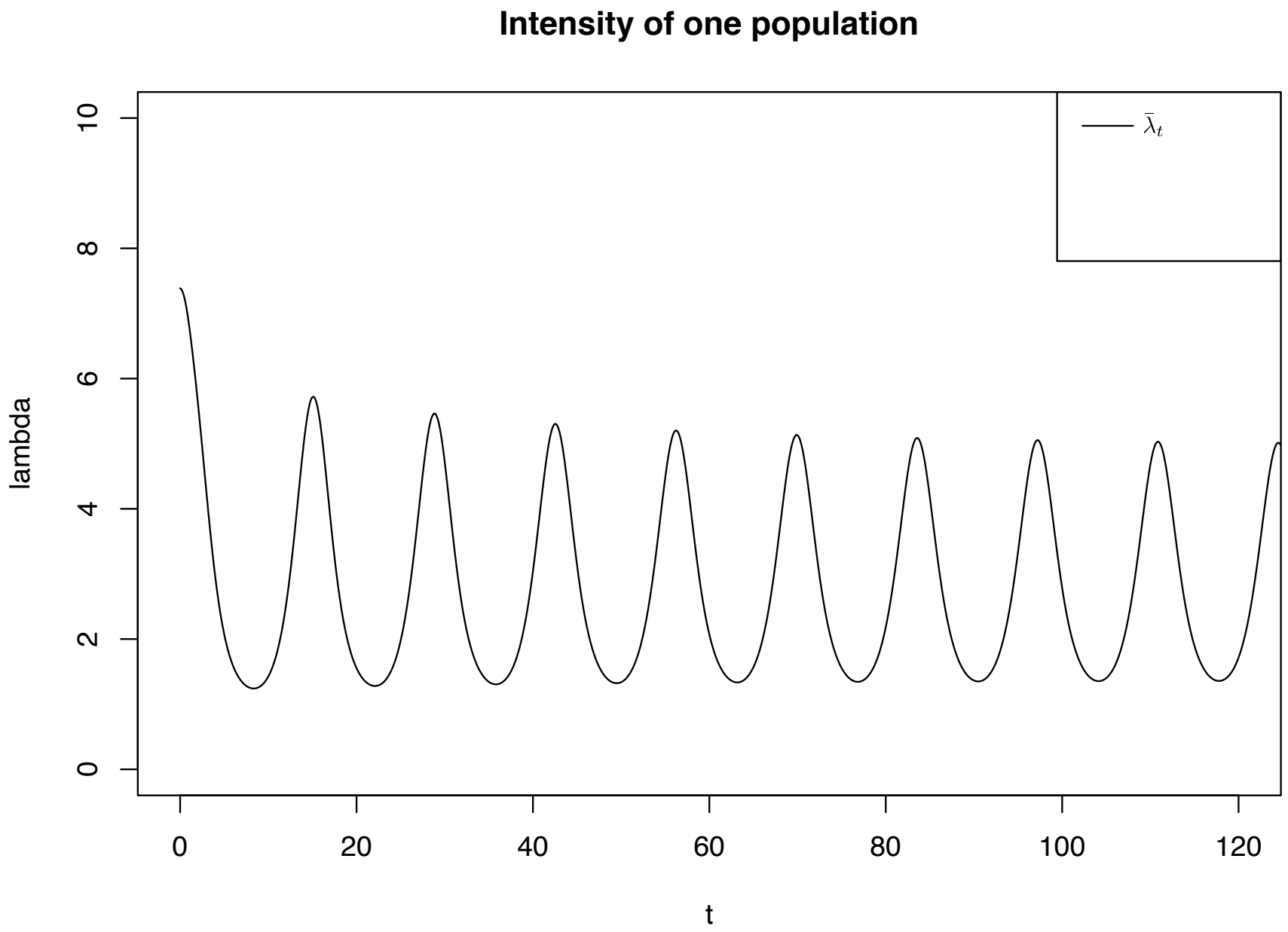


Simulations (oscillatory regime)



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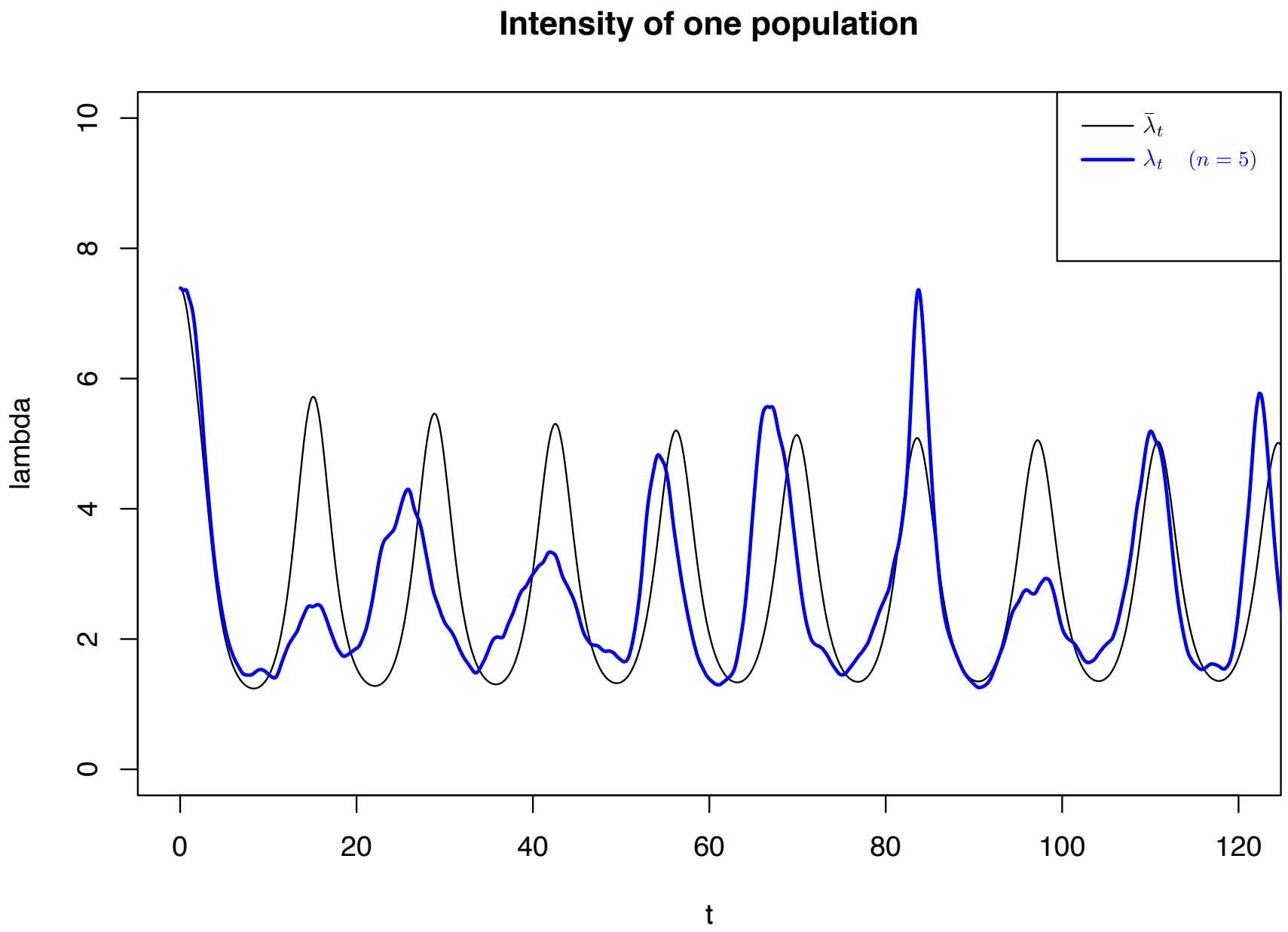

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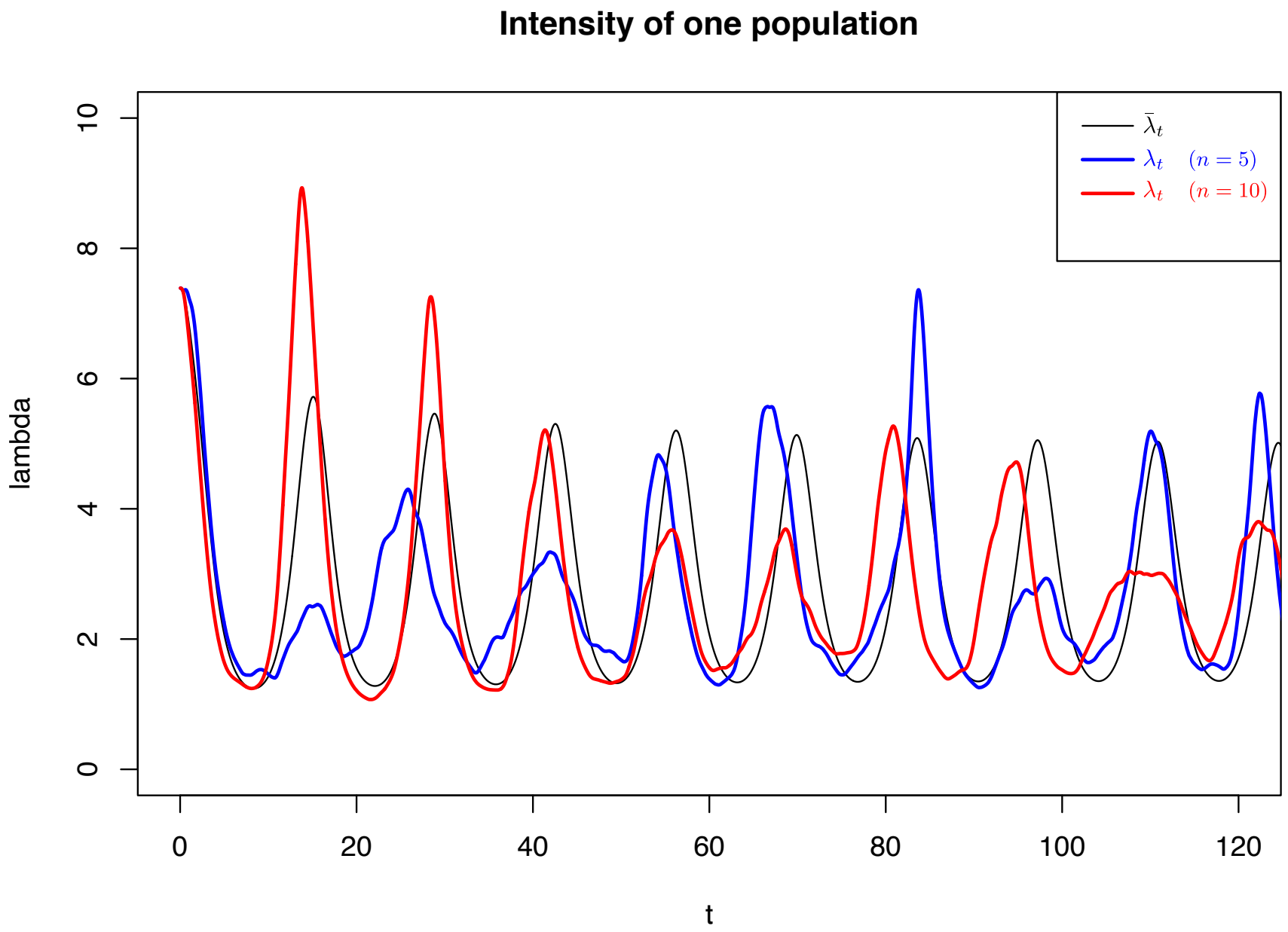

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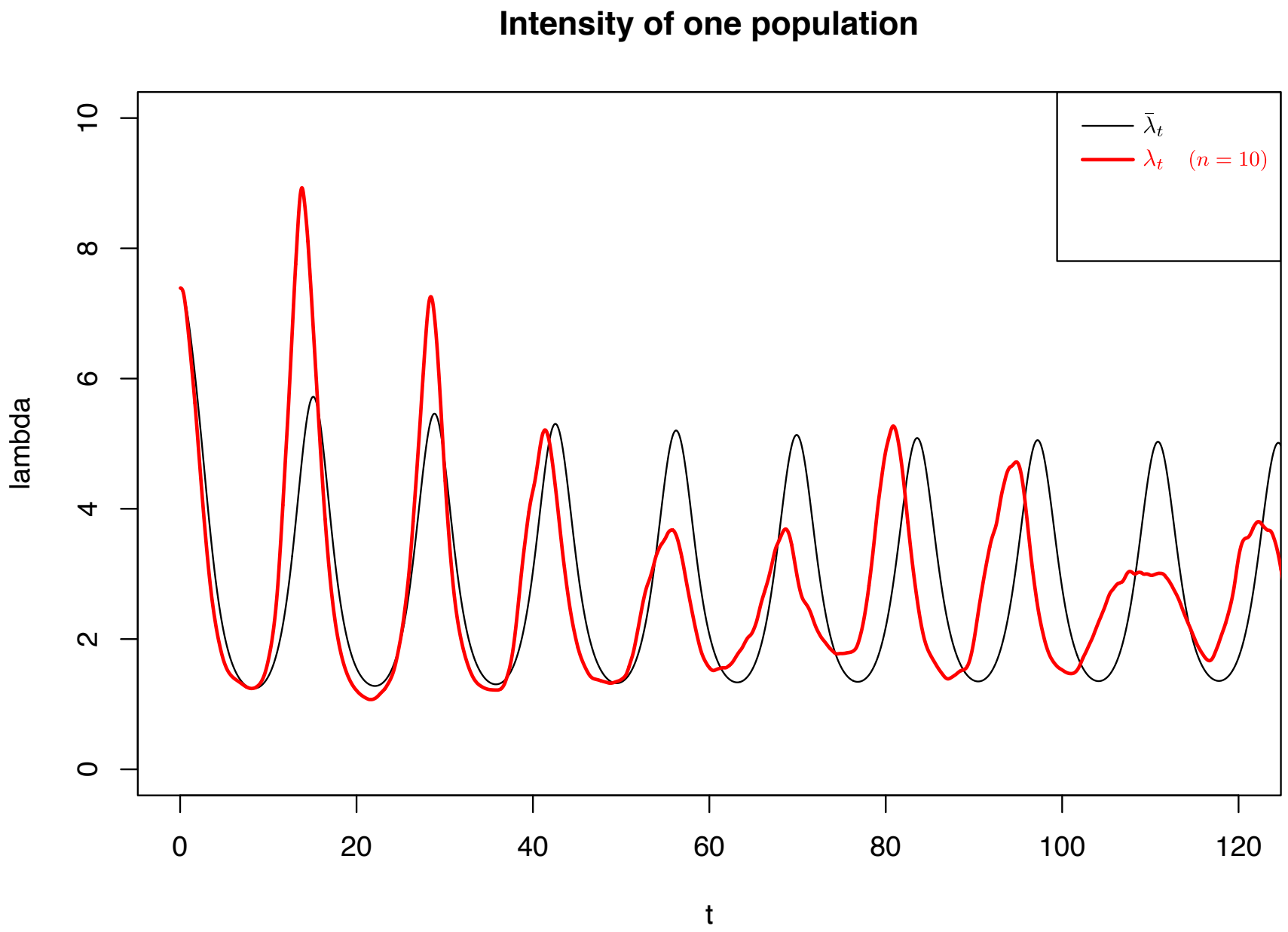

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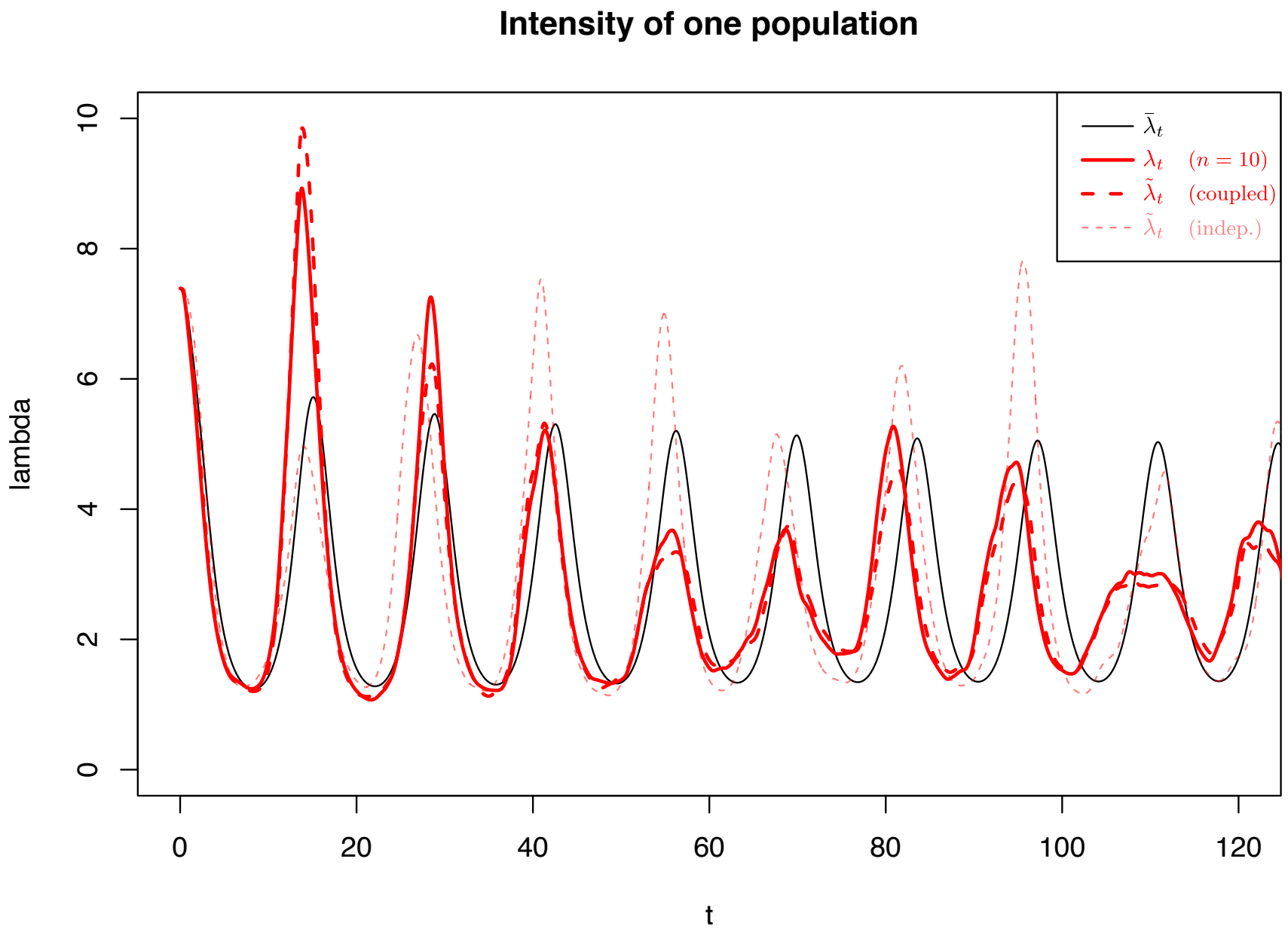
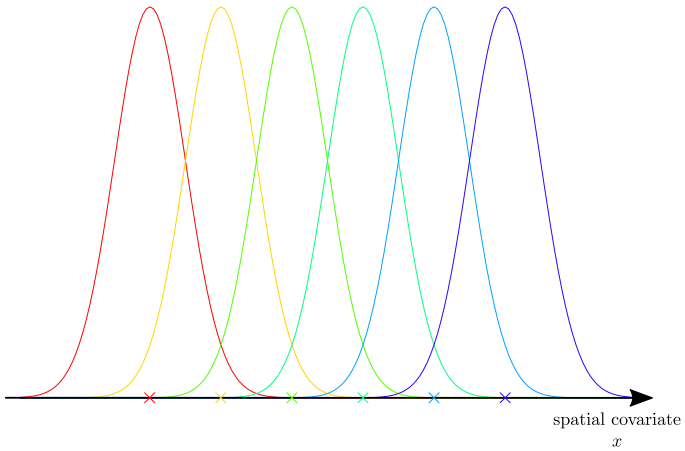


Outline

Two "spatial" structures

Multi-class

(Ditlevsen, Löcherbach, '17)
(Löcherbach, '19)
(Chevallier, Melnykova, Tubikanec, '21)



Continuous spatial covariates

(Chevallier, Duarte, Löcherbach, Ost, '19)
(Chevallier, Ost, '20)

The model - simplified with $x_i = i/n$

- Each neuron $i = 1, \dots, n$ is located at $x_i = i/n \in [0,1]$.
- A Lipschitz continuous spatial kernel $w : [0,1]^2 \to \mathbb{R}_+$.
- A Lipschitz continuous intensity function f.

For each $i = 1, \dots, n$, the spikes of the i-th neuron are described by N^i with intensity $\lambda_t^i = f(U_t^i)$ where

$$U_t^i := e^{-\alpha t} u_0(x_i) + \int_0^t e^{-\alpha(t-s)} \sum_{j=1}^n w\left(x_j, x_i\right) N^j(ds).$$

Exponential simplification

The "potential" variable U_t^i satisfies :

$$\begin{cases} dU_t^i = -\alpha U_t^i + \sum_{j=1}^n w\left(x_j, x_i\right) N^j(dt), \\ U_0^i = u_0(x_i). \end{cases}$$

Micro model

- $x_i = i/n$: (static) position of i.

$$\boxed{\text{Intensity : } \lambda_t^i := f(U_t^i)}$$

$$\begin{cases} dU_t^i = -\alpha U_t^i dt + n^{-1} \sum_{j=1}^{n} w(x_j, x_i) N^j(dt), \\ U_0^i = u_0(x_i). \end{cases}$$

Notation

$N(x_i) := N^i$ and $U(t, x_i) := U_t^i$.

Macro model (Neural Field Equation)

- ρ : Lebesgue measure on $[0,1]$.

$$\boxed{\text{Limit intensity : } \lambda(t,x) = f(u(t,x))}$$

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$\overline{N}(x_i) \sim PP(\lambda(t, x_i))$.

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Theorem (C. et al., '19)

For all $T > 0$,

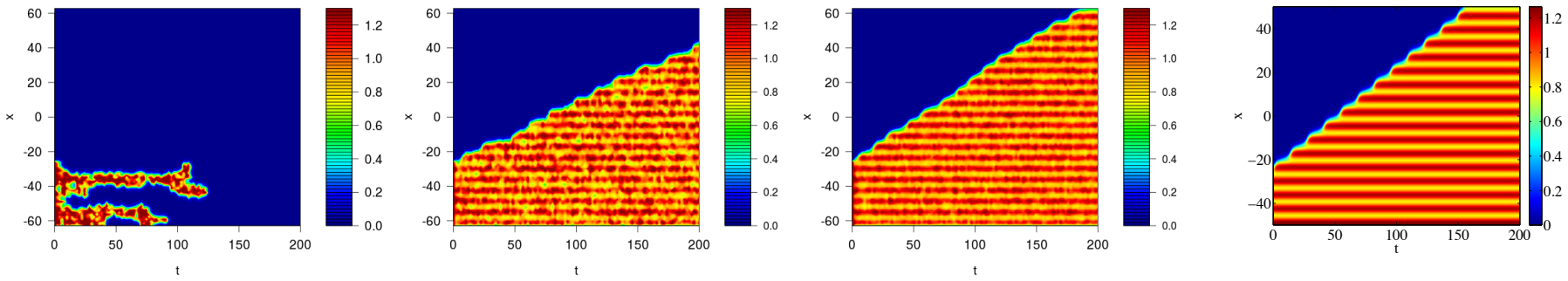
$$\sup_{i=1,\dots,n} \int_0^T |U(t,x_i) - u(t,x_i)| dt \leq C(T) n^{-1/2},$$

$$\sup_{i=1,\dots,n} \mathbb{P}\left(N(x_i)_{[0,T]} \neq \overline{N}(x_i)_{[0,T]} \right) \leq C(T) n^{-1/2}.$$

Simulations

Parameters taken from (Coombes, Laing '11) :

- $\alpha = 1$,
- $f(u) = \mathbf{1}_{[0.3,+\infty[}(u)$,
- $w(x,y) = \exp(-|x-y|/2)\,(1+0.4\sin(y))$.



$n = 100$ $n = 1000$ $n = 5000$ $n = \infty$

Fluctuations study

Fluctuation process

- For each neuron: $\eta_t^i = n^{1/2}\{U(t,x_i) - u(t,x_i)\}$.
- As a distribution: $\Gamma_t^n(dx) = n^{-1}\sum_{i=1}^n \eta_t^i \delta_{x_i}(dx)$.

Additional assumptions

The positions are $x_i = i/n$, f is $\mathscr{C}^\infty(\mathbb{R})$ and lower bounded, $u_0 \in \mathscr{S} = \mathscr{C}^\infty([0,1])$ and $w \in \mathscr{C}^\infty([0,1]^2)$.

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Proposition

- The unique solution of the macro model is in fact smooth (belongs to $\mathscr{C}([0,T],\mathscr{S})$).
- For any $\varphi \in \mathscr{S}$, $\Gamma_t^n(\varphi) = n^{-1}\sum_{i=1}^n \eta_t^i \varphi(x_i)$ satisfies : $(\Gamma^n(\varphi))_n$ is tight in $\mathscr{D}([0,T],\mathbb{R})$.
- $(\Gamma^n)_{n\geq 1}$ is tight in $\mathscr{D}([0,T],\mathscr{S}')$.

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- $(\Gamma^n)_{n\geq 1}$ is tight in $\mathscr{D}([0,T],\mathscr{S}')$.
- Any possible limit Γ is the unique solution of a linear Stochastic PDE.

Non-linear PDE + linear SPDE – Mesoscopic model

$$\begin{cases} \partial_t u = -\alpha u + \int_0^1 w(y,x) f(u(t,y))dy, \\ u(0,x) = u_0(x). \end{cases} \tag{NFE}$$

Linear SPDE

$\hat{U}^n(t,\cdot) := u(t,\cdot) + n^{-1/2}\Gamma_t$ satisfies

$$d_t\hat{U}^n = \left\{ -\alpha\hat{U}^n + \int_0^1 w(y,x)f(u(t,y))dy + n^{-1/2}\Gamma_t\left(w(\cdot,x)f'(u(t,\cdot))\right)\right\}dt + \int_0^1 \sqrt{\frac{w(y,x)^2 f(u(t,y))}{n}}\, W(dt,dy),$$

where W is a centred Gaussian random field.

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Heuristics for a non-linear SPDE

$$\partial_t u = -\alpha u + \int_0^1 w(y,x) f(u(t,y)) dy, \tag{NFE}$$

- The contribution of each location y is $n \times \frac{1}{n} w(y,x) \times f(u(t,y))$: expectation of an empirical mean of n Poisson variables of parameter $f(u(t,y))$.
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Non linear SPDE – Mesoscopic model

$$d_t U^n(t,x) = \left\{ -\alpha U^n(t,x) + \int_0^1 w(y,x) f(U^n(t,y)) dy \right\} dt + \int_0^1 \sqrt{\frac{w(y,x)^2 f(U^n(t,y))}{n}} W(dt,dy)$$

Summary/Comparison

The two SPDE systems can be written as

$$d_t \hat{U}^n(t,x) = -\alpha \hat{U}^n(t,x)dt + \hat{A}^n(dt,x).$$

with

$$\hat{A}^n(dt,x) := \int_0^1 w(y,x)\left[f(u(t,y)) + f'(u(t,y))\frac{\Gamma(t,y)}{\sqrt{n}}\right]dy\,dt + \int_0^1 w(y,x)\frac{\sqrt{f(u(t,y))}}{\sqrt{n}}W(dt,dy),$$

or

$$A^n(dt,s) := \int_0^1 w(y,x)f(U^n(t,y))dy\,dt + \int_0^1 w(y,x)\frac{\sqrt{f(U^n(t,y))}}{\sqrt{n}}W(dt,dy).$$

- Remind that $\hat{U}^n(t,x) := u(t,x) + n^{-1/2}\Gamma(t,x)$.

Theorem (C. and Ost '20)

$$\sup_{t\le T, x\in[0,1]} \mathbb{E}\left[\left|\hat{U}^n(t,x) - U^n(t,x)\right|^2\right]^{1/2} \le C(T)n^{-1}.$$

Take home message

- Zeroth-order approximation $\rightsquigarrow$ non linear ODE or PDE.
- First-order approximation $\rightsquigarrow$ standard CLT, coupled PDE and linear SPDE, non linear SPDE.
- KMT/change-time coupling complements Thinning/Sznitman coupling.

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- Outlook :
  - KMT coupling in higher dimensions. Spatio-temporal Hawkes processes.
  - Finite-size effects perturbs the travelling wave solutions of the NFE: can we predict propagation/no propagation on the mesoscopic model ?