An interacting neuronal network with inhibition: theoretical analysis and perfect simulation

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Workshop : New trends on Hawkes processes

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Introduction

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When a neuron spikes, this spike changes the rate at which other neurons spike. In other words, neurons excite and/or inhibit each other.

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When the neuron i spikes,

- The current state of inhibition of neuron i is replaced by a new value Y^i independently of anything else with distribution F^i.
- The state of inhibition of any neuron $j \neq i$ is increased by a positive value $W_{i \to j}$ at time t.

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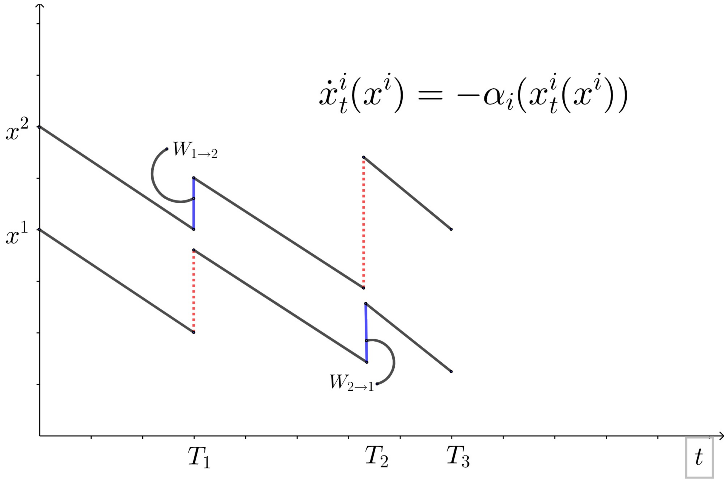

Description of the model

We consider the piecewise deterministic Markov process (PDMP) $X_t^N = (X_t^{1,N}, \cdots, X_t^{N,N}) \in \mathbb{R}_+^N$. For $i \in \{1, \cdots, N\}$, the dynamic of $X_t^{i,N}$ is given by:

$$dX_t^{i,N} = -\alpha_i \left(X_{t-}^{i,N}\right) dt + \int_0^\infty \int_0^\infty (y^i - X_{t-}^{i,N}) \mathbf{1}_{\left\{r \leq \beta_i\left(X_{t-}^{i,N}\right)\right\}} M^i\left(dt, dr, dy^i\right) + \sum_{j \neq i} W_{j \to i} \int_0^\infty \int_0^\infty \mathbf{1}_{\left\{r \leq \beta_j\left(X_{t-}^{j,N}\right)\right\}} M^j\left(dt, dr, dy^j\right), \quad (1)$$

where M^i is a random Poisson measure with intensity $dtdrF^i(dy)$ and for all i, the M^i are all independent.

Infinitesimal generator

The infinitesimal generator associated with this model is given by:

$$G^N V(x) = -\sum_{i=1}^{N} \alpha_i(x^i)\frac{\partial}{\partial x^i}V(x) + \sum_{i=1}^{N} \beta_i(x^i)\int_0^{\infty} F^i(dy^i)[V(x + e^i y^i - e^i x^i + \sum_{j \neq i} e^j W_{i\to j}) - V(x)] \quad (2)$$

where V is a smooth function and e^i is the $i-th$ unit vector.

First jump time

Let S_1 be the first jump time of the first neuron to jump. For all $t > 0$,

$$\mathbb{P}(S_1 > t) = \prod_{i=1}^{N} e^{-\int_0^t \beta_i(x_s^i(x^i))ds}. \tag{3}$$

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Moreover, if $t < \min_i t_0(x^i)$ where

$$t_0(x^i) := \int_0^{x^i} \frac{dy}{\alpha_i(y)}$$

is the time for the neuron i hit 0 starting from x^i,

First jump time

We can write

$$\mathbb{P}(S_1 > t) = \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(x_t^i(x^i))]},$$

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Then, $\mathbb{P}(S_1 = \infty) = 0$ if and only if

Assumption 1

$\Gamma_i(0) = -\infty$ *for all* $1 \leq i \leq N$

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- We explore the past in order to determine all sets of indices and times which affect the value of neuron i at time 0. The set of all such couples (j, s) will be called the clan of ancestors of neuron i. (see Galves and Löcherbach [2], Galves et al. [3])

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- We consider the case where each neuron i has exactly two neighbours so that the neuron i interacts only with the neurons $i + 1$ and $i - 1$.

- T is the time vector.

Algorithm (backward procedure)

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- We simulate , $\forall\, l \in \mathbb{Z}$, $N_t^{l,s}$ and $N_t^{l,p}$ two Poisson processes with respective intensities β_* and $\beta^* - \beta_*$.

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- The jump times $T_n^{l,s}$ will be considered as times of sure jumps (counted by the process $N_t^{l,s}$) and the jump times $T_n^{l,p}$ will be considered as times of possible jumps (counted by the process $N_t^{l,p}$)

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- Let $i \in \mathbb{Z}$ fix and

$$T_1 = \inf\{\, T_1^{i\pm 1,s},\, T_1^{i\pm 1,p},\;\; T_1^{i,s}\;\; T_1^{i,p}\}.$$

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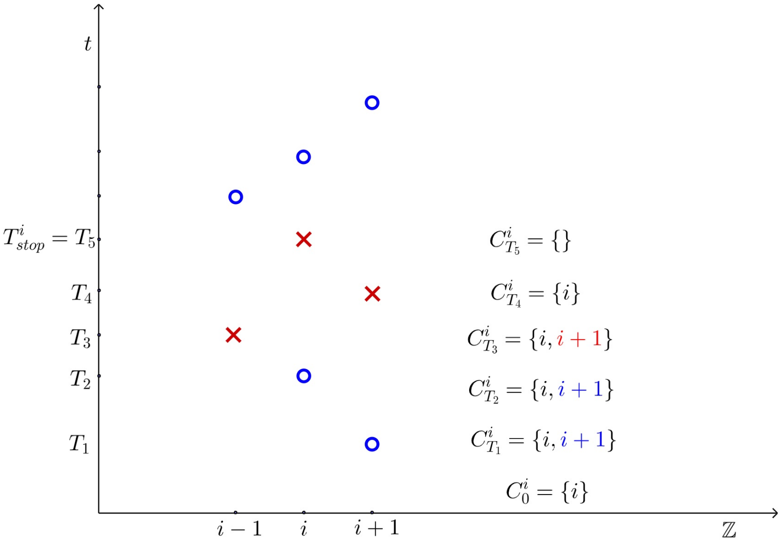
- If $T_1 = T_1^{i,s}$, we set $C^i_{T_1} = \emptyset$ and we stop the algorithm. We put $I_1 = i$.

Algorithm (backward procedure)



Algorithm (backward procedure)

Suppose T_n is the $n-$th jump time of $C^i_{T_n}$. We have:

$$T_{n+1} = \inf\{\, T^{j,s}_m, T^{j,p}_m > T_n : |j - C^i_{T_n}| \leq 1,\ T^{k,s}_m > T_n,\ k \in C^i_{T_n}, \}.$$

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$$\begin{cases} \text{If } j \in C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \\ \text{If } j \notin C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \cup \{j\} \end{cases}$$

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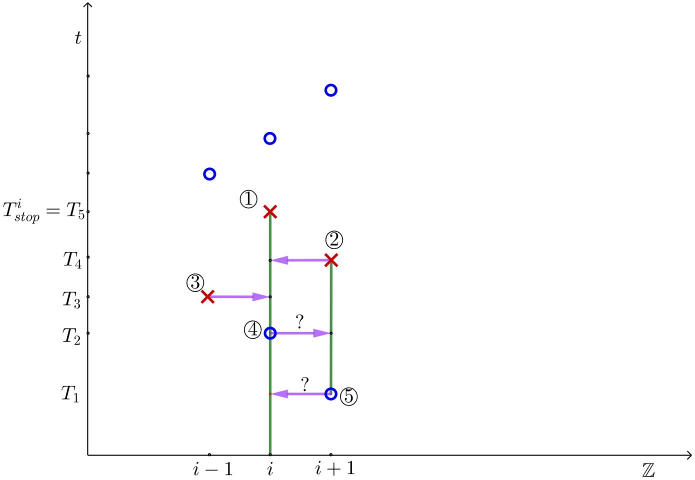
- If $T_{n+1} = T^{k,s}_m$ we set:

$$\begin{cases} \text{If } k \in C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \setminus \{k\} \\ \text{If } k \notin C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \end{cases}$$

Forward procedure



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The procedure will stop at time T^i_{stop} where $T^i_{stop} = \inf\{t : C^i_t = \emptyset\}$.

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Theorem 1

We set $\delta = \frac{\beta_*}{\beta^* - \beta_*}$. *There exists a critical value* $0 < \delta_c < \infty$ *such that:*

- *if* $\delta > \delta_c$, *then the extinction time is finite almost surely that is,* $\mathbb{P}(\forall i,\ \ T^i_{stop} < \infty) = 1$
- *if* $\delta < \delta_c$, *then the extinction time is infinite with a positive probability that is,* $\mathbb{P}(\forall i,\ \ T^i_{stop} = \infty) > 0$.

Sketch of proof

- We first show that $T^i_{stop} < +\infty$ almost surely for sufficiently large δ.

- We can upper bound $|C^i_t|$ by Z_t almost surely for all $t \geq 0$ where $Z_0 = 1$ and $(Z_t)_{t\geq 0}$ is a branching process.

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- The associated infinitesimal generator of $(Z_t)_{t\geq 0}$ as follows :

$$Af(n) = n[(\beta^* - \beta_*)(f(n+1) - f(n)) + \beta_*(f(n-1) - f(n))].$$

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- If we take, $f(n) = n$, then, for $\delta > 1$, we have

$$Af(n) = -cf(n)$$

where $-c = (\beta^* - \beta_*)(1 - \delta)$.

Sketch of proof

- Assume $x_t = \mathbb{E}(f(Z_t))$ and using the Itô formula, we have:

$$x_t = x_0 + \mathbb{E}\int_0^t Af(Z_s)ds = x_0 - c\int_0^t x_s ds = x_0 e^{-ct}.$$

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- When $t \to \infty$, we have $x_t \to 0$.

- If $\delta > 1$, $\mathbb{P}(T_{stop}^i < \infty) \geq \mathbb{P}(\lim_{t\to\infty} Z_t = 0) = 1$ thus ensuring that $\delta_c \leq 1$.

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We show that for all $\delta < \delta_c,\ T^i_{stop} = +\infty$ with positive probability.

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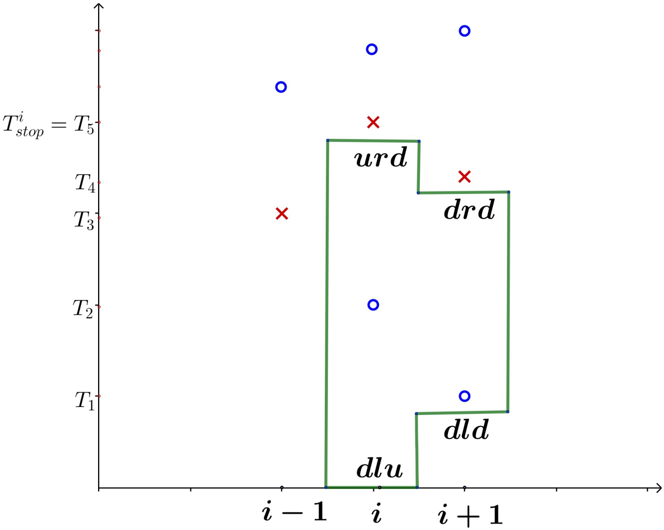
We show that for all $\delta < \delta_c,\ T^i_{stop} = +\infty$ with positive probability.

$T^i_{stop} < \infty$ if and only if $\bar{\mathcal{C}}^i = \cup_{t \geq 0} C^i_t$ is a finite set.

- We show that $\mathbb{P}(T^i_{stop} < \infty) = \mathbb{P}(|\bar{\mathcal{C}}^i| < \infty) < 1$ for sufficiently small values of δ using classical contour techniques. (see Ferrari et al.[1])

- For this, on $|\bar{\mathcal{C}}^i| < \infty,$ we draw the contour of $\bar{\mathcal{C}}^i$ as follow.

Sketch of proof/Contour



Sketch of proof

- Set $N(dlu), N(urd), \cdots$ the number of appearances of the different direction vectors.

$$N(dru) = N(urd) - 1 \leq n/2, \;\; N(drd) + N(dru) + N(uru) + N(urd) = n.$$

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- For $n = 1$, the probability of appearance of a contour of length 4 is equal to $\mathbb{P}(D_1 = urd) \leq \delta$.

Sketch of proof

- For $n = 2$, the probability of appearance of a contour of length 8 is equal to

$$\mathbb{P}(D_1 = ulu,\ D_2 = urd,\ D_3 = drd) + \mathbb{P}(D_1 = ulu,\ D_2 = uru,\ D_3 = urd) + \mathbb{P}(D_1 = uru,\ D_2 = urd,\ D_3 = dld) + \mathbb{P}(D_1 = urd,\ D_2 = drd,\ D_3 = dld) \leq 4\delta^2.$$

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- A very approximate upper bound on the total number of possible triplets is given by $4^{2n} = 16^n$. We get for all $\delta < \frac{1}{(16)^2}$,

$$\mathbb{P}(T^i_{stop} < \infty) \leq \delta + 4\delta^2 + \frac{(16\sqrt{\delta})^3}{1 - 16\sqrt{\delta}} =: \phi(\delta) \to 0 \text{ when } \delta \to 0$$

Bibliography

[1] Pablo Augusto Ferrari et al. "Phase transition for infinite systems of spiking neurons". In: *Journal of Statistical Physics* 172.6 (2018), pp. 1564–1575.

[2] Antonio Galves and Eva Löcherbach. "Infinite systems of interacting chains with memory of variable length- a stochastic model for biological neural nets". In: *Journal of Statistical Physics* 151.5 (2013), pp. 896–921.

[3] Antonio Galves, Eva Löcherbach, and Enza Orlandi. "Perfect simulation of infinite range Gibbs measures and coupling with their finite range approximations". In: *Journal of Statistical Physics* 138.1 (2010), pp. 476–495.

[4] Branda Goncalves. "An interacting neuronal network with inhibition: theoretical analysis and perfect simulation". working paper or preprint. Oct. 2021. URL: https://hal.archives-ouvertes.fr/hal-03403351.

Thank you for your attention!

Questions?