

Multivariate Hawkes processes on inhomogeneous random graphs

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Outlines

Model

N neurons in interaction on a random graph

Well posedness

When $N \rightarrow \infty$

Large time behaviour of λ

Current work

Modelling of N neurons in interaction on a graph

- ▶ The neuron i is located in $x_i \in I \subset \mathbb{R}^d$

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Modelling of N neurons in interaction on a graph

- ▶ The neuron i is located in $x_i \in I \subset \mathbb{R}^d$
- ▶ $(Z_i^{(N)}(t))_{t \geq 0}$ counting process :
 $Z_i^{(N)}(t) = \{\text{number of spikes of the neuron } i \text{ on } [0, t]\}$
- ▶ conditional intensity at time t :

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right). \quad (1)$$

$\Rightarrow (Z_1^{(N)}(t), \dots, Z_N^{(N)}(t))_{t > 0}$ multivariate Hawkes process.

Firing intensity

$$\lambda_i^{(N)}(t) = f \left(u_0(t-, x_i) + \frac{1}{N} \sum_{j=1}^N w_{ij}^{(N)} \int_{]0, t[} h(t-s) dZ_j^{(N)}(s) \right)$$

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- ▶ $w_{ij}^{(N)}$: interaction between the neurons i and j

Graph of interaction

Interaction between neurons i and j : $w_{ij}^{(N)} = \kappa_i^{(N)} \xi_{ij}^{(N)}$ where

- ▶ $\kappa_i^{(N)} \geq 0$ dilution parameter so that the interaction term remains of order 1 as $N \rightarrow \infty$
- ▶ $\forall i, j, \xi_{ij}^{(N)} \in \{0, 1\} \sim \mathcal{B}(W_N(x_i, x_j))$ where
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Definition

Graph of interaction : $\mathcal{G}^{(N)} = \left(\{1, \dots, N\}, \left(\xi_{ij}^{(N)} \right)_{1 \leq i, j \leq N} \right)$

Annealed graph :

$\mathcal{G}_m^{(N)} = \left(\{1, \dots, N\}, \text{edge } j \rightarrow i \text{ with weight } \kappa_i^{(N)} W_N(x_i, x_j) \right)$

Some dense graphs, $I = [0, 1]$, $x_i = \frac{i}{N}$, $\kappa_i^{(N)} = 1$, $i = 1 \dots N$

Erdős-Rényi graph

$$W_N(x, y) = \rho_N \text{ with } \rho_N \rightarrow \rho > 0$$

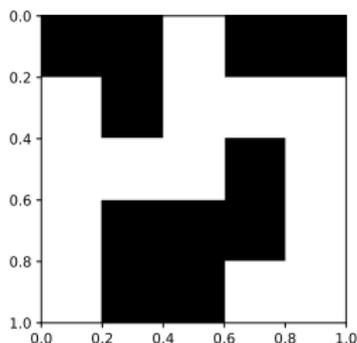


Figure – $\mathcal{G}^{(N)}$ with $\rho_N = 0.5$,
 $N = 5$

P -nearest neighbor

$$W_N(x, y) = 1_{\min(|x-y|, 1-|x-y|) < r}$$

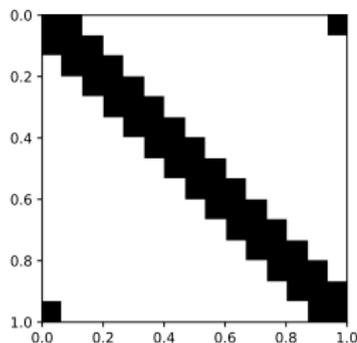


Figure – $\mathcal{G}^{(N)}$ with $r = 0.1$, $N = 15$

Inhomogeneous graph, $I = [0, 1]$, $x_i = \frac{i}{N}$, $\kappa_i^{(N)} = 1$, $i = 1 \dots N$

**Expected Degree
Distribution (EDD)**

$$W_N(x, y) = g(x)k(y)$$

Here : $W_N(x, y) = xy$

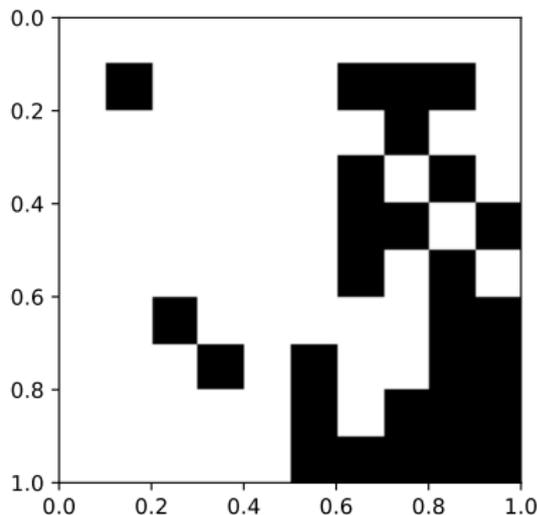


Figure – $\mathcal{G}^{(N)}$ with $N = 10$

$(\pi_i(ds, dz))_{1 \leq i \leq N}$: i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dsdz$. For all $t \geq 0, i \in \llbracket 1, N \rrbracket$:

$$Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds, dz).$$

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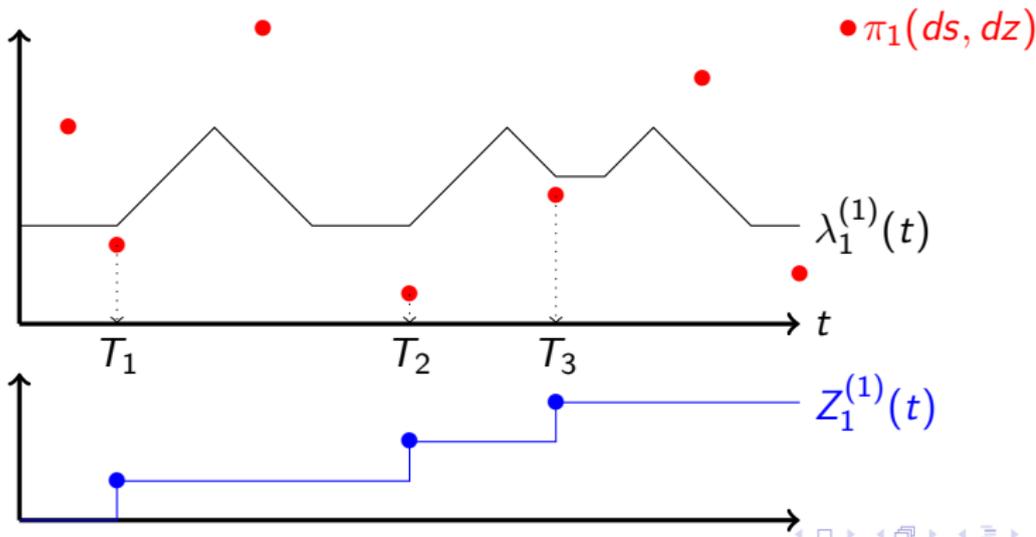
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For instance ($N = 1$) : $\lambda_1^{(1)}(t) = 1 + \int_0^{t-} h(t-s) dZ_1^{(1)}(s)$

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Hypotheses

f Lipschitz continuous, h locally integrable, u_0 bounded, time continuous, Lipschitz continuous in space

Proposition [Delattre et al., 2016, Chevallier et al., 2019]

For a fixed realisation of $(\pi_i)_{1 \leq i \leq N}$, there exists a pathwise unique multivariate Hawkes process

$$\left(Z_1^{(N)}(t), \dots, Z_N^{(N)}(t) \right)_{t > 0} \text{ such that}$$

$$\left(\sup_{1 \leq i \leq N} E[Z_i^{(N)}(t)] \right)_{t \geq 0} \text{ is locally bounded.}$$

Outlines

Model

When $N \rightarrow \infty$

Heuristics

Convergence theorem

Consequences

Large time behaviour of λ

Current work

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right) \xrightarrow{N \rightarrow \infty} ?$$

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About the positions

► $\nu^{(N)} := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dx) \xrightarrow{N \rightarrow \infty} \nu$, a macroscopical distribution

Scenario (1) : $I = [0, 1]$ and $x_i = \frac{i}{N} \Rightarrow \nu(dx) = dx$

Scenario (2) : (x_i) i.i.d. of distribution ν

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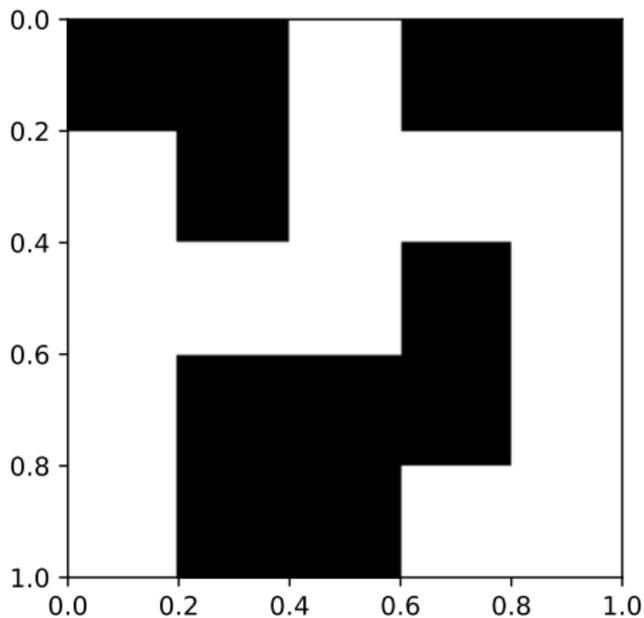
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About the graph

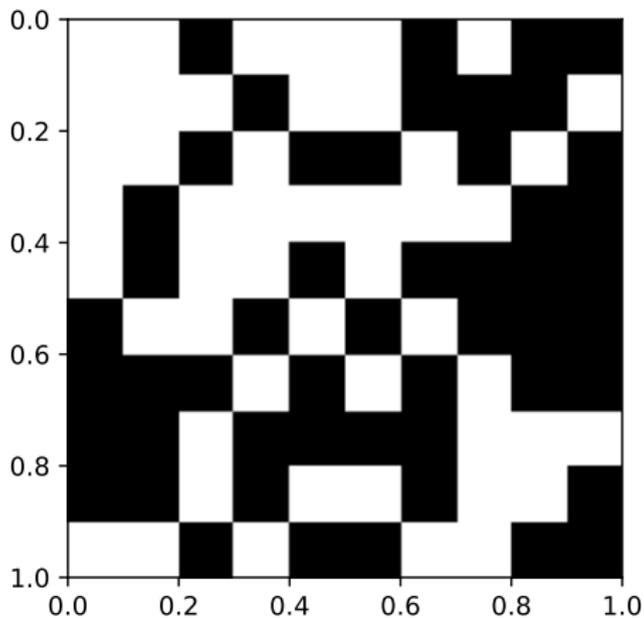
$$\mathcal{G}^{(N)} \xrightarrow{N \rightarrow \infty} ?$$

Example : Erdős-Rényi graph



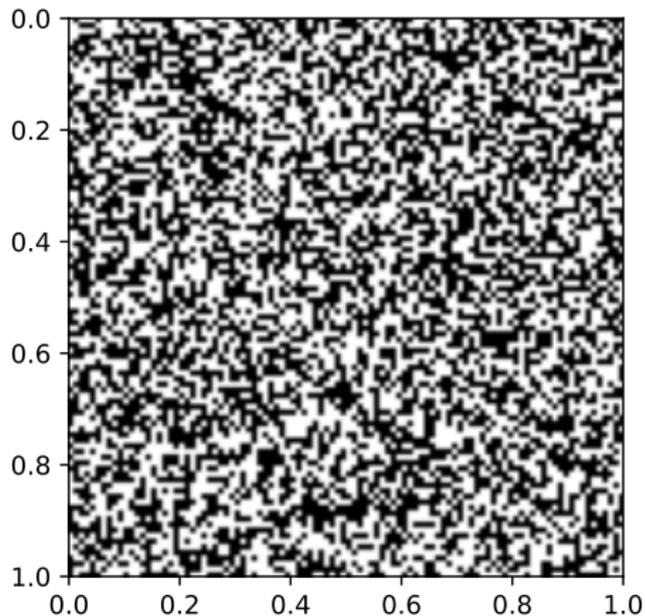
$g^{(N)}, N = 5$

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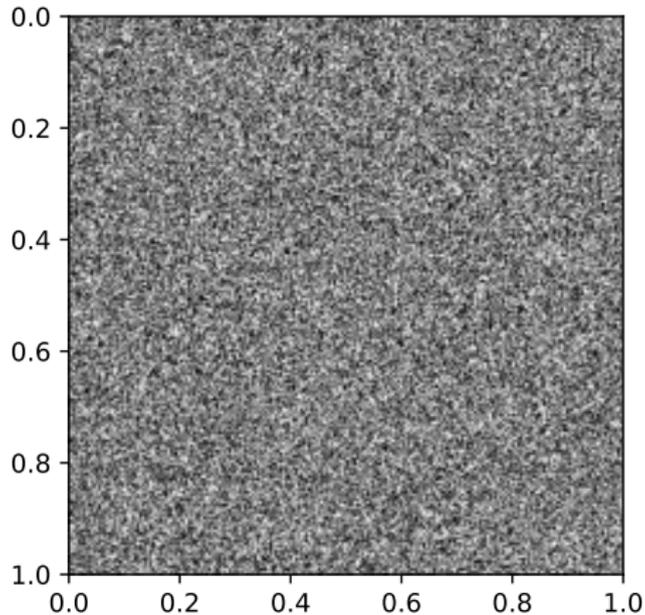
$\mathcal{G}^{(N)}, N = 10$

Example : Erdős-Rényi graph



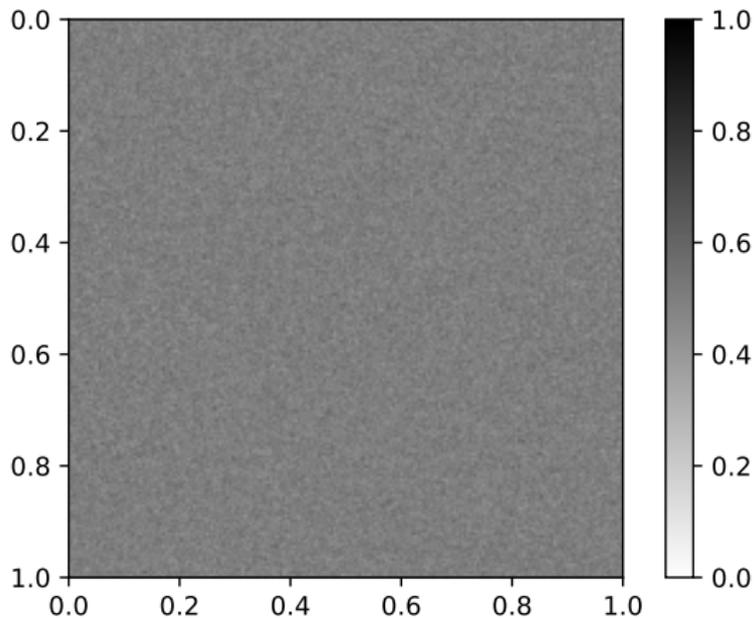
$\mathcal{G}^{(N)}$, $N = 100$

Example : Erdős-Rényi graph



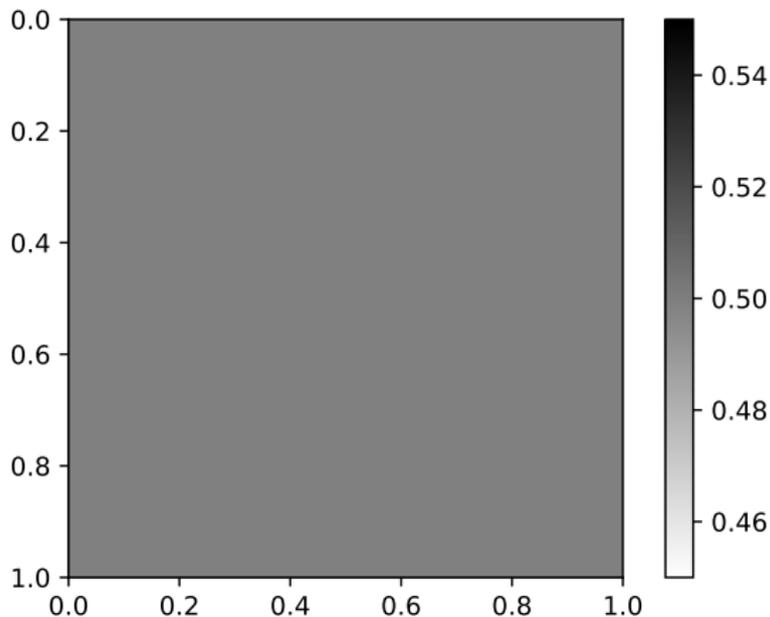
$\mathcal{G}^{(N)}$, $N = 500$

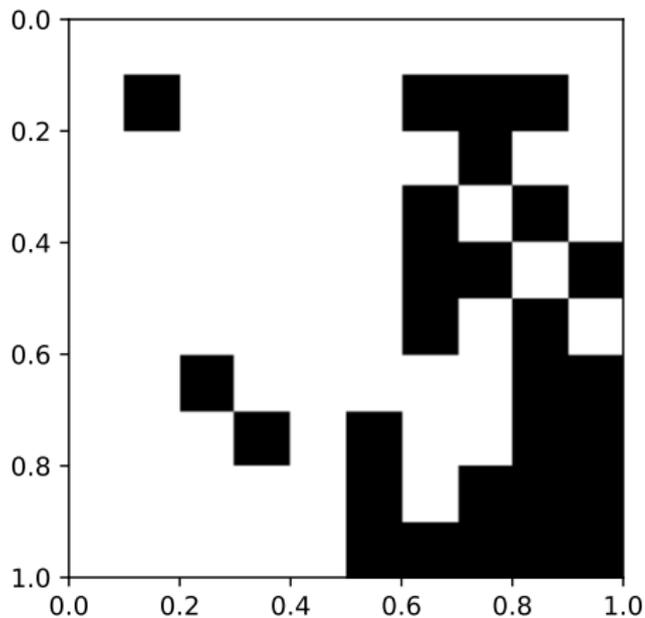
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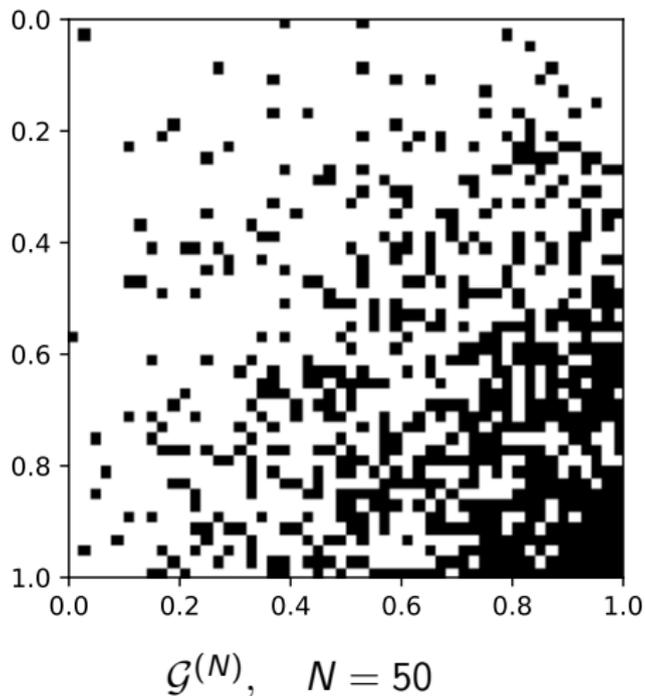
$G^{(N)}, N = 3000$

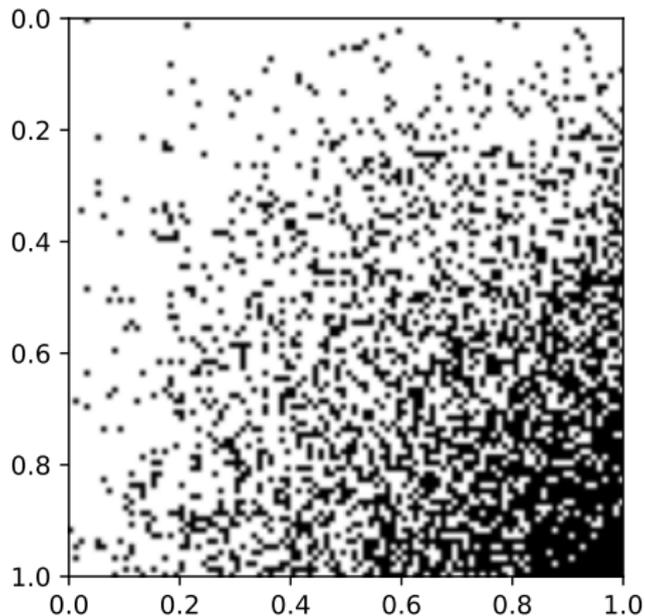
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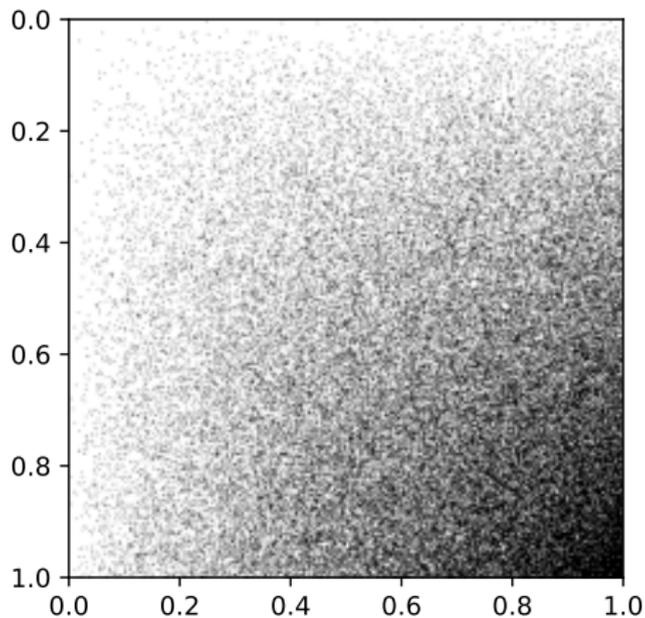


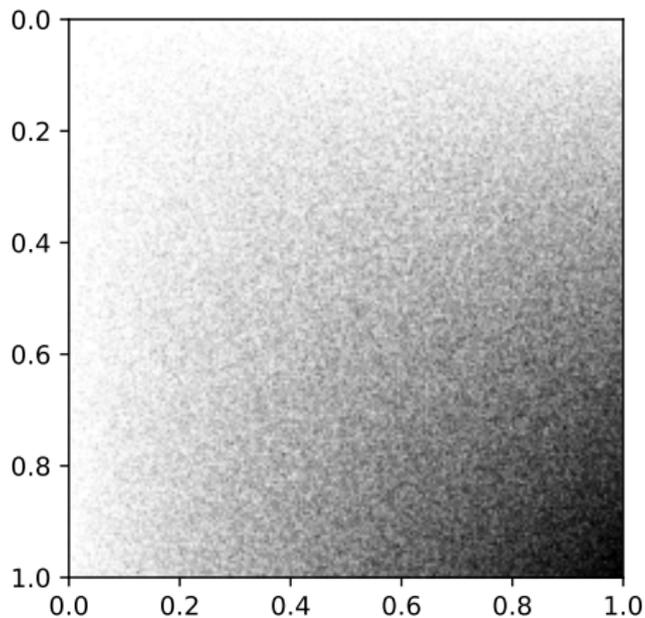
Example : EDD with $W(x, y) = xy$  $\mathcal{G}^{(N)}, N = 10$

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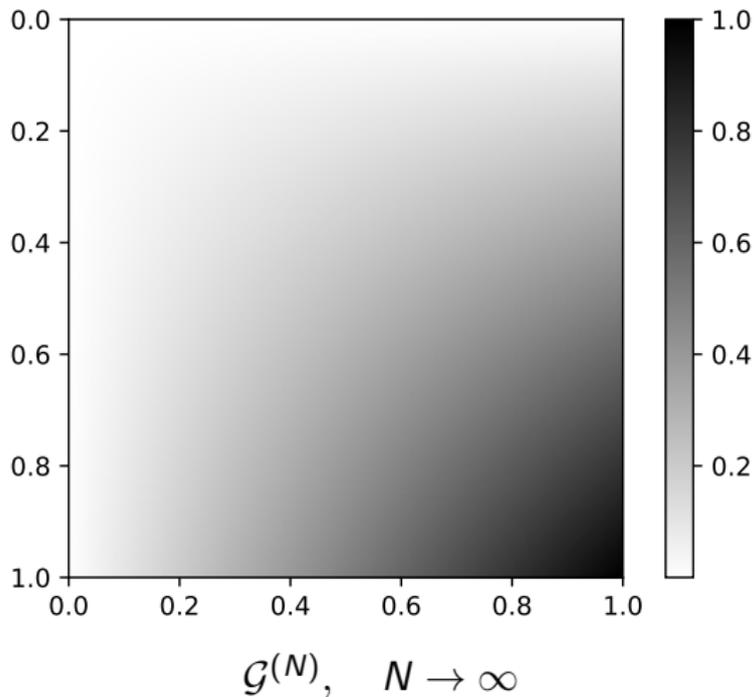


Example : EDD with $W(x, y) = xy$  $\mathcal{G}^{(N)}, N = 100$

Example : EDD with $W(x, y) = xy$  $g^{(N)}, N = 500$

Example : EDD with $W(x, y) = xy$  $\mathcal{G}^{(N)}, N = 1000$

Example : EDD with $W(x, y) = xy$



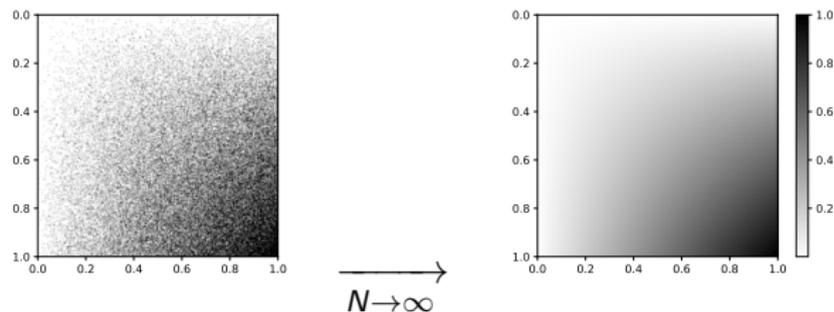
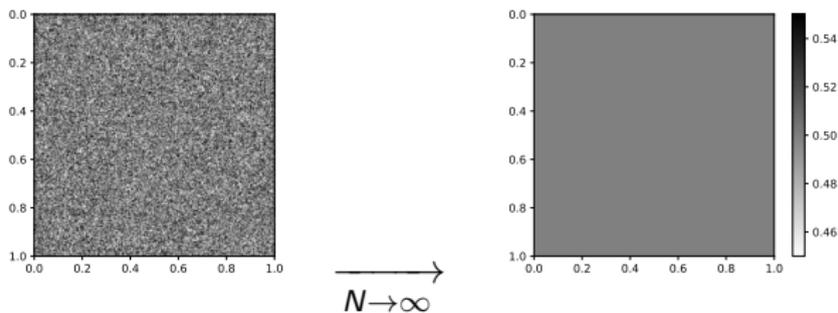


Figure - EDD - $N = 500$ à gauche, graphon $W(x, y) = xy$ à droite

Macroscopic interaction graphon $W : I^2 \rightarrow \mathbb{R}_+$

Assume : $\forall N, \exists I = \bigsqcup_{i=1}^N B_i^{(N)} / \nu(B_i^{(N)}) = \frac{1}{N}$.

- ▶ Graph $\mathcal{G}^{(N)}$ \Rightarrow Connectivity matrix \Rightarrow step-function $W^{\mathcal{G}^{(N)}}$ (graphon) [Lovász, 2012]

$$W^{\mathcal{G}_m^{(N)}}(u, v) := \sum_{i,j=1}^N 1_{B_i^{(N)} \times B_j^{(N)}}(u, v) \kappa_i^{(N)} W_N(x_i, x_j)$$

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- ▶ $W^{\mathcal{G}_m^{(N)}} \xrightarrow[N \rightarrow \infty]{d_{\square, \nu}} W$ a macroscopic interaction graphon with

$$d_{\square, \nu}(W_1, W_2) := \sup_{S, T \subset I} \left| \int_{S \times T} (W_1 - W_2)(x, y) \nu(dx) \nu(dy) \right|$$

Heuristics of the limiting intensity

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right)$$

" $\xrightarrow{N \rightarrow \infty}$ "

$$\lambda(t, x) = f \left(u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy) \right) \quad (2)$$

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Proposition - Let $T > 0$. Assume

- ▶ $D(x) := \int_I W(x, y) \nu(dy)$, $\sup_{x \in I} D(x) < \infty$
- ▶ $\int_I |W(x, y) - W(x', y)| \nu(dy) \leq C \|x - x'\|^\iota$, $\iota \in (0, 1]$

There exists a unique solution λ of (2) continuous and bounded on $[0, T] \times I$.

Coupling [Delattre et al., 2016, Chevallier et al., 2019]

For $t \in [0, T]$, $i \in \llbracket 1, N \rrbracket$ and the same (π_i) :

► $Z_i^{(N)}(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda_i^{(N)}(s)\}} \pi_i(ds, dz)$ with

$$\lambda_i^{(N)}(t) = f \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t^-} h(t-s) dZ_j^{(N)}(s) \right)$$

► $\bar{Z}_i(t) = \int_0^t \int_0^\infty 1_{\{z \leq \lambda(s, x_i)\}} \pi_i(ds, dz)$ with

$$\lambda(t, x) = f \left(u_0(t, x) + \int_{\mathbb{R}^d} W(x, y) \int_0^{t^-} h(t-s) \lambda(s, y) ds \nu(dy) \right).$$

Hypotheses

- ▶ Control of the dilution of the graph $\mathcal{G}^{(N)}$
- ▶ Control of indegrees and outdegrees :

$$\sup_{i \in [1, N]} \frac{1}{N} \sum_{j=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W,$$

$$\sup_{j \in [1, N]} \frac{1}{N} \sum_{i=1}^N \kappa_i^{(N)} W_N(x_i, x_j) \leq C_W$$

Theorem - Let $T > 0$, for \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$ and positions $(\underline{x}_N)_{N \geq 1}$:

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_i^{(N)}(t) - \bar{Z}_i(t) \right| \right] \xrightarrow{N \rightarrow \infty} 0.$$

Empirical measure

Notation

Define the measures on $S := \mathbb{D}([0, T], \mathbb{N}) \times I$:

- ▶ $\mu_N(d\eta, dx) := \frac{1}{N} \sum_{i=1}^N \delta_{(Z_i^{(N)}([0, T]), x_i^{(N)})}(d\eta, dx)$
- ▶ $\mu_\infty(d\eta, dx) := P_{[0, T], \infty}(d\eta | x) \nu(dx),$

for $P_{[0, T], \infty}(\cdot | x)$ distribution of an inhomogeneous Poisson point process with intensity $(\lambda(t, x))_{0 \leq t \leq T}$.

Empirical measure

Proposition

For \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$ and positions $(\underline{x}_N)_{N \geq 1}$,

$$E [d_{BL}(\mu_N, \mu_\infty)] \xrightarrow{N \rightarrow \infty} 0$$

where

$$d_{BL}(\mu, \nu) := \sup_{g, \|g\|_{BL} \leq 1} \left| \int_S g(d\mu - d\nu) \right|$$

$$\|g\|_{BL} := \|g\|_{Lip} + \|g\|_{S, \infty}$$

Spatial profile, with $I = [0, 1]$ and $x_i = \frac{i}{N}$

- ▶ $X_t^{(N)}(x) := \sum_{i=1}^N \left(u_0(t, x_i) + \frac{\kappa_i^{(N)}}{N} \sum_{j=1}^N \xi_{ij}^{(N)} \int_0^{t-} h(t-s) dZ_j^{(N)}(s) \right) \mathbf{1}_{x \in (\frac{i-1}{N}, \frac{i}{N}]}$
- ▶ $X_t(x) := u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy)$

Proposition - For \mathbb{P} -almost realisations of the connectivity sequence $(\xi^{(N)})_{N \geq 1}$,

$$\mathbb{E} \left[\int_0^T \int_0^1 \left| X_t^{(N)}(x) - X_t(x) \right| dx dt \right] \xrightarrow{N \rightarrow \infty} 0.$$

Outlines

Model

When $N \rightarrow \infty$

Large time behaviour of λ

Subcritical case

Supercritical case

Current work

Linear case : $f = Id$

$$\lambda(t, x) = u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy)$$

Without spatial interaction [Delattre et al., 2016]

$$\lambda(t) = u_0 + \int_0^t h(t-s) \lambda(s) ds$$

Phase transition

- ▶ Subcritical case ($\|h\|_1 < 1$) : $\lambda(t) \xrightarrow{t \rightarrow \infty} \frac{u_0}{1 - \|h\|_1}$
- ▶ Supercritical case ($\|h\|_1 > 1$) : $\lambda(t) \sim \alpha e^{\beta t} \rightarrow \infty$ for some $\alpha, \beta > 0$

$$\begin{aligned}\lambda(t, x) &= u_0(t, x) + \int_I W(x, y) \int_0^t h(t-s) \lambda(s, y) ds \nu(dy) \\ &= u_0(t, x) + \int_0^t h(t-s) T_W \lambda(s, \cdot)(x) ds\end{aligned}$$

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Integral operator T_W

$$\begin{aligned}T_W : L^\infty(I) &\longrightarrow L^\infty(I) \\ g &\longmapsto (T_W g : x \longmapsto \int_I W(x, y) g(y) \nu(dy)).\end{aligned}$$

Spectral radius

$$r_\infty := r_\infty(T_W) = \sup_{\sigma \in Sp(T_W)} |\sigma| = \lim_{n \rightarrow \infty} \|T_W^n\|^{\frac{1}{n}}.$$

Subcritical case : $\|h\|_1 r_\infty < 1$

Theorem

In the subcritical case $\|h\|_1 r_\infty < 1$, if $\sup_{x \in I} |u_0(t, x) - u(x)| \xrightarrow{t \rightarrow \infty} 0$, then for any $x \in I$,

$$\lambda(t, x) \xrightarrow{t \rightarrow \infty} \ell(x)$$

where ℓ is the unique continuous and bounded function solving

$$\ell(x) = u(x) + \|h\|_1 \int_I W(x, y) \ell(y) \nu(dy).$$

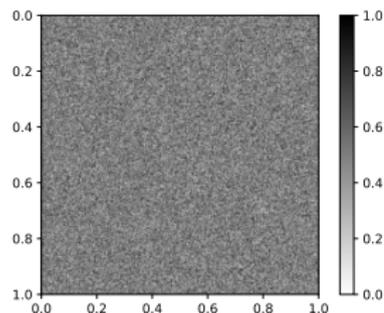
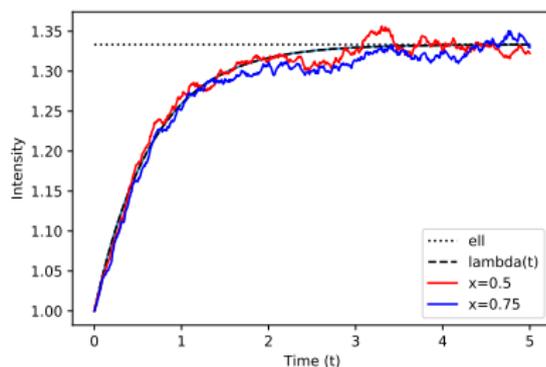
Proposition

If u_0 is constant, ℓ is uniform if and only if the indegree is uniform ($\int_I W(x, y) \nu(dy) = D$ for every $x \in I$). In such case, $r_\infty = D$.

Subcritical case - Erdős-Rényi graph

$$W_N(x, y) = \rho = r_\infty = 0.5$$

$$N = 1000$$

Figure – $\mathcal{G}^{(N)}$ Subcritical condition : $\|h\|_{1\rho} < 1$ 

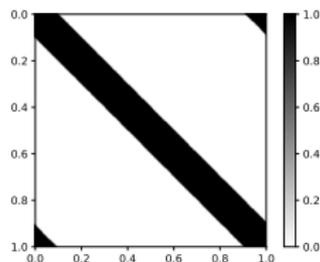
$$h(t) = e^{-2t}, \quad u_0(t, x) = 1$$

$$\lambda(t) = \frac{4}{3} - \frac{1}{3}e^{-\frac{3}{2}t}, \quad \ell = \frac{4}{3}$$

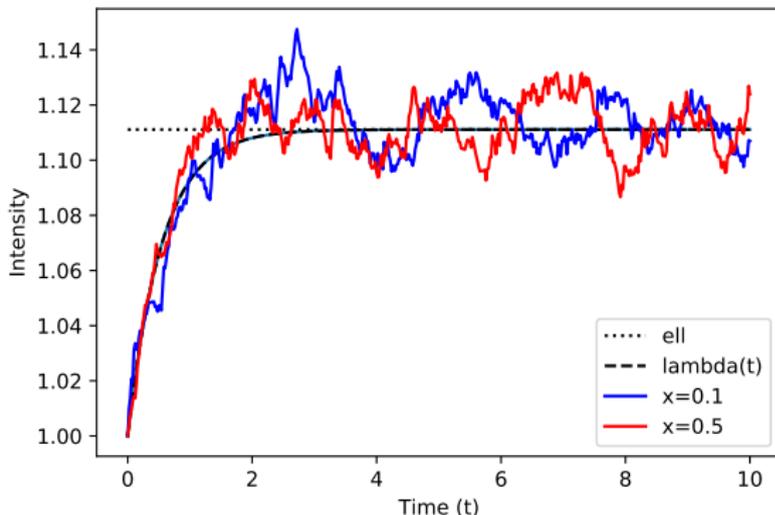
Subcritical case - P -nearest neighbor

$$W_N(x, y) = 1_{\min(|x-y|, 1-|x-y|) < r}, \quad r_\infty = 2r$$

$$h(t) = e^{-2t}, \quad u_0(t, x) = 1, \quad r = 0.1$$

Figure - $\mathcal{G}^{(N)}$ $N = 500$

$$\lambda(t) = \frac{10}{9} - \frac{1}{9}e^{-\frac{9}{5}t}$$



Subcritical case - EDD

f, g densities on I , bounded

$$W_N(x, y) = f(x)g(y), \quad D(x) = f(x), \quad r_\infty = \langle f, g \rangle$$

When $\|h\|_1 \langle f, g \rangle < 1$:

$$\lambda(t, x) \xrightarrow{t \rightarrow \infty} \ell(x) = u(x) + \|h\|_1 \frac{f(x) \langle u, g \rangle}{1 - \|h\|_1 \langle f, g \rangle}.$$

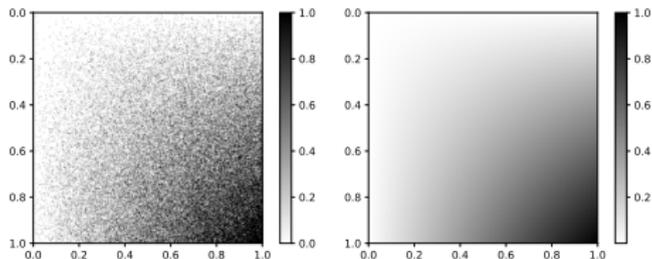


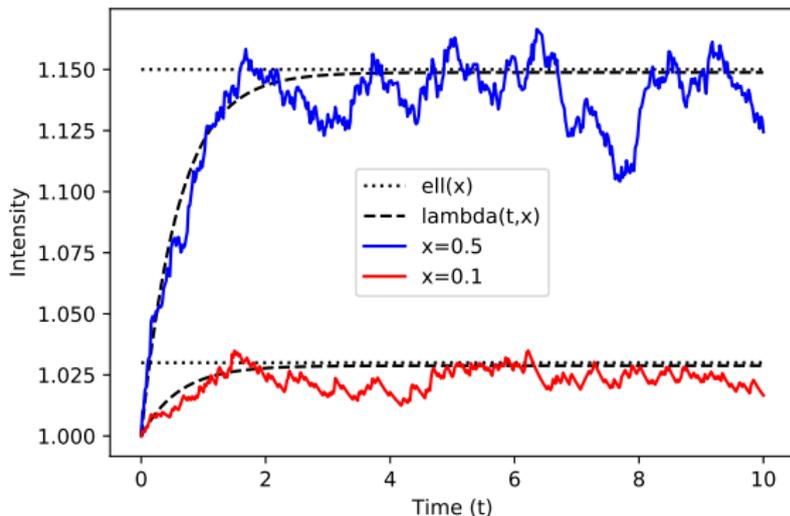
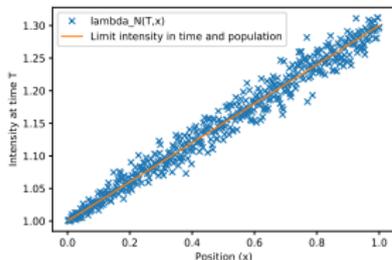
Figure – $\mathcal{G}^{(N)}$ with $N = 500$ and the graphon $W(x, y) = xy$

Subcritical case - EDD $W(x, y) = xy$

$$h(t) = e^{-2t}$$

$$u_0(t, x) = 1$$

$$\ell(x) = 1 + \frac{3}{10}x$$



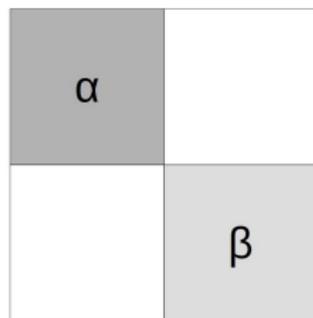
Supercritical case : $\|h\|_1 r_\infty > 1$

$$\|h\|_1 r_\infty > 1 \Rightarrow \lambda(t, x) \xrightarrow[t \rightarrow \infty]{} \infty ?$$

Example

W with 2 disconnected mean-field components, $\alpha > \beta$, $r_\infty = \alpha/2$:

A population can be in the subcritical case and the other in the supercritical case.



(critical parameters : $\alpha_c = \frac{2}{\|h\|_1}$ and $\beta_c = \frac{2}{\|h\|_1}$)

Supercritical case : $\|h\|_1 r_\infty > 1$

Hypotheses

- ▶ $\sup_x \int_I W(x, y)^2 \nu(dy) =: C_{W_2} < \infty,$
- ▶ $\forall (x, y) \in I^2, \quad W(x, y) = W(y, x)$
- ▶ there exists k such that $W^{(k)} > 0$ where
 $W^{(k)}(x, y) := \int_{I \times \dots \times I} W(x, x_1) \cdots W(x_{k-1}, y) dx_1 \cdots dx_{k-1}.$

Proposition

$$\int_I \lambda(t, x)^2 \nu(dx) \xrightarrow[t \rightarrow \infty]{} \infty$$

Outlines

Model

When $N \rightarrow \infty$

Large time behaviour of λ

Current work

Hypotheses

Goal

- ▶ Exponential decay : $h(t) = e^{-\alpha t}$

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 $F(X, u) = u + f(X)$ or $f(u + X)$

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- ▶ Subcritical case $\|F\|_L \|h\|_1 r_\infty < 1$

Then the large time limit of the spatial profile verifies

$$X_t(x) = \int_I W(x, y) \int_0^t e^{-\alpha(t-s)} F(X_s(y), u(s, y)) ds dy$$

(X is then the solution of the scalar neural field equation [Amari, 1977, Wilson - Cowan, 1972, Chevallier et al., 2019])

Work in progress

Let T_N be polynomial in N -i.e., there exists m such that $T_N = \left(\frac{N}{\kappa_N}\right)^m$. Then for any $\varepsilon > 0$,

$$P \left(\sup_{t \in [0, T_N]} \left\| X_t^{(N)} - X_t \right\|_2 > \varepsilon \right) \xrightarrow{N \rightarrow \infty} 0.$$

Tool : the large time limit of X_t satisfying

$$\alpha X_\infty = T_W F(X_\infty, u_\infty)$$

- ▶ Finite time dynamic :

$$P \left(\sup_{t \in [0, T]} \|X_t^{(N)} - X_t\|_2 > \varepsilon \right) \xrightarrow{N \rightarrow \infty} 0.$$

- ▶ Stability around X_∞ for (X_t) :

$$\partial_t (X_t - X_\infty) = -\alpha (X_t - X_\infty) dt + T_W (F(X_t, u_t) - F(X_\infty, u_\infty)) dt$$

- ▶ By an iteration argument in large time $X_t^{(N)}$ remains close to X_∞ as

$$\partial_t (X_t^{(N)} - X_\infty) = \mathcal{L} (X_t^{(N)} - X_\infty) dt + dM_N(t) + r_N(t)dt,$$

with \mathcal{L} an accretive operator, M_N some Poissonian noise and r_N a drift term (from the graph).

Thanks!



Agathe-Nerine, Z. (2021).

Multivariate Hawkes processes on inhomogeneous random graphs.

[arXiv :2106.12259](https://arxiv.org/abs/2106.12259).



Agathe-Nerine, Z. (2022).

Long time behavior for Hawkes processes on random graphs

Work in progress.

References

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-  Delattre, S., Fournier, N., and Hoffmann, M. (2016). Hawkes processes on large networks. *Ann. Appl. Probab.*, 26(1) :216–261.
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