Temps et aléa du Quantique

Alain Connes

Real Variables

Classical formulation

In the classical formulation of real variables as maps from a set X to the real numbers $\mathbb{R}$, the set X has to be uncountable if some variable has continuous range. But then for any other variable with countable range some of the multiplicities are infinite.

This means that discrete and continuous variables cannot coexist in this modern formalism.

Fortunately everything is fine and this problem of treating continuous and discrete variables on the same footing is completely solved using the formalism of quantum mechanics.


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Werner Heisenberg

When manipulating the observables quantities for a microscopic system, the order of terms in a product plays a crucial role. The commutativity of Cartesian coordinates does not hold in the algebra of coordinates on the phase space of a microscopic system.

Langage

(Anagrammes renversantes, d'Etienne Klein et Jacques Perry-Salkow sur le sens caché du monde).

"L'horloge des anges ici-bas"

"Le boson scalaire de Higgs"

On a ici une anagramme parfaite, les deux ensembles de mots donnent le même résultat quand on néglige l'ordre des lettres, à savoir : $a^2bcde^3g^2hi^2l^2no^2rs^3$. On voit clairement que passer au commutatif est une perte de sens.

Quantum formalism

The first basic change of paradigm has indeed to do with the classical notion of a "real variable" which one would classically describe as a real valued function on a set X, ie as a map from this set X to real numbers. In fact quantum mechanics provides a very convenient substitute. It is given by a self-adjoint operator in Hilbert space. Note that the choice of Hilbert space is irrelevant here since all separable infinite dimensional Hilbert spaces are isomorphic.

All the usual attributes of real variables such as their range, the number of times a real number is reached as a value of the variable etc... have a perfect analogue in the quantum mechanical setting. The range is the spectrum of the operator, and the spectral multiplicity gives the number of times a real number is reached. In the early times of quantum mechanics, physicists had a clear intuition of this analogy between operators in Hilbert space (which they called q-numbers) and variables.

Discrete and continuous coexist

It is only because one drops commutativity that variables with continuous range can coexist with variables with countable range.

Thus it is the uniqueness of the separable infinite dimensional Hilbert space that cures the above problem, $L^2[0,1]$ is the same as $\ell^2(\mathbb{N})$, and variables with continuous range coexist happily with variables with countable range, such as the infinitesimal ones. The only new fact is that they do not commute, and the real subtlety is in their algebraic relations. For instance it is the lack of commutation of the line element ds with the coordinates that allows one to measure distances in a noncommutative space given as a spectral triple.

Newton

One striking point is the role that “variables” play in Newton’s approach, while Leibniz introduced the term “infinitesimal” but did not use variables. According to Newton :

“In a certain problem, a variable is the quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order”

“A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number”

Infinitesimals

What is surprising is that the new set-up immediately provides a natural home for the "infinitesimal variables" and here the distinction between "variables" and numbers (in many ways this is where the point of view of Newton is more efficient than that of Leibniz) is essential.

Indeed it is perfectly possible for an operator to be "smaller than epsilon for any epsilon" without being zero. This happens when the norm of the restriction of the operator to subspaces of finite codimension tends to zero when these subspaces decrease (under the natural filtration by inclusion). The corresponding operators are called "compact" and they share with naive infinitesimals all the expected algebraic properties. Indeed they form a two-sided ideal of the algebra of bounded operators in Hilbert space and the only property of the naive infinitesimal calculus that needs to be dropped is the commutativity.

Space X	Algebra $\mathcal{A}$
Real variable x^μ	Self-adjoint operator H
Infinitesimal dx	Compact operator ϵ
Integral of infinitesimal	$\int \epsilon =$ coefficient of $\log(\Lambda)$ in $\mathrm{Tr}_\Lambda(\epsilon)$
Line element $\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$	$D^{-1} =$ Fermion propagator

Variability

At the philosophical level there is something quite satisfactory in the variability of the quantum mechanical observables. Usually when pressed to explain what is the cause of the variability in the external world, the answer that comes naturally to the mind is just : the passing of time.

But precisely the quantum world provides a more subtle answer since the reduction of the wave packet which happens in any quantum measurement is nothing else but the replacement of a "q-number" by an actual number which is chosen among the elements in its spectrum. Thus there is an intrinsic variability in the quantum world which is so far not reducible to anything classical. The results of observations are intrinsically variable quantities, and this to the point that their values cannot be reproduced from one experiment to the next, but which, when taken altogether, form a q-number.

How can time emerge

from quantum variability ?

As we shall see the study of subsystems as initiated by von Neumann leads to a potential answer.

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Factorizations

Let the Hilbert space $\mathcal{H}$ factor as a tensor product :

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Von Neumann investigated the meaning of such a factorization at the level of operators.

A factor is an algebra of operators which has all the obvious properties of the algebra of operators of the form $T_1 \otimes 1$ acting in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

4. Another interpretation of $(\overline{D}_5)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6), pp. 55–60, and (2C), p. 167. We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1$, $\mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1$, $\mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1$, $\mathbf{M}_2$.

(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6), pp. 11–14 and 76, or (20), pp. 117–121). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.

(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1$, $\mathfrak{S}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

Thus our problem of solving $(\overline{\mathrm{D}}_5)$ corresponds to the quantum mechanical problem of dividing a system $\mathfrak{S}$ into two subsystems $\mathfrak{S}_1$, $\mathfrak{S}_2$; and in particular the solutions **M** of $(\overline{\mathrm{D}}_5)$ correspond to the complete rings of all observables of suitable quantum mechanical systems.

This interpretation of $(\overline{\mathrm{D}}_5)$ suggests of course strongly the surmise formulated at the end of §2.2: It should be possible to describe $\mathfrak{H}$ as (isomorphic to) the space of all two variable functions $f(x, y)$, ($\iint |f(x, y)|^2\, dx\, dy$ finite), **M** operating on x only, and **M′** on y only. In this case $\mathfrak{S}_1$, $\mathfrak{S}_2$ would be explicitly given: $\mathfrak{S}_1$ being described by the coordinate x, and $\mathfrak{S}_2$ by the coordinate y.

The fact that the surmise of §2.2 is not true, is therefore the more remarkable; particularly so because certain features of the "exceptional" rings **M** seem to make them even better suited for quantum mechanical purposes than the customary **B**. We will now discuss these properties of **M**.

Three types

Type I, if the Hilbert space $\mathcal{H}$ factor as a tensor product :

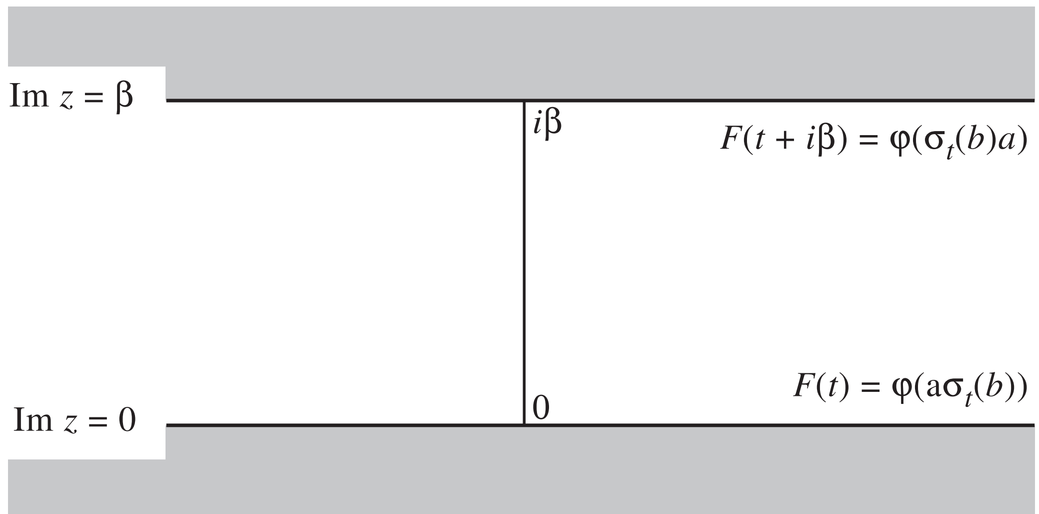
$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Von Neumann found two other types :

Type II : The classification of subspaces gives an interval $[0, 1]$ or $[0, \infty]$; continuous dimensions !

Type III : All that remains.

KMS Condition



$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \quad F_{x,y}(t+i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}.$$

Tomita–Takesaki

Theorem

Let M be a von Neumann algebra and φ a faithful normal state on M, then there exists a unique

$$\sigma_t^\varphi \in \mathrm{Aut}(M)$$

which fulfills the KMS condition for $\beta = 1$.

Thesis (1971–1972)

Theorem (ac)

$$1 \to \mathrm{Int}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}) \to \mathrm{Out}(\mathcal{M}) \to 1,$$

The class of σ_t^{φ} in $\mathrm{Out}(\mathcal{M})$ does not depend on φ.

Thus a von Neumann algebra $\mathcal{M}$, has a canonical evolution

$$\mathbb{R} \xrightarrow{\delta} \mathrm{Out}(\mathcal{M}).$$

Noncommutativity $\Rightarrow$ Evolution

Classification of factors

New invariants and reduction of type III to type II and automorphisms.

The Module $S(M)$: It is a closed subgroup of $\mathbb{R}_+^*$,

Factors of type III$_\lambda$, $\lambda \in [0, 1]$

Periods : It is a subgroup of $\mathbb{R}$, $T(M) \subset \mathbb{R}$.

Thermodynamical origin of time

Many mathematical corollaries but what about physics ?

Carlo Rovelli had found for philosophical reasons that rather than the usual determination of an equilibrium state from the time evolution one should reverse the correspondence and obtain the time evolution from the statistical state. Two papers in 92 and 93.

Statistical mechanics of gravity and the thermodynamical origin of time

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Abstract. The lack of a statistical and thermodynamical theory of the gravitational field is pointed out. The possibility of developing such a theory is discussed, as well as its theoretical and cosmological relevance. The connection between this problem and the problem of physical time in (compact space) general relativity is emphasized. The idea that a preferred physical time variable is singled out by the statistical properties of the state is proposed. A scheme for a generally covariant statistical thermodynamics is put forward, by extending the Gibbs formalism to the presymplectic, or constrained, dynamical systems. This scheme is based on three equations. The first characterizes any statistical state; the second is the definition of a vector flow in terms of the statistical state; the third is an intrinsic definition of equilibrium. The vector flow of an equilibrium state is denoted thermodynamical time, and the suggestion is made that thermodynamical time is the internal time that carries all the 'common sense' characterizations of the notion of time. Finally, it is suggested that the issue of the second law of the thermodynamics could be reconsidered within this generally covariant framework.

Mechanical time and common sense time are identified in non-relativistic physics. In general relativistic physics, there is no preferred mechanical time. What about the second domain, namely the common sense properties of time? We propose the idea that in thermal relativistic physics there *is* a preferred time variable to which we may associate the common sense properties of time, namely a preferred common sense time. This common sense time has thermodynamical origin, and it is not unique in the theory but is determined by the state. If one knows how to define a suitable thermodynamical time, *the common sense time is the thermodynamical time defined below.* In the rest of this paper we construct a covariant statistical thermodynamics in which this idea is implemented.

Next, consider a peculiar statistical state, namely the equilibrium Gibbs distribution ρ_e, defined in (1). Now, it is easy to verify that

$$\omega(X_t) = -\frac{1}{\beta}\,\mathrm{d}\ln\rho_e\,. \tag{14}$$

Namely, there is a relation between the time vector field X_t and the equilibrium probability distribution. Thus, we obtain the following crucial result, which is extremely relevant for what follows: *the Hamiltonian vector field of the logarithm of an equilibrium distribution is proportional to the Hamiltonian time flow.*

The key idea concerns the second of these results, namely the relation between an equilibrium statistical state ρ_e and the time vector field X_t: in the general case, we propose to read this equation as the *definition* of the preferred time vector field, in terms of a given statistical state. We put forth the physical hypothesis that in the general case it is X_t determined in this way that captures our physical notion of 'time flow', including in particular its common sense features. Namely, *we postulate that the common sense time, in a given equilibrium statistical state, is determined by the statistical state itself via equation (14).*

Algebra of observables for gravitation

We interpret time as a one parameter group of automorphisms of the algebra of observables for gravitation.

"Where are we ?"

The answer is spectral

It is well known since a famous one page paper of John Milnor that the spectrum of operators, such as the Laplacian, does not suffice to characterize a compact Riemannian space. But it turns out that the missing information is encoded by the relative position of two abelian algebras of operators in Hilbert space. Due to a theorem of von Neumann the algebra of multiplication by all measurable bounded functions acts in Hilbert space in a unique manner, independent of the geometry one starts with. Its relative position with respect to the other abelian algebra given by all functions of the Laplacian suffices to recover the full geometry, provided one knows the spectrum of the Laplacian. For some reason which has to do with the inverse problem, it is better to work with the Dirac operator.

The unitary (CKM) invariant of Riemannian manifolds

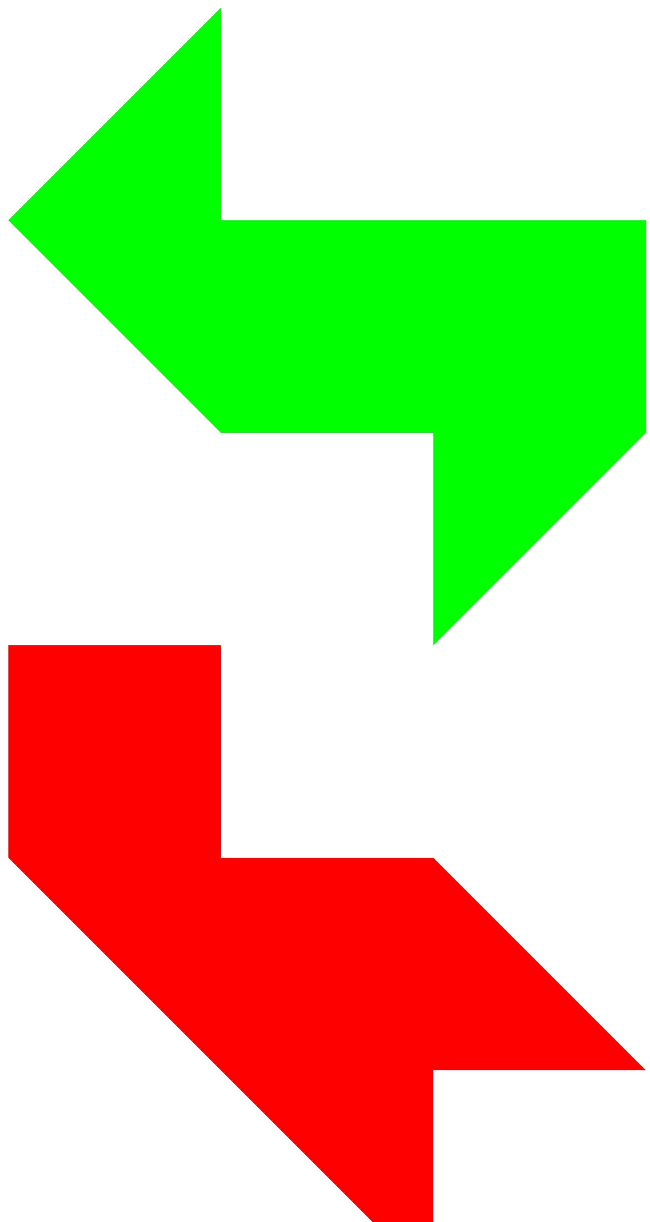
The invariants are :

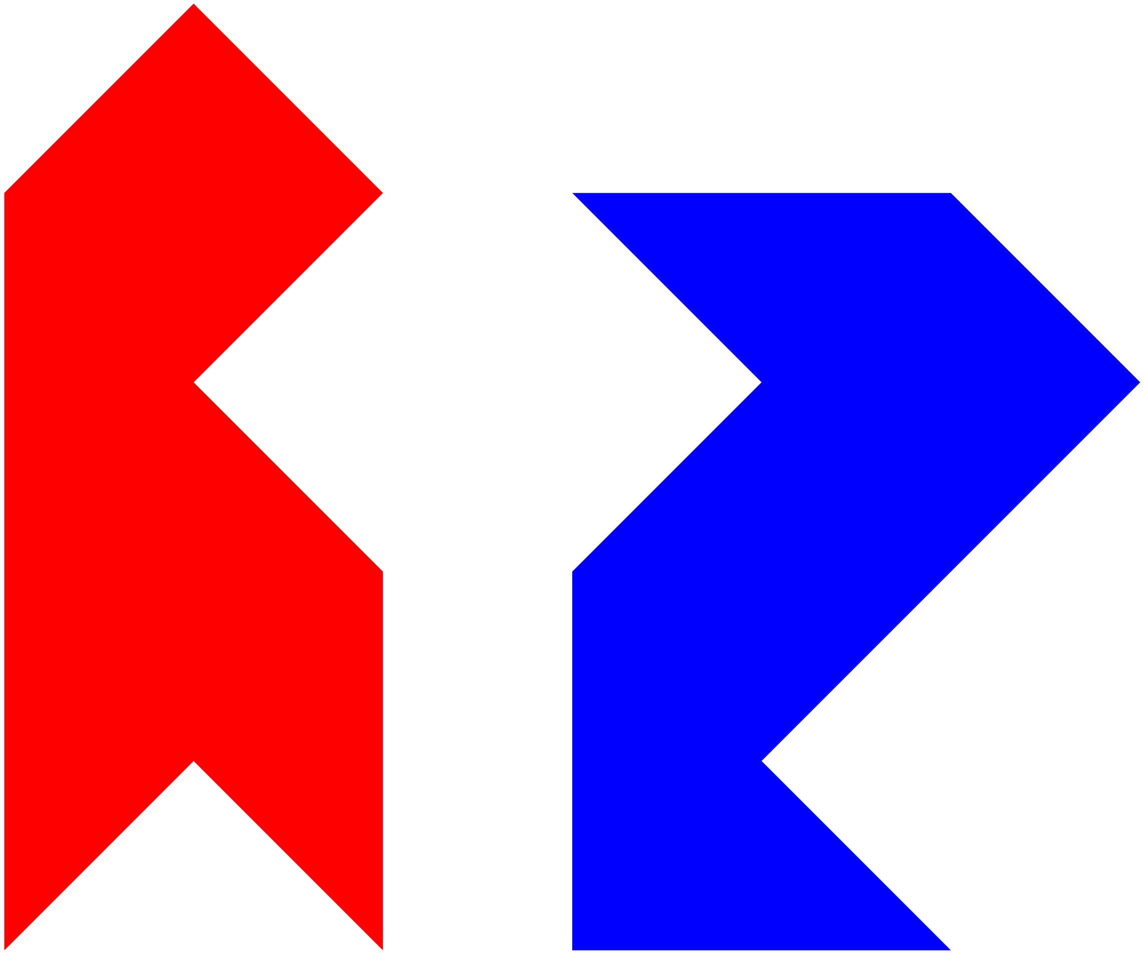
— The spectrum Spec(D).

— The relative spectrum $\text{Spec}_N(M)$ $(N = \{f(D)\})$.

Gordon, Web, Wolpert

Gordon, C. ; Webb, D. ; Wolpert, S. (1992), ''Isospectral plane domains and surfaces via Riemannian orbifolds'', Inventiones mathematicae

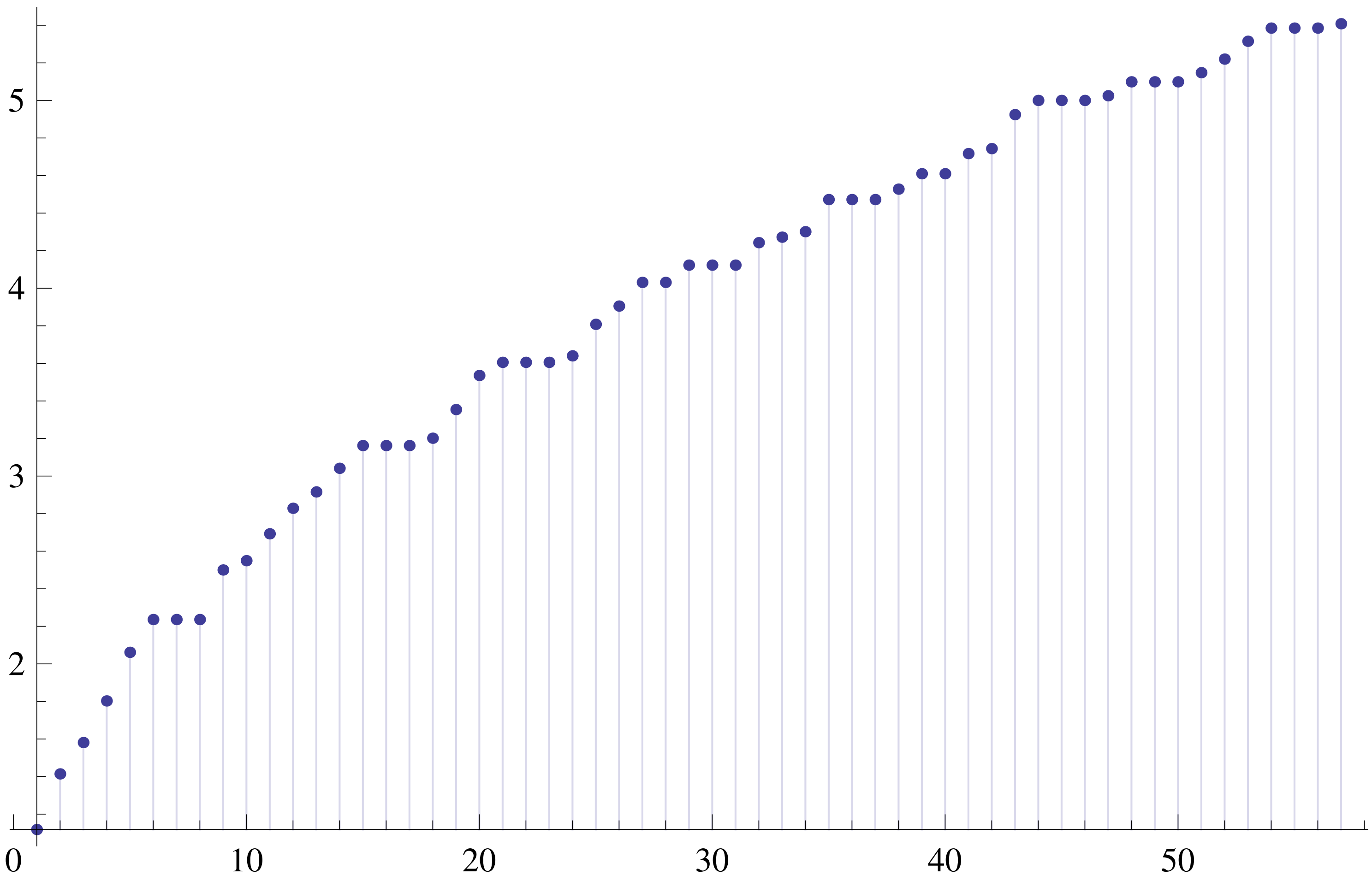




Two shapes with same spectrum (Chapman).

Shape I

Shape II


5
4
3
2
0
10
20
30
40
50

Spectrum $= \{\sqrt{x} \mid x \in S\}$,

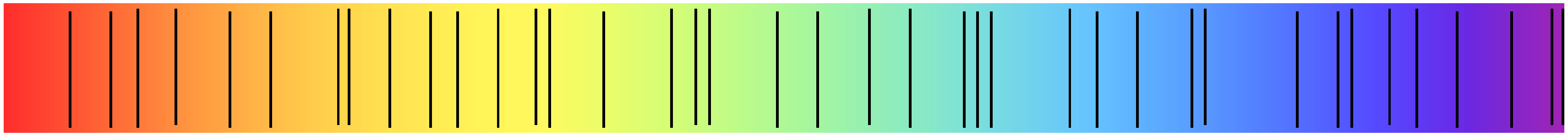
$$S = \{\frac{5}{4}, 2, \frac{5}{2}, \frac{13}{4}, \frac{17}{4}, 5, 5, 5, \frac{25}{4}, \frac{13}{2}, \frac{29}{4}, 8, \frac{17}{2}, \frac{37}{4}, 10, 10, 10, \frac{41}{4}, \frac{45}{4}, \frac{25}{2},$$
$$13, 13, 13, \frac{53}{4}, \frac{29}{2}, \frac{61}{4}, \frac{65}{4}, \frac{65}{4}, 17, 17, 17, 18, \frac{73}{4}, \frac{37}{2}, 20, 20, 20,$$
$$\frac{41}{2}, \frac{85}{4}, \frac{85}{4}, \frac{89}{4}, \frac{45}{2}, \frac{97}{4}, 25, 25, 25, \frac{101}{4}, 26, 26, 26, \frac{53}{2},$$
$$\frac{109}{4}, \frac{113}{4}, 29, 29, 29, \frac{117}{4}, \ldots$$

Same spectrum

$$\{a^2 + b^2 \mid a, b > 0\} \cup \{c^2/4 + d^2/4 \mid 0 < c < d\}$$

$$=$$

$$\{e^2/4 + f^2 \mid e, f > 0\} \cup \{g^2/2 + h^2/2 \mid 0 < g < h\}$$



Three classes of notes

One looks at the fractional part

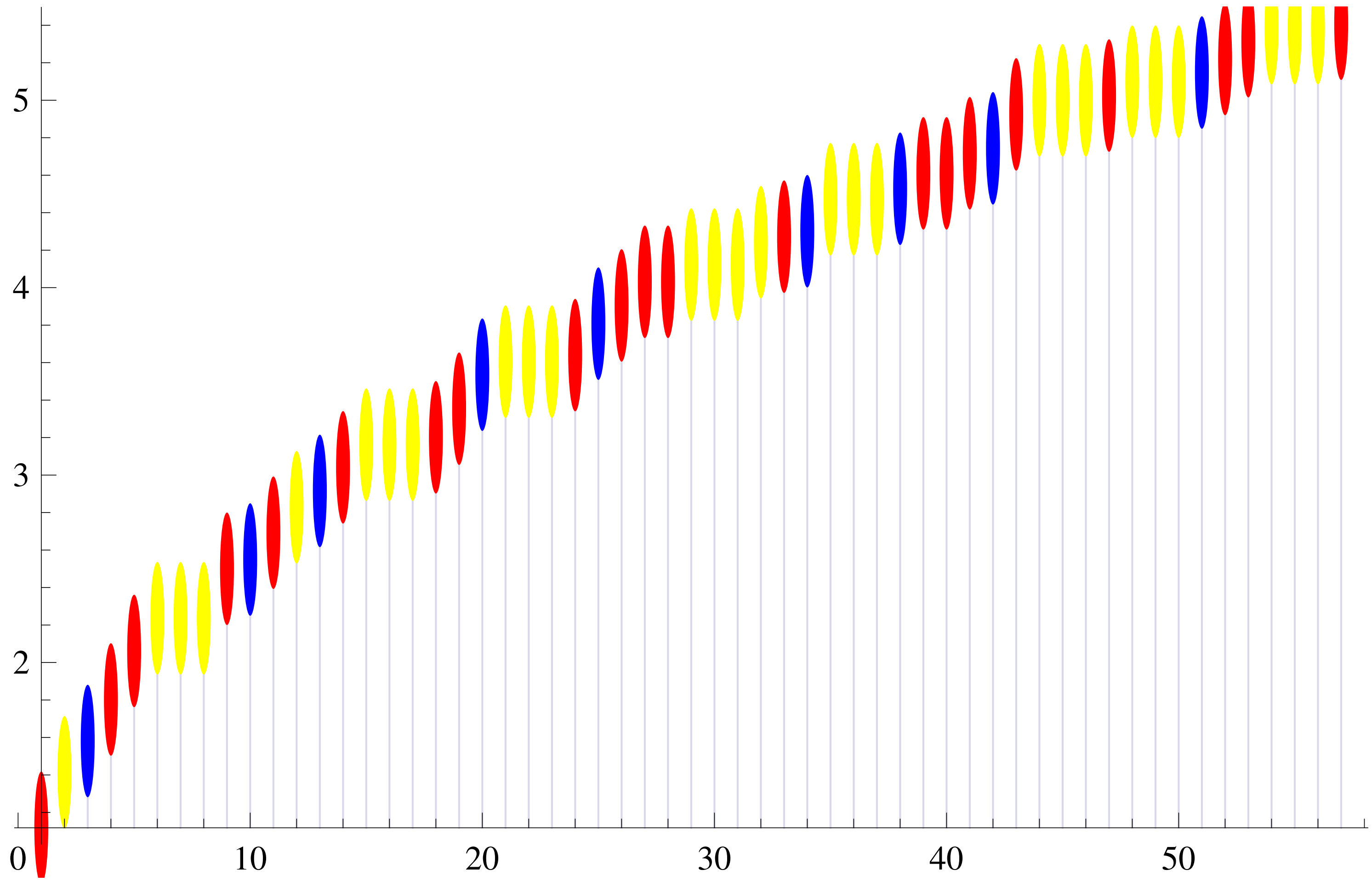
$\frac{1}{4}$: $\{e^2/4 + f^2\}$ with $e, f > 0 = \{c^2/4 + d^2/4\}$ with $c + d$ odd.

$\frac{1}{2}$: The $c^2/4 + d^2/4$ with c, d odd and $g^2/2 + h^2/2$ with $g + h$ odd.

0 : $\{a^2 + b^2 \mid a, b > 0\} \cup \{4c^2/4 + 4d^2/4 \mid 0 < c < d\}$ et $\{4e^2/4 + f^2 \mid e, f > 0\} \cup \{g^2/2 + h^2/2 \mid 0 < g < h\}$ with $g + h$ even.

Possible chords

The possible chords are not the same. Blue–Red is not possible for shape II the one which contains the rectangle.

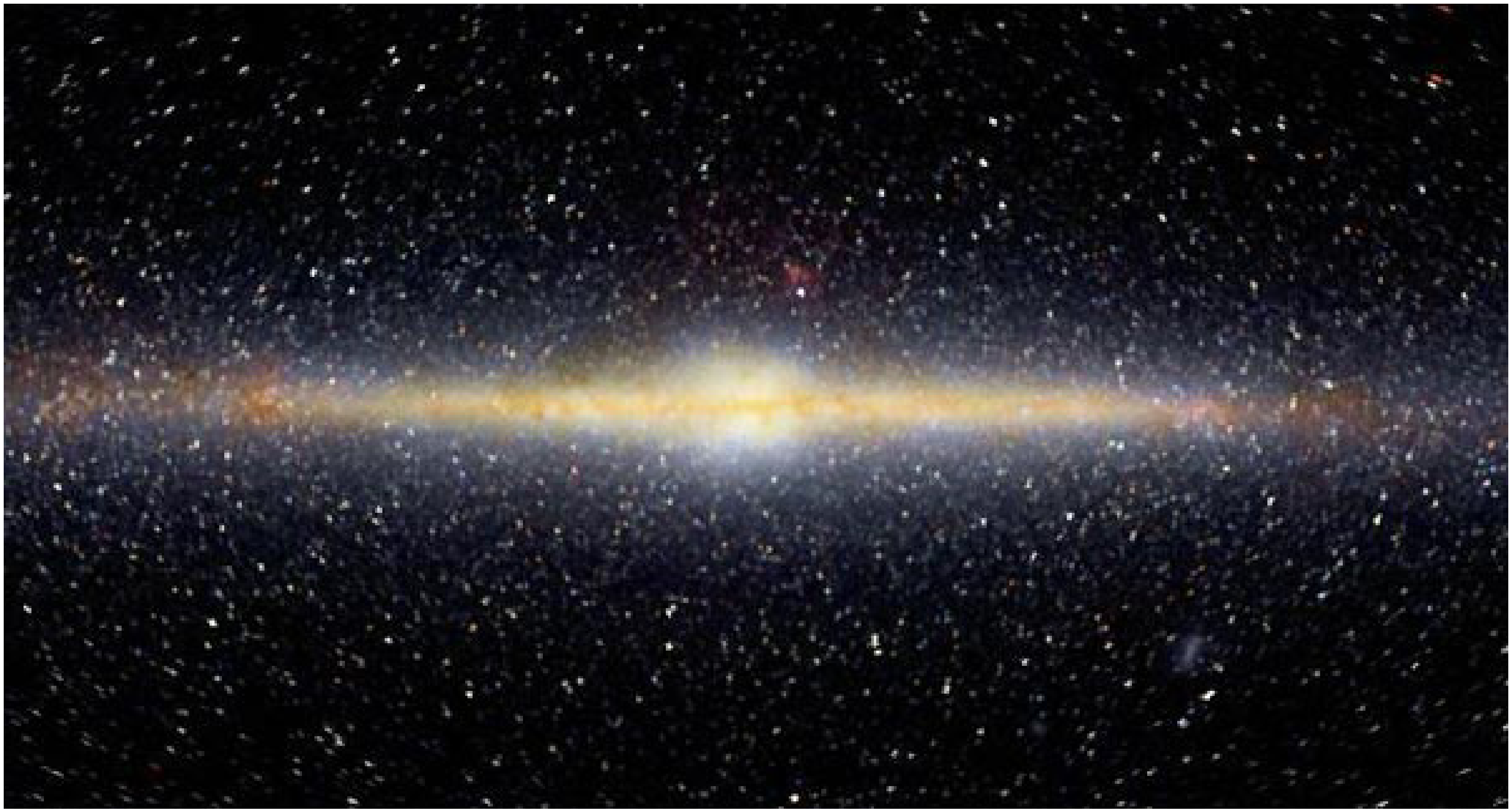


5
4
3
2
0
10
20
30
40
50

Points

The missing invariant should be interpreted as giving the probability for correlations between the possible frequencies, while a "point" of the geometric space X can be thought of as a correlation, *i.e.* a specific positive hermitian matrix $\rho_{\lambda\kappa}$ (up to scale) which encodes the scalar product at the point between the eigenfunctions of the Dirac operator associated to various frequencies *i.e.* eigenvalues of the Dirac operator.

It is rather convincing also that our faith in outer space is based on the strong correlations that exist between different frequencies, as encoded by the matrix $g_{\lambda\mu}$, so that the picture in infrared of the milky way is not that different from its visible light counterpart, which can be seen with a bare eye on a clear night.



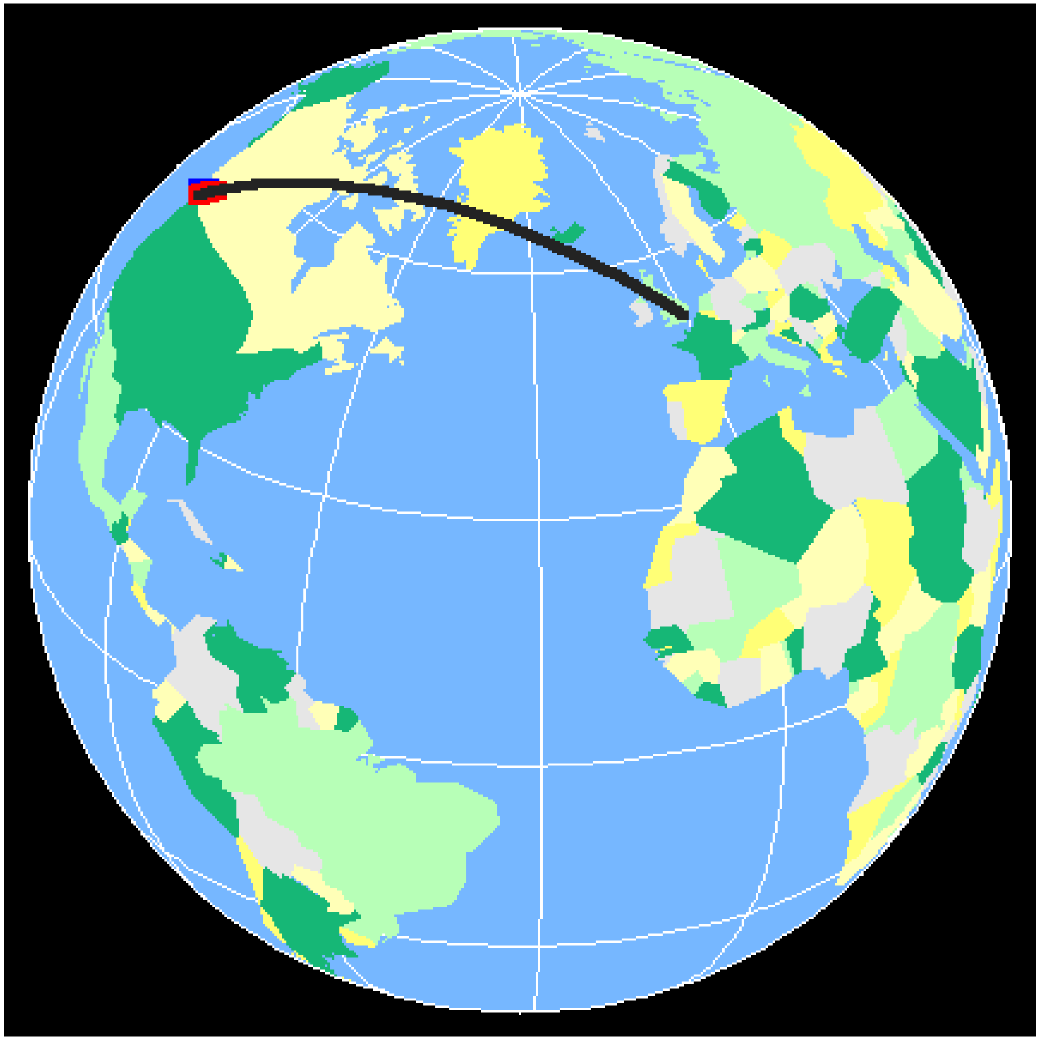
Geometry

Developing geometry for spaces whose coordinates do not commute leads to a spectral version of geometry intimately related to the formalism of quantum mechanics.

Riemannian paradigm

The Riemannian paradigm is based on the Taylor expansion in local coordinates of the square of the line element and in order to measure the distance between two points one minimizes the length of a path joining the two points

$$d(a,b) = \mathrm{Inf} \int_\gamma \sqrt{g_{\mu\,\nu}\, dx^\mu\, dx^\nu}$$



Dirac

Spectral paradigm

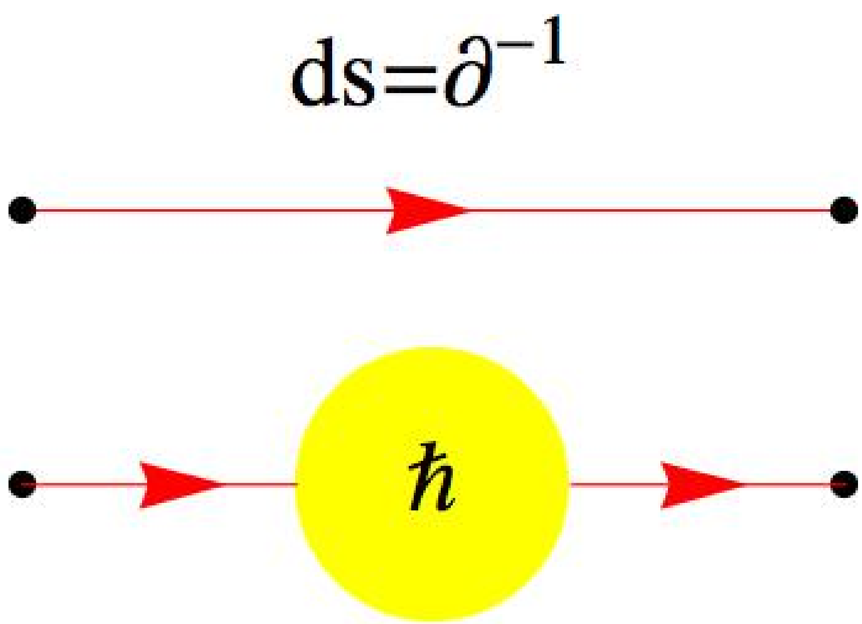
P. Dirac showed how to extract the square root of the Laplacian and this provides a direct connection with the quantum formalism : the line element is the propagator

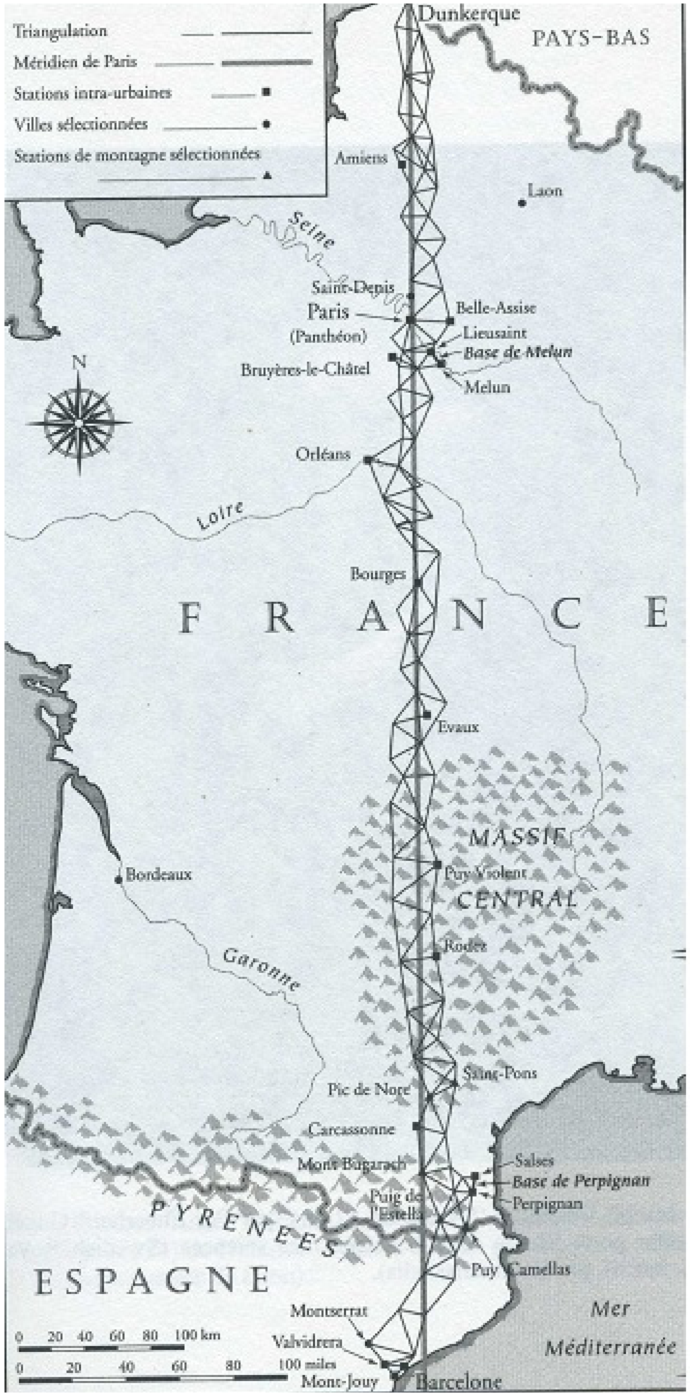
$$ds = D^{-1}$$

$$d(a,b) = \text{Sup}\,|f(a) - f(b)| \mid \|[D,f]\| \leq 1.$$

This is a "Kantorovich dual" of the usual formula.

Line Element





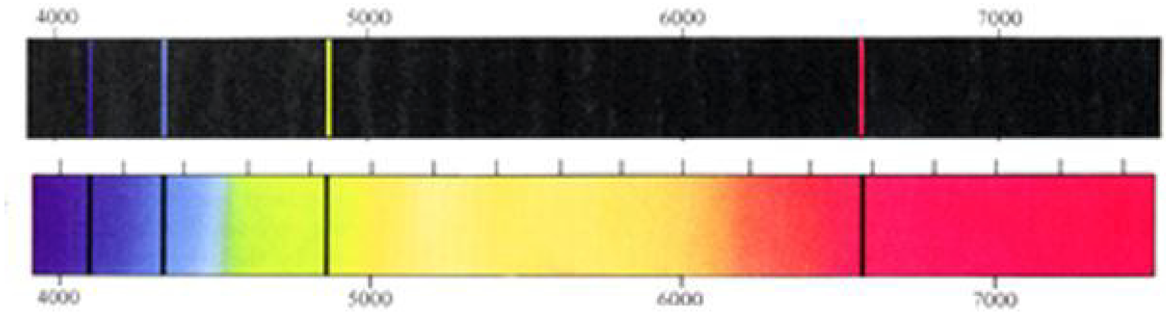
J-B. J. DELAMBRE

P. F. A. MECHAIN

1792--1799

DUNKERQUE--BARCELONE

Change of unit of length, 1967, 1984



Meter $\rightarrow$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Gauge transfos = Int($\mathcal{A}$)

Let us consider the simplest example

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

Algebra of $n \times n$ matrices of smooth functions on manifold M.

The group Int($\mathcal{A}$) of inner automorphisms is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group $SU(n)$

$$1 \to \mathrm{Int}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

Einstein–Yang-Mills

We have shown that the study of pure gravity on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.

Geometry

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Spectral paradigm

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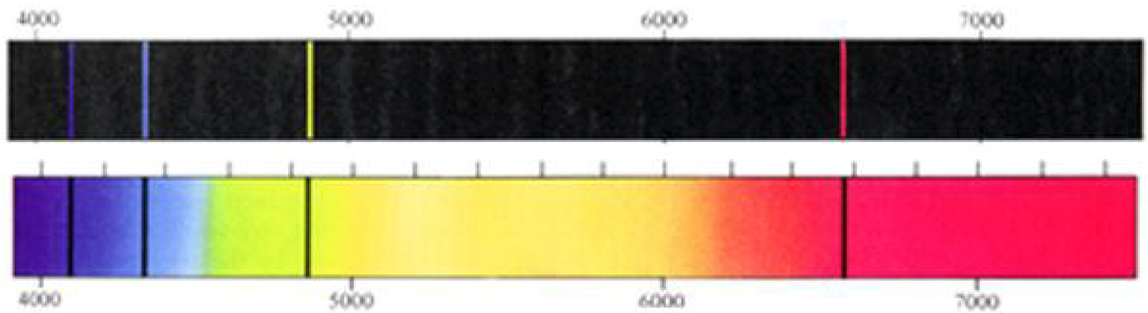
Line element $\sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$	$D^{-1} =$ Fermion propagator

Spectral triples

$$(\mathcal{A}, \mathcal{H}, D)\,, \quad ds = D^{-1}\,,$$

$$d(A,B) = \text{Sup}\,\{|f(A) - f(B)|\,;\, f \in \mathcal{A}\,,\ \|[D,f]\| \leq 1\,\}$$



Meter $\to$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Spectral Geometry

$$d(a,b) = \text{Sup}\,\{|f(a) - f(b)|\,;\; f \in \mathcal{A}\,,\;\; \|[D,f]\| \leq 1\,\}$$

Once we know the spectrum Λ of D, the missing information is contained in the relative position $\text{Spec}_N(M)$ of the two von Neumann algebras : $N = \{h(D)\}$ and M all measurable (bounded) functions. The pair $(M, \mathcal{H})$ does not depend upon the manifold.

The relative spectrum $\text{Spec}_N(M)$ gives the probability for correlations between the possible frequencies, while a "point" of the geometric space X can be thought of as a correlation, *i.e.* a specific positive hermitian matrix $\rho_{\lambda\kappa}$.

Spectral Geometry

Manifold $\leftrightarrow$ Poincaré duality in KO-homology

$$(\mathcal{A}, \mathcal{H}, D)\,, \quad ds = D^{-1}\,,\ J,\ \gamma$$

$$[a, b^0] = 0\,,\ [[D, a], b^0] = 0\,,\ b^0 = J b^* J^{-1}$$

$$J^2 = \varepsilon\,,\ DJ = \varepsilon' JD, \quad J\,\gamma = \varepsilon'' \gamma J, \quad D\gamma = -\gamma D$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

In Physics

— $\mathcal{H}$: one particle Euclidean Fermions
— D : inverse propagator
— J : charge conjugation
— γ : chirality

Reconstruction Theorem

The restriction to spin manifolds is obtained by requiring a *real structure i.e.* an antilinear unitary operator J acting in $\mathcal{H}$ which plays the same role and has the same algebraic properties as the charge conjugation operator in physics.

In the even case the chirality operator γ plays an important role, both γ and J are decorations of the spectral triple.

The following further relations hold for D, J and γ

$$J^2 = \varepsilon\,, \;\; DJ = \varepsilon' JD, \quad J\,\gamma = \varepsilon''\gamma J, \quad D\gamma = -\gamma D$$

The values of the three signs $\varepsilon, \varepsilon', \varepsilon''$ depend only, in the classical case of spin manifolds, upon the value of the dimension n modulo 8 and are given in the following table :

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Metric dimension and KO-dimension

In the classical case of spin manifolds there is thus a relation between the metric (or spectral) dimension given by the rate of growth of the spectrum of D and the integer modulo 8 which appears in the above table. For more general spaces however the two notions of dimension (the dimension modulo 8 is called the KO-dimension because of its origin in K-theory) become independent since there are spaces F of metric dimension 0 but of arbitrary KO-dimension.

Fine Structure

Starting with an ordinary spin geometry M of dimension n and taking the product $M \times F$, one obtains a space whose metric dimension is still n but whose KO-dimension is the sum of n with the KO-dimension of F.

As it turns out the Standard Model with neutrino mixing favors the shift of dimension from the 4 of our familiar space-time picture to $10 = 4 + 6 = 2$ modulo 8.

Finite spaces

In order to learn how to perform the above shift of dimension using a 0-dimensional space F, it is important to classify such spaces. This was done in joint work with A. Chamseddine. We classified there the *finite* spaces F of given KO-dimension. A space F is finite when the algebra $\mathcal{A}_F$ of coordinates on F is finite dimensional. We no longer require that this algebra is commutative.

Classification

We classified the irreducible $(\mathcal{A}, \mathcal{H}, J)$ and found out that the solutions fall into two classes. Let $\mathcal{A}_{\mathbb{C}}$ be the complex linear space generated by $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, the algebra of operators in $\mathcal{H}$. By construction $\mathcal{A}_{\mathbb{C}}$ is a complex algebra and one only has two cases :

1. The center $Z\,(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$, in which case $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ for some k.
2. The center $Z\,(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$ and $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ for some k.

Moreover the knowledge of $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ shows that $\mathcal{A}$ is either $M_k(\mathbb{C})$ (unitary case), $M_k(\mathbb{R})$ (real case) or, when $k = 2\ell$ is even, $M_\ell(\mathbb{H})$, where $\mathbb{H}$ is the field of quaternions (symplectic case). This first case is a minor variant of the Einstein-Yang-Mills case described above.

It turns out by studying their $\mathbb{Z}/2$ gradings γ, that these cases are incompatible with KO-dimension 6 which is only possible in case (2).

KO-dimension 6

If one assumes that one is in the “symplectic–unitary” case and that the grading is given by a grading of the vector space over $\mathbb{H}$, one can show that the dimension of $\mathcal{H}$ which is $2k^2$ in case (2) is at least 2×16 while the simplest solution is given by the algebra $\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. This is an important variant of the Einstein-Yang-Mills case because, as the center $Z\left(\mathcal{A}_{\mathbb{C}}\right)$ is $\mathbb{C} \oplus \mathbb{C}$, the product of this finite geometry F by a manifold M appears, from the commutative standpoint, as two distinct copies of M.

Reduction to SM gauge group

We showed that requiring that these two copies of M stay a finite distance apart reduces the symmetries from the group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ of inner automorphisms of the even part of the algebra to the symmetries $U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ of the Standard Model. This reduction of the gauge symmetry occurs because of the order one condition

$$[[D, a], b^0] = 0\,, \quad \forall\, a, b \in \mathcal{A}$$

Inner fluctuations

Joint work with **Ali Chamseddine**
and **Walter van Suijlekom**

Breaking to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

This leads us to address the issue of the breaking from the natural algebra $\mathcal{A}$ which results from the classification of irreducible finite geometries of KO-dimension 6 (modulo 8), to the algebra corresponding to the SM. This breaking was effected using the requirement of the first order condition on the Dirac operator. This condition was used as a mathematical requirement to select the maximal subalgebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

which is compatible with the first order condition and is the main reason behind the unique selection of the SM.

Fluctuations without order one condition

Our point of departure is that one can extend inner fluctuations to the general case, *i.e.* without assuming the order one condition. It suffices to add a quadratic term which only depends upon the universal 1-form $\omega \in \Omega^1(\mathcal{A})$ to the formula and one restores in this way,

— The gauge invariance under the unitaries $U = uJuJ^{-1}$

— The fact that inner fluctuations are transitive, *i.e.* that inner fluctuations of inner fluctuations are themselves inner fluctuations.

$$D_A = D + A_{(1)} + \widetilde{A}_{(1)} + A_{(2)}$$

where

$$A_{(1)} = \sum_i a_i \left[D, b_i\right]$$

$$\widetilde{A}_{(1)} = \sum_i \widehat{a}_i \left[D, \widehat{b}_i\right], \qquad \widehat{a}_i = J a_i J^{-1}, \quad \widehat{b}_i = J b_i J^{-1}$$

$$A_{(2)} = \sum_{i,j} \widehat{a}_i a_j \left[\left[D, b_j\right], \widehat{b}_i\right] = \sum_{i,j} \widehat{a}_i \left[A_{(1)}, \widehat{b}_i\right].$$

Clearly $A_{(2)}$ which depends quadratically on the fields in $A_{(1)}$ vanishes when the first order condition is satisfied.

Semigroup of inner fluctuations

We show moreover that the resulting inner fluctuations come from the action on operators in Hilbert space of a semi-group Pert($\mathcal{A}$) of *inner perturbations* which only depends on the involutive algebra $\mathcal{A}$ and extends the unitary group of $\mathcal{A}$.

The map η to $\Omega^1(\mathcal{A})$

(i) The following map η is a surjection

$$\eta : \{\sum a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \mid \sum a_j b_j = 1\} \to \Omega^1(\mathcal{A}),$$

$$\eta(\sum a_j \otimes b_j^{\mathrm{op}}) = \sum a_j \delta(b_j).$$

(ii) One has

$$\eta\left(\sum b_j^* \otimes a_j^{*\mathrm{op}}\right) = \left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)^*$$

(iii) One has, for any unitary $u \in \mathcal{A}$,

$$\eta\left(\sum u a_j \otimes (b_j u^*)^{\mathrm{op}}\right) = \gamma_u\left(\eta\left(\sum a_j \otimes b_j^{\mathrm{op}}\right)\right)$$

where γ_u is the gauge transformation of potentials.

Semigroup Pert($\mathcal{A}$)

(i) The self-adjoint normalized elements of $\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$ form a semi-group Pert($\mathcal{A}$) under multiplication.

(ii) The transitivity of inner fluctuations (*i.e.* the fact that inner fluctuations of inner fluctuations are inner fluctuations) corresponds to the semi-group law in the semi-group Pert($\mathcal{A}$).

(iii) The semi-group Pert($\mathcal{A}$) acts on real spectral triples through the homomorphism

$$\mu : \mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$$

given by

$$A \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \mapsto \mu(A) = A \otimes \hat{A} \in \left(\mathcal{A} \otimes \hat{\mathcal{A}}\right) \otimes \left(\mathcal{A} \otimes \hat{\mathcal{A}}\right)^{\mathrm{op}}$$

(iv) Let $A = \sum a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ normalized by the condition $\sum a_j b_j = 1$. Then the operator $D' = D(\eta(A))$ is equal to the inner fluctuation of D with respect to the algebra $\mathcal{A} \otimes \widehat{\mathcal{A}}$ and the 1-form $\eta(A \otimes \widehat{A})$, that is

$$D' = D + \sum a_i \widehat{a}_j [D, b_i \widehat{b}_j]$$

(v) An inner fluctuation of an inner fluctuation of D is still an inner fluctuation of D, and more precisely one has, with A and A' normalized elements of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ as above,

$$(D(\eta(A)))\,(\eta(A')) = D(\eta(A'A))$$

where the product $A'A$ is taken in the tensor product algebra $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.