Quantum field theory and gravitation

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Introduction

Since about a century, the relation between quantum physics and gravitation is not fully understood.

Quantum physics: very successful in nonrelativistic physics where precise mathematical results can be compared with experiments,

somewhat less successful in elementary particle physics where theory delivers only the first terms of a formal power series, but up to now also excellent agreement with experiments.

General relativity: also excellent confirmation by astronomical data. Deviations can explained by plausible assumptions (dark matter, dark energy).

Essentially open: Consistent theory which combines general relativity and quantum physics.

Plan of the lecture: A systematic development of quantum field theory on curved spacetimes with an attempt to include also perturbative quantum gravity.

Surprises in QFT on curved backgrounds

Classical field theory, e.g. Maxwell's theory of the electromagnetic field, can easily be formulated on generic Lorentzian manifolds. The electromagnetic field strength is considered as a 2-form

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu$$

and Maxwell's equation take the form

$$dF = 0$$

with the exterior differentiation d and

$$\delta F = j$$

where $\delta = \star^{-1} d \star$ is the coderivative, and j is a conserved current.

Structures which enter are

- Spacetime as a smooth orientable manifold M, hence differential forms and exterior derivative are well defined.
- A nondegenerate metric g and an orientation in terms of which the Hodge dual can be defined:

$$\star : \Lambda^k(M) \to \Lambda^{n-k}(M), \ n = \mathrm{dim} M$$

The standard formalism of QFT, however, relies heavily on Poincaré symmetry:

- Particles are defined as irreducible representations of the (covering of the) Poincaré group.
- There is a distinguished state, the vacuum, understood as the state with all particles absent.
- The main physical observable is the S-matrix, describing the transition from incoming to outgoing particle configurations.
- Momentum space (as the dual of the subgroup of translations) plays an important role for calculations.
- Transition to imaginary time (euclidean QFT) is often helpful in order to improve convergence.

None of these features is present for QFT on generic Lorentzian spacetimes:

- Generically, the group of spacetime symmetries is trivial.
- Accordingly, the very concept of particles is no longer available.
- In particular, the concept of the vacuum as state without particles becomes meaningless.
- Transition to imaginary times (and a corresponding transition to a Riemannian space) is possible only in special cases.
- Calculations relying on momentum space cannot be done.
- There is no unique definition of the Feynman propagator.

First observations:

- "Particle creation": In free field theory, one might introduce a particle concept appropriate for some spatial hypersurface. But comparison of the particle numbers on different hypersurfaces yields particle creation (typically infinite changes).
- "Hawking radiation": In the analysis of a scalar field in the field of a collapsing star one finds that an initial ground state (in the static situation before the collapse) evolves into a state with thermal radiation after the collapse.
- "Unruh effect": Even on Minkowski space, for a uniformly accelerated observer, the vacuum gets thermal properties.

Strategy for the formulation of QFT on curved spacetime:

- Decouple local features (field equations, commutation relations) from nonlocal features (correlations).
  This amounts to construct, in a first step, the algebra of observables as an abstract algebra and consider afterwards representations by Hilbert space operators ("algebraic approach to QFT" (Haag-Araki-Kastler)).
- Find a local version of the spectrum condition ("positivity of energy") which is the most important structural impact of the Hilbert space representation of QFT on Minkowski space.

Plan of the lecture:

1. Lorentzian geometry and field equations
2. Quantization
3. Microlocal spectrum condition
4. Renormalization
5. Covariance
6. Gauge theories and gravity
7. Outlook

A spacetime is a smooth manifold equipped with a metric g with Lorentzian signature $(+---)$.
A causal curve is a smooth curve γ with a tangent vector $\dot{\gamma}$ which is timelike or lightlike ,

$$g(\dot{\gamma},\dot{\gamma}) \geq 0 \ .$$

We assume that our spacetime is time orientable,
i.e. there exists a smooth vector field v which is everywhere timelike.
The choice of such a vector field induces a time orientation:
A timelike or lightlike tangent vector ξ at some spacetime point x is called future directed if

$$g(\xi, v(x)) > 0 \ .$$

A causal curve is called future directed if all its tangent vectors are future directed.
This allows to introduce the future $J_+(x)$ of a point x as the set of points which can be reached from x by a future directed causal curve:

$$J_+(x) = \{y \in M | \exists \gamma : x \to y \text{ future directed}\}$$

In an analogous way the past J_- of a point is defined.
Note that future and past are in general not closed.

Crucial for the following is the concept of global hyperbolicity.

Definition

A spacetime is called globally hyperbolic if it does not contain closed causal curves and if for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.

Globally hyperbolic spacetimes have many nice properties:

- They have a Cauchy surface, i.e. a smooth spacelike hypersurface which is hit exactly once by each nonextendible causal curve.
- They have even a foliation by Cauchy surfaces, and all Cauchy surfaces are diffeomorphic, i.e. globally hyperbolic spacetimes are diffeomorphic to $\Sigma \times \mathbb{R}$ with Cauchy surfaces $\Sigma \times \{t\}, t \in \mathbb{R}$.

- Normally hyperbolic linear partial differential equations, i.e. with principal symbol the inverse metric, considered as a function on the cotangent bundle, have a well posed Cauchy problem. In particular they have unique retarded and advanced Green's functions.

The simplest example is the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0$$

with the d'Alembertian

$$\Box = |\det g|^{-\frac{1}{2}}\partial_\mu (g^{-1})^{\mu\nu} |\det g|^{\frac{1}{2}}\partial_\nu$$

in local coordinates.
Let $\Delta_{R/A}$ denote the retarded and advanced propagators,

$$\Delta_{R/A} : \mathcal{D} \to \mathcal{E} \ ,$$

$\mathcal{E}$ space of smooth functions, $\mathcal{D}$ subspace of functions with compact support.

They are uniquely characterized by the conditions

$$(\Box + m^2) \circ \Delta_{R/A} = \Delta_{R/A} \circ (\Box + m^2) = \mathrm{id}_{\mathcal{D}}$$

and the support condition

$$\mathrm{s}upp\ \Delta_{R/A} f \subset J_{\pm}(\mathrm{s}upp\ f)$$

with the future (past) $J_{\pm}$ of a subset of spacetime.

The difference

$$\Delta = \Delta_R - \Delta_A$$

(often called the causal propagator, and later named the commutator function) has the property

- The bilinear form on $\mathcal{D}$

$$\sigma(f,g) = \int f(x)(\Delta g)(x) d\mathrm{vol}(x) \ ,$$

is antisymmetric, and $\sigma(f,g) = 0$ for all $g \in \mathcal{D}$ iff $f = (\Box + m^2)h$ for some h with compact support.

This leads to the Poisson bracket

$$\{\varphi(x), \varphi(y)\} = \Delta(x,y)$$

for the classical Klein Gordon field φ.

2. Quantization

The space of field configurations for a scalar field is the space $\mathcal{E}$ of smooth functions. Observables are functionals $F : \mathcal{E} \to \mathbb{C}$.
Examples:

- Regular linear functionals

$$F(\varphi) = \int \varphi(x) f(x)$$

  with a test density f.

- Regular polynomials

$$F(\varphi) = \sum_{n=0}^{N} \int \varphi(x_1) \ldots \varphi(x_n) f_n(x_1, \ldots, x_n)$$

  with test densities f_n in n variables.

- Local functionals ("Lagrangians")

$$F(\varphi) = \int f(x, \varphi(x), \partial\varphi(x), \dots)$$

with a density valued function f on the jet space of M.

Functional derivatives: $F^{(n)} \equiv \frac{\delta F}{\delta \varphi^n}$ are compactly supported distributional densities in n variables, symmetrical under permutations of arguments, determined by

$$\langle F^{(n)}(\varphi), \psi^{\otimes n} \rangle = \frac{d^n}{d\lambda^n} F(\varphi + \lambda\psi)|_{\lambda=0}$$

Canonical structure: Poisson bracket on functionals of field configurations by the Peierls bracket:

$$\{F, G\} = \int \frac{\delta F}{\delta\varphi(x)} \Delta(x, y) \frac{\delta G}{\delta\varphi(y)}$$

Deformation Quantization: Find an $\hbar$-dependent associative product $*_\hbar$ on the space of functionals such that in the limit $\hbar \to 0$

$$(F *_\hbar G)(\varphi) \to F(\varphi)G(\varphi) \ ,$$

$$\frac{1}{i\hbar}(F *_\hbar G - G *_\hbar F) \to \{F, G\} \ .$$

First approach: Weyl-Moyal quantization.

Define the $*$-product in terms of the commutator function

$$(F * G)(\varphi) = e^{\frac{i\hbar}{2} \int \frac{\delta}{\delta\varphi_1(x)} \Delta(x,y) \frac{\delta}{\delta\varphi_2(y)}} F(\varphi_1)G(\varphi_2)|_{\varphi_1 = \varphi_2 = \varphi}$$

For linear functionals $F(\varphi) = \int \varphi f, G(\varphi) = \int \varphi g$ we find

$$F * G(\varphi) = \int \varphi(x)\varphi(y)f(x)g(y) + \frac{i\hbar}{2}\int f(x)\Delta(x,y)g(y)$$

Problem: Product is ill defined on nonlinear local functionals.
Example: Formally

$$\int \varphi(x)^2 f(x) * \int \varphi(y)^2 g(y) =$$

$$\int f(x)g(y)\left(\varphi(x^2)\varphi(y)^2 + 2i\hbar\Delta(x,y)\varphi(x)\varphi(y) - \frac{\hbar^2}{2}\Delta(x,y)^2\right)$$

Δ^2 is not a well defined distribution.

3. Microlocal spectrum condition

The conditions on deformation quantization,

$$(F *_\hbar G)(\varphi) \to F(\varphi)G(\varphi) \ ,$$

$$\frac{1}{i\hbar}(F *_\hbar G - G *_\hbar F) \to \{F, G\} \ ,$$

do not fix the $*$-product.
We can replace in the formula

$$(F * G)(\varphi) = e^{\frac{i\hbar}{2}\int \frac{\delta}{\delta\varphi_1(x)}\Delta(x,y)\frac{\delta}{\delta\varphi_2(y)}} F(\varphi_1)G(\varphi_2)|_{\varphi_1=\varphi_2=\varphi}$$

the distribution $\frac{i}{2}\Delta$ by any distribution H with

$$H(x, y) - H(y, x) = i\Delta(x, y) \ .$$

The new $*_H$-product is equivalent to the previous one:

Namely, let $\Gamma_H = e^{\frac{\hbar}{2}\int H(x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}}$. Then

$$\Gamma_H F *_H \Gamma_H G = \Gamma_H(F * G)\ .$$

Wish list for H:

1. H is a bisolution of the Klein-Gordon equation, hence the functionals F vanishing on solutions form an ideal.
2. Pointwise products of H exist, such that $*_H$-products of polynomial local functionals are well defined.
3. H is a distribution of positive type,

$$\int \overline{f}(x)H(x,y)f(y) \geq 0$$

   for all complex valued test densities f.
   Then the evaluation maps

$$F \to F(\varphi)$$

   are states, i.e. $(\overline{F} *_H F)(\varphi) \geq 0$.

4. H should locally select positive frequencies, thus fulfilling the requirements on positivity of energy in the small.

Example:
On Minkowski space, the positive frequency part Δ_+ of Δ (the Wightman 2-point function),

$$\Delta_+(x,y) =$$

$$\int \frac{d^3\mathbf{p}}{2\omega(\mathbf{p})} e^{-i\omega(\mathbf{p})(x^0-y^0)+i\mathbf{p}(\mathbf{x}-\mathbf{y})} \ ,$$

with $\omega(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}$, fulfills all conditions.

The conditions can be discussed in terms of the wave front sets of the arising distributions.

The wave front set of Δ is

$$\mathrm{WF}\Delta =$$

$$\{(x, x'; k, k') | \exists \text{ a null geodesic } \gamma : x \to x',$$
$$\text{such that } k || g(\dot{\gamma}(0), \cdot) \text{ and } P_\gamma k + k' = 0\}$$

Here P_γ denotes the parallel transport along γ.

Wave front set of Δ_+:

$$\mathrm{WF}\Delta_+ = \{(x, x'; k, k') \in \mathrm{WF}\Delta, \langle k, v(x) \rangle > 0\}$$

This is one of the smallest possible wave front sets under the condition $2\mathrm{Im}\Delta_+ = \Delta$ (the other is obtained by reversing the sign in $\langle k, v(x) \rangle > 0\rangle$ and is the wave front set of $\Delta_- = -\overline{\Delta}_+$).
But Δ_- is not of positive type.
(Lucky coincidence of positivity of states and positivity of energy.)

Definition of Hadamard functions (Radzikowski):

Definition

A bisolution H of the Klein-Gordon equation is called a Hadamard function if

1. $2\mathrm{Im}H = \Delta$
2. $\mathrm{WF}H = \{(x, x'.k, k') \in \mathrm{WF}\Delta, \langle k, v(x)\rangle > 0\rangle\}$ (microlocal spectrum condition (μSC))
3. H is of positive type.

Before the work of Radzikowski, Hadamard functions were characterized by their singular structure. In 4 dimensions one finds for x and y in a geodesically convex neighborhood V, and x and y spacelike separated

$$H(x,y) = \frac{u(x,y)}{\sigma(x,y)} + v(x,y) \log \sigma(x,y) + w(x,y)$$

with smooth function u, v, w and

$$\sigma(x,y) = l(\gamma_{x,y})^2$$

with the length l of the (unique) geodesic which connects x and y within the neighborhood V.

Radzikowski proved that his condition (later called the microlocal spectrum condition) is equivalent to the previous one (which was precisely formulated by Kay and Wald immediately before Radzikowski's work).

The new definition makes the machinery of microlocal analysis available for QFT on curved spacetimes.

Immediate results:

- Wick polynomials as operator valued distributions (Brunetti,F,Köhler):
  Correlation functions of Wick polynomials are expressed in terms of products of (derivatives of) H. The wave front sets of all these products satisfy the condition that their sum never hits the zero section. As a consequence, the fluctuations of the smeared energy momentum tensor are finite.
- "Quantum energy inequalities" (Fewster). The energy density has a finite expectation value in a Hadamard state, but can become arbitrarily negative (this holds even for the vacuum state on Minkowski space). But after smearing with a square of a real test function it becomes bounded from below.
- UV renormalization (Brunetti,F; Hollands, Wald)

4. Renormalization

Program of causal perturbation theory (Stückelberg, Bogoliubov, Epstein-Glaser):

Define the time ordered products of Wick products of free fields as operator valued distributions on Fock space.

Here the time ordered products are supposed to satisfy a few axioms, the most important one being that the time ordered product coincides with the operator product if the arguments are time ordered.

Epstein and Glaser (1973) succeeded in proving that solutions satisfying the axioms exist and that the ambiguity is labeled by the known renormalization conditions. The solution can be either constructed directly or via one of the known methods (BPHZ, Pauli-Villars, momentum cutoff and counter terms,etc.)

Application of causal perturbation theory to curved spacetimes requires a completely local reformulation. In particular

- No reference to Fock space
- No use of translation symmetry
- New concept of the adiabatic limit
- Universal renormalization conditions

Crucial ingredient is the time ordering operator $T : \mathcal{F}_{\mathrm{mloc}} \to \mathcal{F}_{\mu\mathrm{c}}$

$$T = e^{\frac{1}{2}\langle H_F, \frac{\delta^2}{\delta\varphi^2}\rangle}$$

with the Feynman propagator determined by H, $H_F = H + i\Delta_R$.

$\mathcal{F}_{\mathrm{mloc}}$ unital algebra (with respect to the pointwise (classical) product) generated by local functionals.
$\mathcal{F}_{\mathrm{loc}}$ space of compactly supported local functionals which vanish at $\varphi = 0$.
$\mathcal{F}_{\mu\mathrm{c}}$ space of microcausal functionals, i.e. compactly supported smooth functionals with $\mathrm{WF} F^{(n)} \cap M \times (\overline{V}_+^n \cup \overline{V}_-^n) = \emptyset$.

Example: Let $F(\varphi) = \int \varphi(x)^2\varphi(y)^2 f(x)g(y)$. Then

$$TF(\varphi) =$$

$$F(\varphi)+4\hbar \int H_F(x,y)\varphi(x)\varphi(y)f(x)g(y)+2\hbar^2 \int H_F(x,y)^2 f(x)g(y)$$

The wave front set of H_F is the union of the wave front set of H and the wave front set of the δ-function.

$$\mathrm{WF}\delta = \{(x,x;k,-k), k \neq 0\}$$

Hence the wave front set of H_F does satisfy the criterion for multiplication of distributions only outside of the diagonal $\mathrm{diag}(M) = \{(x,x), x \in M\}$.

Mathematical problem: Extend a distribution which is defined outside of a submanifold to the full manifold.
Can be reduced to the extension problem of distributions on $\mathbb{R}^n \setminus \{0\}$. The criterion is Steinmann's scaling degree.

Definition

Let t be a distribution on an open cone in $\mathbb{R}^n$. The scaling degree is defined as

$$\mathrm{sd}(t) = \inf\{\rho \in \mathbb{R} | \lambda^\rho t(\lambda \cdot) \to 0\}$$

Theorem

Let t be a distribution on $\mathbb{R}^n \setminus \{0\}$.

- *If $\mathrm{sd} t < \infty$ then t has extensions with the same scaling degree. Two such extensions differ by a derivative of the δ-function of order $\leq \mathrm{sd} t - n$.*
- *If $\mathrm{sd} = \infty$ then t cannot be extended to a distribution on $\mathbb{R}^n$.*

This theorem replaces completely the standard regularization techniques. They may however be used for getting an explicit choice among the possible extensions. The relation to regularization techniques is as follows: In case of a finite scaling degree $\geq n$ the distribution can be uniquely extended to all test functions which vanish at zero with order $\omega = [\mathrm{sd}t - n]$. Choose a projection W on a complementary subspace of $\mathcal{D}(\mathbb{R}^n)$. Then an extension t_W is given by

$$t_W = t \circ (1 - W)$$

All extensions are of this form. W is a projection on a finite dimensional space and has the form

$$W = \sum |w_\alpha\rangle\langle\partial^\alpha\delta|$$

where the functions w_α form a basis of $W\mathcal{D}(\mathbb{R}^n)$ which is dual to $\{\partial^\alpha\delta, |\alpha| \leq \omega\}$, i.e. $\partial^\alpha w_\beta(0) = (-1)^{|\alpha|}\delta^\alpha_\beta$.

Now let t_n be a sequence of distributions on $\mathbb{R}^n$ which converges to t on the space of test functions which vanish at zero of order ω.
Then

$$\begin{aligned} t_W = t \circ (1 - W) &= \lim_n \quad t_n \circ (1 - W) \\ &= \lim_n \quad t_n - \sum \langle t_n, w_\alpha\rangle\langle\partial^\alpha\delta| \end{aligned}$$

This shows the occurence of divergent counter terms.

Construction of T:

- Let $\mathrm{S}\mathcal{F}_{\mathrm{loc}}$ be the symmetric Fock space over the space of local functionals. The multiplication map

$$m : \mathrm{S}\mathcal{F}_{\mathrm{loc}} \to \mathcal{F}_{mloc} \ , \ F_1 \otimes \cdots \otimes F_n \mapsto F_1 \dots F_n$$

  is bijective.

- Define n-linear maps

$$T_n : \mathrm{S}_n\mathcal{F}_{\mathrm{loc}} \to \mathcal{F}_{\mu\mathrm{c}}$$

  by

$$T_n = m \circ \exp \sum_{i<j} D_{ij}$$

  with $D_{ij} = \langle \frac{\delta}{\delta\varphi_i} H_F \frac{\delta}{\delta\varphi_j} \rangle$.

- The definition of T_n involves renormalization and is performed inductively in n by extending distributions. Define T by

$$T = \sum T_n \circ m^{-1} \ .$$

The time ordering operator T can be formally understood in terms of path integrals. Namely

$$TF(\varphi) = \int d\mu_{H_F}(\psi) F(\varphi - \psi)$$

where μ_{H_F} is the oscillating Gaussian "measure" with covariance H_F.

The renormalized time ordered product is now defined on the space $T\mathcal{F}_{\mathrm{mloc}}$ by

$$F \cdot_T G = T(T^{-1}F \cdot T^{-1}G)$$

where $\cdot$ denotes the classical (pointwise) product. We have the following theorem (F-Rejzner)

Theorem

The renormalized time ordered product $\cdot_T$ is equivalent to the classical product. It is in particular commutative and associative.

The formal S-matrix for a local functional L is now defined as the time ordered exponential:

$$S(L) = T \circ \exp \circ T^{-1} L \equiv \exp_T L \ .$$

Note that T acts trivially on local functionals. This corresponds to doing normal ordering with respect to H. If we change H then L is transformed.

Bogoliubov's formula gives rise to a Moeller map from the free to the interacting theory

$$R_L(F) = S(L)^{-1} * (S(L) \cdot_T F)$$

Here $T^{-1}F$ is multilocal.

Renormalization group:

The ambiguity in the construction of T can be described in terms of the renormalization group $\mathcal{R}$ of Stückelberg and Petermann. It is related, but not identical to the renormalization group of Wilson (which is not a group). $\mathcal{R}$ is the set of all analytic bijections Z of $\mathcal{F}_{\mathrm{loc}}$ which are local, i.e.

$$Z(F+G) = Z(F) + Z(G)$$

if F and G have disjoint support, and have $Z^{(1)}(0) = \mathrm{id}$.
The main theorem of renormalization (Stora-Popineau, Pinter, Hollands-Wald, Dütsch-F,Brunetti-F-Dütsch) states

Theorem

Let S be a formal S-matrix. Then any other formal S-matrix $\hat{S}$ is obtained by

$$\hat{S} = S \circ Z$$

for a unique $Z \in \mathcal{R}$.

5. Covariance

In the construction of the renormalized theory the extensions of distributions had to be done independently for every point. This induces a huge ambiguity in the theory and would it make difficult to compare results obtained at different points of spacetime. The difficulty is the absence of isometries in the generic situations. In highly symmetric situations, e.g. for de Sitter spacetime, the problem does not appear.

The solution of this problem is to construct the theory on all spacetimes of a given class simultaneously in a coherent way.

It is best formulated in the language of algebraic field theory. There one associates to suitable subregions of a given spacetime unital *-algebras such that certain axioms (the Haag-Kastler axioms) are satisfied.

We may consider these subregions as spacetimes in their own right and generalize the Haag-Kastler framework in the following way:

- Associate to every globally hyperbolic, contractible, orientable and time oriented manifold M of a given dimension a unital (C)*-algebra $\mathfrak{A}(M)$.
- For every isometric, causality and orientation preserving embedding $\chi : M \to N$ there existe an injective homomorphism $\mathfrak{A}\chi : \mathfrak{A}(M) \to \mathfrak{A}(N)$.
- If $\chi : M \to N$ and $\psi : N \to L$ are admissible embeddings as characterized above then $\mathfrak{A}\psi \circ \chi = \mathfrak{A}\psi \circ \mathfrak{A}\chi$.
- If $\chi : M \to L$ and $\psi : N \to L$ are admissible embeddings such that $\chi(M)$ is spacelike separated from $\psi(N)$, then

$$[\mathfrak{A}\chi(\mathfrak{A}(M)), \mathfrak{A}\psi(\mathfrak{A}(N))] = \{0\} \ .$$

- If $\chi(M)$ contains a Cauchy surface of N, then $\mathfrak{A}\chi$ is an isomorphism.

In other words, the quantum field theory is considered to be a functor
from the category of spacetimes (with admissible embeddings as morphisms)
to the category of unital (C)*-algebras (with injective homomorphisms as morphisms)
(locally covariant QFT (Brunetti-F-Verch)).
In this framework, one can define fields independently of the choice of a spacetime.
Namely a locally covariant quantum field is a natural transformation between the functor of test function spaces and the QFT functor. This means that a field Φ is a family of maps $\Phi_M : \mathcal{D}(M) \to \mathfrak{A}(M)$ such that for $\chi : M \to N$

$$\Phi_N \circ \chi_* = \mathfrak{A}\chi \circ \Phi_M$$

i.e.

$$\Phi_N(\chi(x)) = \mathfrak{A}\chi(\Phi_M(x)) \ .$$

Note that the last equation expresses, in the case $N = M$, the covariance of the field under the symmetry χ.
Thus the condition can be considered as a generalization of covariance to the situations where spacetimes may have no nontrivial symmetries.

Application to renormalization:
Define the time ordering operator as a natural transformation, i.e. if M is an admissible subregion of N then the time ordering T_M must coincide with T_N restricted to functionals with support in M.
Obstruction: there does not exist a natural Hadamard function, i.e. a family H_M such that $H_N(\chi(x), \chi(y)) = H_M(x, y)$. This is related to the nonexistence of a vacuum.
Solution was obtained by Hollands and Wald.

6. Gauge theories and gravity

Up to now, only scalar theories are considered.
Dirac and Majorana fields create no fundamentally new problems, but require some work on consistent choice of signs and factors i.

In gauge theories and gravity, however, new phenomena occur since the field equations have no well posed Cauchy problem, due to the gauge symmetry.

In classical field theory one usually fixes the gauge such that the Cauchy problem becomes well posed.

In quantum field theory, and also in the canonical formalism of classical field theory this is not directly possible since the expressions one would like to set to 0 have nonvanishing commutators (or Poisson brackets, respectively).

One therefore chooses another way and extends the theory such that the Cauchy problem becomes well posed, and then extracts from this the physically relevant theory.

In electrodynamics, e.g., one introduces the so-called Nakanishi-Lautrup field B and adds a term $B\delta A + \frac{\lambda}{2}B^2$ to the Lagrangian.
After construction of the theory, one considers the commutant of B, divides out the ideal generated by B and obtains at the end the algebra of observables.

In Yang Mills theories one needs in addition to add scalar fermionic fields, the Faddeev-Popov ghosts and antighosts. One finds a graded derivation, the BRST transformation s which satisfies $s^2 = 0$. The cohomology of s, i.e.$\mathrm{Ke}s/\mathrm{Ra}s$ yields then the algebra of observables.

In gravity, an analogous procedure leads to a trivial cohomology, due to the absence of local observables.

Way out: Use dynamical fields as coordinates (possible for generic field configurations).

Quantization: Choose a generic globally hyperbolic metric g_0. Expand the extended action around g_0 to second order in $h = (g - g_0), b_\mu, c^\mu, \bar{c}_\mu$ and choose a Hadamard solution for the linearized field equation.

Construct the time-ordered product such that BRST invariance holds.

Show, that infinitesimal changes of the background do not change the theory (principle of perturbative agreement (Hollands-Wald)).

Construct states around solutions of the classical field equations.

Conclusions and outlook

- The functorial approach to quantum field theory, originally developed for the purposes of renormalization on curved spacetimes, offers a framework for a background independent approach to quantum gravity.
- The problem of nonrenormalizability is open. An interpretation as an effective field theory seems to be possible.
- The flow of the renormalization group should be studied and compared with the results of Reuter et al.
- Applications to physical phenomena are urgently needed.
- The restriction to generic backgrounds is a practical obstacle for the discussion of concrete examples. A way out might be the addition of fields like the dust fields (Brown-Kuchar) as used e.g. in loop quantum gravity.