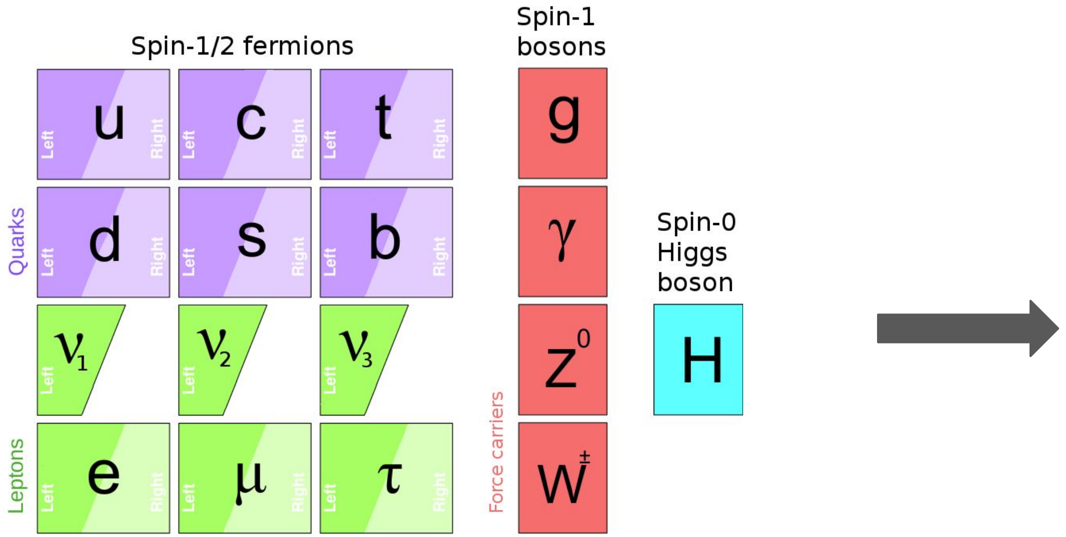
Consistent propagation of spin-3/2 and spin-2 in electromagnetic background

Wenqi Ke (LPTHE)

LPTHE
LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

RPP, Tours, 2022



Standard Model: spin ≤ 1

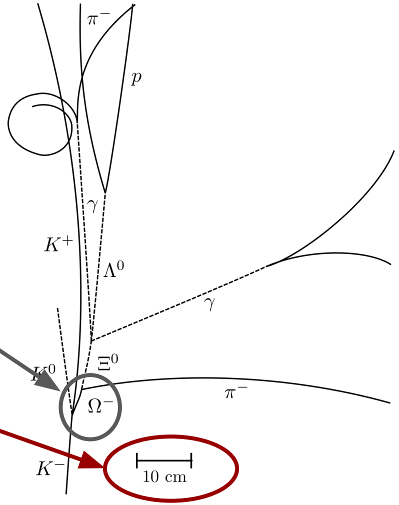
Why are we interested in spin > 1?

Why are we interested in spin > 1?

- **A mathematical problem**
  (Irreducible representations of the Poincaré group are classified by **<u>spin</u>** and **<u>mass</u>**)

- **Important for gravity**
  (Graviton: spin-2, “gravitino” in sugra: spin-3/2)

- **Composite states of spin>1 exist in Nature!**
  e.g. Ω^- baryon (spin-3/2)



We can take the point particle limit

$\Longrightarrow$ **Consistent Lagrangian and EoM will be very useful**

Massive free particles: some early progress

- **1939** — Fierz, Pauli
- **1941** — Rarita, Schwinger
- **1974** — Singh, Hagen

Massive spin-2 $\Rightarrow$ symmetric tensor h_{mn}

$$\mathcal{L} = \frac{1}{2} h^{mn}\left(\partial^2 - M^2\right) h_{mn} - \frac{1}{2} h \left(\partial^2 - M^2\right) h + h_{mn}\partial^m \partial^n h + \partial^n h_{mn} \partial_k h^{mk}$$

Linear expansion of Einstein-Hilbert+mass terms

$$\left(\partial^2 - M^2\right) h_{mn} = 0$$ **EoM**

Constraints:

- $h = 0$ — Trace — – 1 DoF
- $\partial^n h_{mn} = 0$ — Divergence — – 4 DoF

On-shell DoF = 10-1-4 = **5** $\Leftrightarrow$ helicity states -2, -1, 0, 1 ,2

Massive free particles: some early progress

1939
Fierz, Pauli

1941
Rarita, Schwinger

1974
Singh, Hagen

Massive spin-3/2 $\Rightarrow$ vector-spinor $\psi_\alpha{}^m$

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}_m \gamma^{mnk}\partial_n \psi_k - \frac{\mathrm{i}}{2} M \bar{\psi}_m \gamma^{mn}\psi_n$$

$$\left(\mathrm{i}\partial\!\!\!/ + M\right)\psi_m = 0$$ **EoM**

$$\gamma^m \psi_m = 0$$ γ-Trace – 2 DoF

$$\partial^m \psi_m = 0$$ Divergence – 2 DoF

Constraints

On-shell DoF = 8-2-2 = **4** $\Leftrightarrow$ helicity states -3/2, -1/2, 1/2, 3/2

Massive free particles: some early progress

1939
Fierz, Pauli

1941
Rarita, Schwinger

1974
Singh, Hagen

♦ Systematic Lagrangian formulation for **arbitrary** spin

♦ The constraint equations are fixed by **auxiliary lower spins**

Lagrangian formulation for arbitrary spin. I. The boson case*†

L. P. S. Singh and C. R. Hagen

Lagrangian formulation for arbitrary spin. II. The fermion case*†

L. P. S. Singh and C. R. Hagen

♦ Bosons of spin s≥2: need auxiliary spin s–2, s–3, ... , 0

♦ Fermions of spin s≥3/2: need auxiliary tensor-spinors of rank s–3/2, s–5/2, ... , 0

Massive charged particles: an old problem

1939
Fierz, Pauli

"…
the most immediate method of taking into account the effect of the **electromagnetic** field, proposed by Dirac, leads to **inconsistent** equations as soon as the **spin is greater than 1**."

1969~1972
Velo, Zwanziger

"The Velo-Zwanziger Problem"

Massive charged particles of spin>1 **minimally coupled** to electromagnetism suffer from **acausality**

Constant or dynamical

Massive charged particles: an old problem

🤔 How to write a **consistent Lagrangian** of **charged massive** spin-2 and spin-3/2 in a **constant** EM background?

🤔 What do their **EoM and constraints** look like?

Massive charged particles: an old problem

How to write a **consistent Lagrangian** of **charged massive** spin-2 and spin-3/2 in a **constant** EM background?

What do their **EoM and constraints** look like?

The essential requirements are:

1) No ghost: correct **on-shell DoF**

   DoF(spin-3/2)=4 , DoF(spin-2)=5

2) Gyromagnetic ratio **g=2** Ferrara, Porrati, Telegdi (1992)
3) No superluminal propagation

Some history of charged massive spin-3/2

(2000)

Massive Spin 3/2 Electrodynamics

S. Deser[♯], V. Pascalutsa[♭] and A. Waldron[♯]

Included a large class of non-minimal couplings, but *none of them* escaped acausality

(2001)

Inconsistencies of Massive Charged Gravitating Higher Spins

S. Deser[♯] and A. Waldron[♭]

N=2 sugra with charged gravitino: *causal* if gravitino has *Planckian mass*

(2009)

Causal Propagation of a Charged Spin 3/2 Field in an External Electromagnetic Background

Massimo Porrati and Rakibur Rahman[1]

A consistent Lagrangian a priori *exists*, with coefficients given implicitly by a *recursive* relation

Some history of charged massive spin-3/2

(2000) Massive Spin 3/2 Electrodynamics

Included a large class of non-minimal couplings, but *none* escaped acausality

(2001) Inc

a with charged *causal* if gravitino has *mass*

No explicit Lagrangian
for massive charged spin-3/2
has been written yet!

(2009) Causal Propagation of a Charged Spin 3/2 Field in an External Electromagnetic Background

Massimo Porrati and Rakibur Rahman[1]

A consistent Lagrangian a priori *exists*, with coefficients given implicitly by a *recursive* relation

Some history of charged massive spin-2

(1961) Federbush added a non-minimal term

Fierz-Pauli → **Federbush**

$$\mathcal{L} = \frac{1}{2} h^{mn}\left(\partial^2 - M^2\right) h_{mn} - \frac{1}{2} h \left(\partial^2 - M^2\right) h + h_{mn}\partial^m \partial^n h + \partial^n h_{mn} \partial_k h^{mk}$$

U(1) covariant derivative

$$\mathcal{L} = \bar{h}^{mn}\left(\mathfrak{D}^2 - M^2\right) h_{mn} - \bar{h}\left(\mathfrak{D}^2 - M^2\right) h + \left(\bar{h}_{mn}\mathfrak{D}^m \mathfrak{D}^n h + \text{h.c.}\right) + 2\mathfrak{D}^n \bar{h}_{mn} \mathfrak{D}_k h^{mk} + 2iegTr\left(h \cdot F \cdot \bar{h}\right)$$

charge

Constant EM field strength

Some history of charged massive spin-2

(1961) Federbush added a non-minimal term

Federbush

$$\mathcal{L} = \bar{h}^{mn}\left(\mathfrak{D}^2 - M^2\right)h_{mn} - \bar{h}\left(\mathfrak{D}^2 - M^2\right)h + \left(\bar{h}_{mn}\mathfrak{D}^m\mathfrak{D}^n h + \text{h.c.}\right) + 2\mathfrak{D}^n\bar{h}_{mn}\mathfrak{D}_k h^{mk} + 2ieg\,\text{Tr}\left(h\cdot F\cdot\bar{h}\right)$$

Trace equation: $h \propto \underbrace{(2g-1)}\, F^{mn}\mathfrak{D}_m\mathfrak{D}^k h_{kn} + (\text{no derivative terms})$

This equation is a **constraint** only for **g=1/2** !

Federbush Lagrangian

- \+ The ONLY available 4D Lagrangian of massive charged spin-2, without ghost, without coupling to other spins
- − No ghost only if g=1/2
- − SUPERLUMINAL propagation !

Porrati, Rahman, Sagnotti (2010)

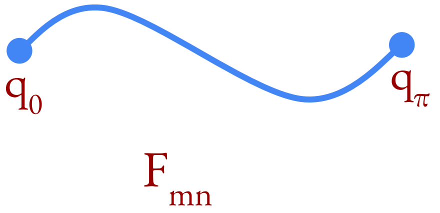
Some history of charged massive spin-2

(1989~1990) Argyres, Nappi derived a Lagrangian using open bosonic string theory

- Originally a model of hadronic resonances
- Presence of a tower of higher spins
- g=2 for all charged states

Coupling an open string to an EM background:

Abouelsaood, Callan, Nappi, Yost (1987)



Total charge of the string: $Q=q_0+q_\pi$

First massive level of open bosonic string:

$$|\Phi\rangle = h_{mn}(x) a_1^{\dagger m} a_1^{\dagger n}|0\rangle + \sqrt{2} i B_m(x) a_2^{\dagger m}|0\rangle$$

Some history of charged massive spin-2

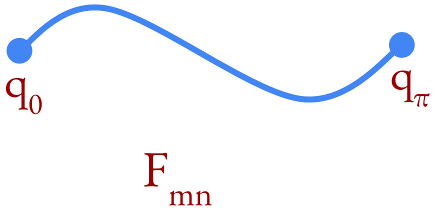
(1

$$|\Phi\rangle = \underbrace{h_{mn}(x)}_{\text{Massive spin-2}} a_1^{\dagger m} a_1^{\dagger n}|0\rangle + \sqrt{2}\mathrm{i}\underbrace{B_m(x)}_{\text{Massive spin-1}} a_2^{\dagger m}|0\rangle$$

Stückelberg field

The only physical field is the massive spin-2

Coupling an open string to an EM background: A saood, Callan, Nappi, Yost (1987)



Total charge of the string: $Q=q_0 + q$

First massive level of open bosonic stri

$$|\Phi\rangle = h_{mn}(x)a_1^{\dagger m}a_1^{\dagger n}|0\rangle + \sqrt{2}\mathrm{i}B_m(x)a_2^{\dagger m}|0\rangle$$

Some history of charged massive spin-2

The Argyres-Nappi Lagrangian

$$\mathcal{L}_{\rm AN} = \bar{\mathcal{H}}_{mn}\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\,\mathrm{Tr}\,\epsilon^2\right)h^{mn} - 2\mathrm{i}\bar{\mathcal{H}}_{mn}(\epsilon h - h\epsilon)^{mn} - \bar{\mathcal{H}}\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\,\mathrm{Tr}\,\epsilon^2\right)\mathcal{H}$$

$$- \bar{\mathcal{H}}_{mn}\left\{\mathfrak{D}^m\mathfrak{D}^k[(1+\mathrm{i}\epsilon)h]_k{}^n - \frac{1}{2}\mathfrak{D}^m\mathfrak{D}^n\mathcal{H} + (m \leftrightarrow n)\right\} + \bar{\mathcal{H}}\mathfrak{D}^m\mathfrak{D}^n\mathcal{H}_{mn}$$

$$\epsilon = \frac{\Lambda^2}{\pi}\left[\operatorname{arctanh}\left(\frac{\pi q_0 F}{\Lambda^2}\right) + \operatorname{arctanh}\left(\frac{\pi q_\pi F}{\Lambda^2}\right)\right]$$

Constant antisymmetric tensor

$$\mathcal{H} \equiv \mathcal{H}^m{}_m$$

Trace

$$\mathcal{H}_{mn} \equiv (1+\mathrm{i}F)_{mk}\,(1+\mathrm{i}F)_{nl}\,h^{kl}$$

Rescaled spin-2 field

$$\mathcal{M}D \equiv \mathfrak{D}, \quad \text{with} \quad \mathcal{M}\mathcal{M}^T = \frac{\epsilon}{QF}$$

Dressed covariant derivative

- ♦ In the free limit, we recover the **Fierz-Pauli Lagrangian**
- ♦ A number of **non-minimal terms**

Some history of charged massive spin-2

The Argyres-Nappi Lagrangian

$$\mathcal{L}_{\text{AN}} = \bar{\mathcal{H}}_{mn}\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\operatorname{Tr}\epsilon^2\right)h^{mn} - 2\mathrm{i}\bar{\mathcal{H}}_{mn}(\epsilon h - h\epsilon)^{mn} - \bar{\mathcal{H}}\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\operatorname{Tr}\epsilon^2\right)\mathcal{H}$$
$$- \bar{\mathcal{H}}_{mn}\left\{\mathfrak{D}^m\mathfrak{D}^k[(1+\mathrm{i}\epsilon)h]_k{}^n - \frac{1}{2}\mathfrak{D}^m\mathfrak{D}^n\mathcal{H} + (m \leftrightarrow n)\right\} + \bar{\mathcal{H}}\mathfrak{D}^m\mathfrak{D}^n\mathcal{H}_{mn}$$

– (mass)^2 in the units with **α'=1/2**

on shell

$$\begin{cases} \left(\mathfrak{D}^2 - 2 - \frac{1}{2}\operatorname{Tr}\epsilon^2\right)\mathcal{H}_{\mu\nu} - 2\mathrm{i}\left(\epsilon\cdot\mathcal{H} - \mathcal{H}\cdot\epsilon\right)_{\mu\nu} = 0 \\ \mathcal{H} = 0 \\ \mathfrak{D}^n\mathcal{H}_{mn} = 0 \end{cases}$$

The mass of the spin-2 depends on the EM background

Some history of charged massive spin-2

The Argyres-Nappi Lagrangian

$$\mathcal{L}_{\rm AN} = \bar{\mathcal{H}}_{mn}\left(\mathfrak{D}^2 - 2 - \ldots \right.$$

$$\ldots - \bar{\mathcal{H}}\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\,\mathrm{Tr}\,\epsilon^2\right)\mathcal{H}$$

$$- \bar{\mathcal{H}}_{mn}\left\{\mathfrak{D}^{m}\ldots \ldots n)\right\} + \bar{\mathcal{H}}\mathfrak{D}^m\mathfrak{D}^n\mathcal{H}_{mn}$$

free limit:

$$\left(\partial^2 - 2\right) h_{mn} = 0$$

$$h = 0$$

$$\partial^n h_{mn} = 0$$

on shell

$$\left(\mathfrak{D}^2 - 2 - \frac{1}{2}\,\mathrm{Tr}\,\epsilon^2\right)\mathcal{H}_{\mu\nu} - 2\mathrm{i}\left(\epsilon\cdot\mathcal{H} - \mathcal{H}\cdot\epsilon\right)_{\mu\nu} = 0$$

$$\mathcal{H} = 0$$

$$\mathfrak{D}^n\mathcal{H}_{mn} = 0$$

Manifestly consistent:

Hyperbolic EoM

Correct DoF (no ghost)

Correct gyromagnetic ratio g=2

Some history of charged massive spin-2

The Argyres-Nappi Lagrangian

BUT:

Consistent only in **D=26** !

(2011) Porrati, Rahman:

- Compactify the Argyres-Nappi Lagrangian to **D<26**
- The Lagrangian of spin-2 is **coupled** to an extra scalar
- The spin-2 and the scalar can be **decoupled on shell**

Correct DoF (no ghost)

Correct gyromagnetic ratio g=2

Back to spin-3/2

Previously:
The bosonic open string leads to a consistent Lagrangian for charged massive **spin-2**

Now:
Does string theory help to obtain a consistent Lagrangian for charged massive **spin-3/2**?

Back to spin-3/2

Previously:
The bosonic open string leads to a consistent Lagrangian for charged massive **spin-2**

Now:
Does string theory help to obtain a consistent Lagrangian for charged massive **spin-3/2**?

Important results, **without** EM background

4D open superstring first mass states: spin-2 h_{mn}, spin-3/2 $\{\chi_m{}^{\alpha}, \lambda_m{}^{\alpha}\}$ ... Berkovits, Leite (1997)

Back to spin-3/2

Previously:
The bosonic open string leads to a consistent Lagrangian for charged massive **spin-2**

Now:
Does string theory help to obtain a consistent Lagrangian for charged massive **spin-3/2**?

Important results, **without** EM background

4D open superstring first mass states: spin-2 h_{mn}, spin-3/2 $\{\chi_m{}^\alpha, \lambda_m{}^\alpha\}$ ... Berkovits, Leite (1997)

A 4D superspace action of the first massive level is derived

On shell, this action describes a **massive spin-2 multiplet**
and two massive scalar multiplets Berkovits, Leite (1998)

Back to spin-3/2

Previously:
The bosonic open string leads to a consistent Lagrangian for charged massive **spin-2**

Now:
Does string theory help to obtain a consistent Lagrangian for charged massive **spin-3/2**?

YES, by generalising to the **charged** case, the studies in

Berkovits, Leite (1997)

Berkovits, Leite (1998)

Higher-Spin States of the Superstring in an Electromagnetic Background

Karim Benakli*[i], Nathan Berkovits†[ii], Cassiano A. Daniel†[iii] and Matheus Lize†[iv]

Back to spin-3/2

(2021)

Higher-Spin States of the Superstring in an Electromagnetic Background

Karim Benakli[*i], Nathan Berkovits[†ii], Cassiano A. Daniel[†iii] and Matheus Lize[†iv]

Main results:

- ♦ **4D superspace action** of the first massive level of charged superstring
- ♦ **12** complex on-shell DoF for bosons, and **12** for fermions
- ♦ The EoM and constraints for spin-2 and spin-3/2 are **consistent**

We have the on-shell equations, where is the **Lagrangian**?

Benakli, Daniel, WK (2022)

The superspace action of the charged states

$$
\begin{aligned}
S = \frac{1}{16}\int d^4x\, p_0^2\bar p_0^2 \Big\{ & V_n^\dagger(\eta^{nm} - i\varepsilon^{nm})\Big[-\{d_0^2, \bar d_0^2\}V_m + 16\Pi_0^n\Pi_{n0}V_m - 32(\eta_{mp} - i\varepsilon_{mp})V^p \\
& - 32\Big((\partial\bar\theta_0\bar d_0)V_m + (\partial\theta_0 d_0)V_m\Big) + 8\bar\sigma_m^{\dot\alpha\alpha}\Big(d_{\alpha 0}\bar U_{2\dot\alpha} - \bar d_{\dot\alpha 0}U_{1\alpha}\Big) + 32\Pi_{m0}B \\
& + 24\bar\sigma_m^{\dot\alpha\alpha}[\bar d_{\dot\alpha 0}, d_{\alpha 0}]C\Big] + U_2^\alpha\Big[-8\sigma_{\alpha\dot\alpha}^n(\eta_{nm} - i\varepsilon_{nm})\bar d_0^{\dot\alpha}V^m + 4\bar d_{\dot\alpha 0}d_{\alpha 0}\bar U_2^{\dot\alpha} - 4\bar d_0^2 U_{1\alpha} \\
& + d_{\alpha 0}\bar d_0^2(-2iB + 18C) + \partial\theta_{\alpha 0}(-32iB - 96C) - 48i\Pi_{\alpha\dot\alpha 0}\bar d_0^{\dot\alpha}C\Big] \\
& - \bar U_{1\dot\alpha}\Big[-8\bar\sigma^{n\dot\alpha\alpha}(\eta_{nm} - i\varepsilon_{nm})d_{\alpha 0}V^m + 4d_0^2\bar U_2^{\dot\alpha} - 4d_0^\alpha\bar d_0^{\dot\alpha}U_{1\alpha} - \bar d_0^{\dot\alpha}d_0^2(2iB + 18C) \\
& + \partial\bar\theta_0^{\dot\alpha}(-32iB + 96C) + 48i\Pi_0^{\dot\alpha\alpha}d_{\alpha 0}C\Big] + B^\dagger\Big[-32\Pi_0^n(\eta_{nm} - i\varepsilon_{nm})V^m \\
& + (\{d_0^2, \bar d_0^2\} - 64)B + 3i[d_0^2, \bar d_0^2]C - i\Big(2d_0^2\bar d_{\dot\alpha 0} + 32\partial\bar\theta_{\dot\alpha 0}\Big)\bar U_2^{\dot\alpha} + i\Big(2\bar d_0^2 d_0^\alpha + 32\partial\theta_0^\alpha\Big)U_{1\alpha}\Big] \\
& + 3C^\dagger\Big[-8\bar\sigma^{n\dot\alpha\alpha}[d_{\alpha 0}, \bar d_{\dot\alpha 0}](\eta_{nm} - i\varepsilon_{nm})V^m - \Big(6d_0^\alpha\bar d_0^2 + 8i\Pi_0^{\dot\alpha\alpha}\bar d_{\dot\alpha 0}\Big)U_{1\alpha} \\
& - \Big(6\bar d_{\dot\alpha 0}d_0^2 + 8i\Pi_{\alpha\dot\alpha 0}d_0^\alpha\Big)\bar U_2^{\dot\alpha} - [d_0^2, \bar d_0^2]iB \\
& - \Big(-11\{d_0^2, \bar d_0^2\} + 128\Pi_0^n\Pi_{n0} - 256\partial\bar\theta_{\dot\alpha 0}\bar d_0^{\dot\alpha} - 256\partial\theta_0^\alpha d_{\alpha 0} - 64\Big)C\Big]\Big\},
\end{aligned}
$$

5 superfields
80 off-shell cplx DoF

The superspace action of the charged states

$$S=\frac{1}{16}\int d^4x\, p_0^2\bar p_0^2\Big\{V_n^\dagger(\eta^{nm}-i\varepsilon^{nm})\Big[-\{d_0^2,\bar d_0^2\}V_m+16\Pi_0^n\Pi_{n0}V_m-32(\eta_{mp}-i\varepsilon_{mp})V^p$$

$$-32\Big((\partial\bar\theta_0\bar d_0)V_m+(\partial\theta_0 d_0)V_m\Big)+8\bar\sigma_m^{\dot\alpha\alpha}\Big(d_{\alpha0}\bar U_{2\dot\alpha}-\bar d_{\dot\alpha0}U_{1\alpha}\Big)+32\Pi_{m0}B$$

$$+24\bar\sigma_m^{\dot\alpha\alpha}[\bar d_{\dot\alpha0},d_{\alpha0}]C\Big]+U^\alpha\Big[\ \ldots\ -4\bar d_0^2U_{1\alpha}$$

$$+d_{\alpha0}\bar d_0^2(-2iB\ \ldots$$

$$-\bar U_{1\dot\alpha}\Big[\ \ldots\ \Big)$$

$$+\partial\bar\theta_0^{\dot\alpha}(-\ \ldots$$

$$+(\{d_0^2,\bar d_0^2\}-64)B\ \ldots\ \Big)\bar U_2+i\Big(2\bar d_0^2d_0^\alpha+32\partial\theta_0^\alpha\Big)U_{1\alpha}\Big]$$

$$+3C^\dagger\Big[-8\bar\sigma^{n\dot\alpha\alpha}[d_{\alpha0},\bar d_{\dot\alpha0}](\eta_{nm}-i\varepsilon_{nm})V^m-\Big(6d_0^\alpha\bar d_0^2+8i\Pi_0^{\dot\alpha\alpha}\bar d_{\dot\alpha0}\Big)U_{1\alpha}$$

$$-\Big(6\bar d_{\dot\alpha0}d_0^2+8i\Pi_{\alpha\dot\alpha0}d_0^\alpha\Big)\bar U_2^{\dot\alpha}-[d_0^2,\bar d_0^2]iB$$

$$-\Big(-11\{d_0^2,\bar d_0^2\}+128\Pi_0^n\Pi_{n0}-256\partial\bar\theta_{\dot\alpha0}\bar d_0^{\dot\alpha}-256\partial\theta_0^\alpha d_{\alpha0}-64\Big)C\Big]\Big\},$$

5 superfields
80 off-shell cplx DoF

It’s time to expand...

The superfields...

$$
\begin{aligned}
V_m =& C_m + \mathrm{i}\left(\theta\chi_{1m}\right) - \mathrm{i}\left(\bar{\theta}\bar{\chi}_{2m}\right) + \mathrm{i}\left(\theta\theta\right) M_{1m} - \mathrm{i}\left(\bar{\theta}\bar{\theta}\right) \bar{M}_{2m} + \left(\theta\sigma^n\bar{\theta}\right) h_{mn} \\
& + \mathrm{i}\left(\theta\theta\right)\left(\bar{\theta}\bar{\lambda}_{1m}\right) - \mathrm{i}\left(\bar{\theta}\bar{\theta}\right)\left(\theta\lambda_{2m}\right) + \left(\theta\theta\right)\left(\bar{\theta}\bar{\theta}\right) D_m \\
\mathcal{B} =& \varphi + \mathrm{i}(\theta\gamma_1) - \mathrm{i}(\bar{\theta}\bar{\gamma}_2) + \mathrm{i}(\theta\theta)N_1 - \mathrm{i}(\bar{\theta}\bar{\theta})\bar{N}_2 + (\theta\sigma^m\bar{\theta})c_m \\
& + \mathrm{i}(\theta\theta)(\bar{\theta}\bar{\rho}_1) - \mathrm{i}(\bar{\theta}\bar{\theta})(\theta\rho_2) + (\theta\theta)(\bar{\theta}\bar{\theta})G \\
\mathcal{C} =& \phi + \mathrm{i}\left(\theta\xi_1\right) - \mathrm{i}\left(\bar{\theta}\bar{\xi}_2\right) + \mathrm{i}\left(\theta\theta\right) M_1 - \mathrm{i}\left(\bar{\theta}\bar{\theta}\right) \bar{M}_2 + \left(\theta\sigma^m\bar{\theta}\right) a_m \\
& + \mathrm{i}\left(\theta\theta\right)\left(\bar{\theta}\bar{\psi}_1\right) - \mathrm{i}\left(\bar{\theta}\bar{\theta}\right)\left(\theta\psi_2\right) + \left(\theta\theta\right)\left(\bar{\theta}\bar{\theta}\right) D \\
U_{1\alpha} =& v_{1\alpha} + \theta_\alpha s_1 - (\sigma^{mn}\theta)_\alpha s_{1mn} + (\sigma^m\bar{\theta})_\alpha w_{1m} + (\theta\theta)\eta_{1\alpha} + (\bar{\theta}\bar{\theta})\zeta_{1\alpha} + (\theta\sigma^m\bar{\theta})r_{1m\alpha} \\
& + (\theta\theta)(\sigma^m\bar{\theta})_\alpha q_{1m} + (\bar{\theta}\bar{\theta})\theta_\alpha t_1 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha t_{1mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\mu_{1\alpha} \\
\bar{U}_1^{\dot{\alpha}} =& \bar{v}_1^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{s}_1 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{s}_{1mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{w}_{1m} + (\bar{\theta}\bar{\theta})\bar{\eta}_1^{\dot{\alpha}} + (\theta\theta)\bar{\zeta}_1^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{r}_{1m}^{\dot{\alpha}} \\
& - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{q}_{1m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{t}_1 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{t}_{1mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\mu}_1^{\dot{\alpha}} \\
U_{2\alpha} =& v_{2\alpha} + \theta_\alpha s_2 - (\sigma^{mn}\theta)_\alpha s_{2mn} + (\sigma^m\bar{\theta})_\alpha w_{2m} + (\theta\theta)\eta_{2\alpha} + (\bar{\theta}\bar{\theta})\zeta_{2\alpha} + (\theta\sigma^m\bar{\theta})r_{2m\alpha} \\
& + (\theta\theta)(\sigma^m\bar{\theta})_\alpha q_{2m} + (\bar{\theta}\bar{\theta})\theta_\alpha t_2 - (\bar{\theta}\bar{\theta})(\sigma^{mn}\theta)_\alpha t_{2mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\mu_{2\alpha} \\
\bar{U}_2^{\dot{\alpha}} =& \bar{v}_2^{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}\bar{s}_2 - (\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{s}_{2mn} - (\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{w}_{2m} + (\bar{\theta}\bar{\theta})\bar{\eta}_2^{\dot{\alpha}} + (\theta\theta)\bar{\zeta}_2^{\dot{\alpha}} + (\theta\sigma^m\bar{\theta})\bar{r}_{2m}^{\dot{\alpha}} \\
& - (\bar{\theta}\bar{\theta})(\bar{\sigma}^m\theta)^{\dot{\alpha}}\bar{q}_{2m} + (\theta\theta)\bar{\theta}^{\dot{\alpha}}\bar{t}_2 - (\theta\theta)(\bar{\sigma}^{mn}\bar{\theta})^{\dot{\alpha}}\bar{t}_{2mn} + (\theta\theta)(\bar{\theta}\bar{\theta})\bar{\mu}_2^{\dot{\alpha}}
\end{aligned}
$$

and the Lagrangian in components

$$
\begin{aligned}
\mathcal{L}_1 =& \frac{1}{2}\bar{q}_1^m q_{1m} + \left[\mathrm{i}\bar{q}_1^m\left(M_{1m} - \mathrm{i}\epsilon_{mn}M_1^n + \frac{1}{2}\mathfrak{D}^n s_{1mn} + \frac{1}{4}\mathfrak{D}_m s_1 - 3\mathrm{i}\mathfrak{D}_m M_1 + \mathfrak{D}_m N_1\right) + \text{h.c.}\right] \\
&+ \frac{1}{2}\bar{q}_2^m q_{2m} + \left[\mathrm{i}\bar{q}_2^m\left(M_{2m} + \mathrm{i}\epsilon_{mn}M_2^n + \frac{1}{2}\mathfrak{D}^n s_{2mn} + \frac{1}{4}\mathfrak{D}_m s_2 - 3\mathrm{i}\mathfrak{D}_m M_2 + \mathfrak{D}_m N_2\right) + \text{h.c.}\right] \\
&- \frac{1}{8}\bar{s}_1\mathfrak{D}^2 s_1 + \left[\frac{1}{2}\mathfrak{D}^m\bar{s}_1\left(M_{1m} - \mathrm{i}\epsilon_{mn}M_1^n - 3\mathrm{i}\mathfrak{D}_m M_1 + \mathfrak{D}_m N_1\right) - \frac{1}{8}\mathrm{i}\epsilon^{mn}\bar{s}_1 s_{1mn} + \text{h.c.}\right] \\
&- \frac{1}{8}\bar{s}_2\mathfrak{D}^2 s_2 + \left[\frac{1}{2}\mathfrak{D}^m\bar{s}_2\left(M_{2m} + \mathrm{i}\epsilon_{mn}M_2^n - 3\mathrm{i}\mathfrak{D}_m M_2 + \mathfrak{D}_m N_2\right) + \frac{1}{8}\mathrm{i}\epsilon^{mn}\bar{s}_2 s_{2mn} + \text{h.c.}\right] \\
&- 3\bar{M}_1\left(3\mathfrak{D}^2 + 4\right)M_1 + \left[6\mathrm{i}\mathfrak{D}^m\bar{M}_1\left(M_{1m} - \mathrm{i}\epsilon_{mn}M_1^n + \frac{1}{2}\mathfrak{D}_m N_1\right) + \frac{3}{2}\epsilon^{mn}\bar{M}_1 s_{1mn} + \text{h.c.}\right] \\
&- 3\bar{M}_2\left(3\mathfrak{D}^2 + 4\right)M_2 + \left[6\mathrm{i}\mathfrak{D}^m\bar{M}_2\left(M_{2m} + \mathrm{i}\epsilon_{mn}M_2^n + \frac{1}{2}\mathfrak{D}_m N_2\right) - \frac{3}{2}\epsilon^{mn}\bar{M}_2 s_{2mn} + \text{h.c.}\right] \\
&- \bar{N}_1\left(\mathfrak{D}^2 - 4\right)N_1 + \left[2\mathfrak{D}^m\bar{N}_1\left(M_{1m} - \mathrm{i}\epsilon_{mn}M_1^n\right) - \frac{1}{2}\mathrm{i}\epsilon^{mn}\bar{N}_1 s_{1mn} + \text{h.c.}\right] \\
&- \bar{N}_2\left(\mathfrak{D}^2 - 4\right)N_2 + \left[2\mathfrak{D}^m\bar{N}_2\left(M_{2m} + \mathrm{i}\epsilon_{mn}M_2^n\right) + \frac{1}{2}\mathrm{i}\epsilon^{mn}\bar{N}_2 s_{2mn} + \text{h.c.}\right] \\
&+ \frac{1}{2}\mathfrak{D}^n s_{1mn}\mathfrak{D}_k\bar{s}_1^{mk} + \left[\bar{s}_{1mn}\left(\mathfrak{D}^m M_1^n - \mathrm{i}\epsilon^{nk}\mathfrak{D}^m M_{1k}\right) + \text{h.c.}\right] \\
&+ \frac{1}{2}\mathfrak{D}^n s_{2mn}\mathfrak{D}_k\bar{s}_2^{mk} + \left[\bar{s}_{2mn}\left(\mathfrak{D}^m M_2^n + \mathrm{i}\epsilon^{nk}\mathfrak{D}^m M_{2k}\right) + \text{h.c.}\right] \\
&+ 2\left[\left(\eta^{mk} - \mathrm{i}\epsilon^{mk}\right)M_{1k}\right]^\dagger\left[(\eta_{mn} - \mathrm{i}\epsilon_{mn})M_1^n\right] + 2\left[\left(\eta^{mk} + \mathrm{i}\epsilon^{mk}\right)M_{2k}\right]^\dagger\left[(\eta_{mn} + \mathrm{i}\epsilon_{mn})M_2^n\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_2 =& \frac{1}{8}\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)C_n\right]^\dagger\mathfrak{D}^4 C_m - \frac{1}{2}\left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)C_n\right]^\dagger\mathfrak{D}^2 D_m + \text{h.c.}\right\} \\
&+ 2\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)D_n\right]^\dagger D_m + 2\left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)C_n\right]^\dagger\left[\left(\eta_{mk} - \mathrm{i}\epsilon_{mk}\right)D^k\right] + \text{h.c.}\right\} \\
&- \left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)h_{nk}\right]^\dagger\left[\left(\eta_{ml} - \mathrm{i}\epsilon_{ml}\right)h^{lk} + \mathrm{i}\epsilon^{lk}h_{ml} - \frac{1}{2}\mathfrak{D}^2 h_m{}^k\right] \\
&+ \frac{1}{2}\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)\mathfrak{D}^k h_{nk}\right]^\dagger\mathfrak{D}^l h_{ml} + 3\left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)\mathfrak{D}^k h_{nk}\right]^\dagger\mathfrak{D}_m\phi + \text{h.c.}\right\} \\
&+ \left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)C_n\right]^\dagger\left(\frac{3}{2}\mathfrak{D}^2 a_m - 3\mathfrak{D}_m\mathfrak{D}_k a^k + 3\mathrm{i}\epsilon_{mk}a^k + 3\mathrm{i}\bar{\epsilon}_{mk}\mathfrak{D}^k\phi + 2\mathfrak{D}_m G + \mathrm{i}\bar{\epsilon}_{mk}c^k\right.\right. \\
&\left.\left.+ \frac{1}{2}\mathfrak{D}_m\tau_2 + \mathfrak{D}^k\tau_{2mk} - \frac{1}{4}\mathrm{i}\epsilon_{mk}\omega_1^k + \frac{1}{4}\mathrm{i}\bar{\epsilon}_{mk}\omega_2^k\right) + \text{h.c.}\right\} \\
&+ \left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)h_n{}^k\right]^\dagger\varepsilon_{mkpq}\left(\frac{1}{4}\mathfrak{D}^p\omega_1^q - 3\mathfrak{D}^p a^q + \frac{1}{2}\tau_2^{pq}\right) + \text{h.c.}\right\} \\
&+ \left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)h_{nk}\right]^\dagger\left(-\mathfrak{D}_m c^k + \mathrm{i}\bar{\epsilon}_m{}^k\varphi - 3\mathrm{i}\epsilon_m{}^k\phi - \frac{1}{4}\mathfrak{D}_m\omega_2^k - \frac{1}{4}\mathfrak{D}^k\omega_{2m}\right) + \text{h.c.}\right\} \\
&+ \left\{\left[\left(\eta^{mn} + \mathrm{i}\epsilon^{mn}\right)h_{mn}\right]^\dagger\left(-6D + \frac{3}{2}\mathfrak{D}^2\phi + \frac{1}{4}\mathfrak{D}^k\omega_{2k} + \frac{1}{2}\tau_1\right) + \text{h.c.}\right\} \\
&+ \left\{\left[\left(\eta^{mn} - \mathrm{i}\epsilon^{mn}\right)D_n\right]^\dagger\left(2\mathfrak{D}_m\varphi - 6a_m + \omega_{1m}\right) + \text{h.c.}\right\} - \frac{1}{2}\mathfrak{D}^m\bar{c}_m\mathfrak{D}^n c_n - 2\bar{c}_m c^m \\
&+ \left[\mathfrak{D}^m\bar{c}_m\left(-3D - \frac{3}{4}\mathfrak{D}^2\phi + \frac{1}{2}\tau_1 - \frac{1}{4}\mathfrak{D}^n\omega_{2n}\right) + \text{h.c.}\right] - 66\bar{D}D - \frac{1}{8}\bar{\varphi}\mathfrak{D}^4\varphi \\
&- \frac{33}{8}\bar{\phi}\mathfrak{D}^4\phi - 12\bar{a}^m\mathfrak{D}^2 a_m - \frac{33}{2}\mathfrak{D}^m\bar{a}_m\mathfrak{D}^n a_n - 24\mathrm{i}\epsilon^{mn}\bar{a}_m a_n + 6\bar{a}^m a_m \\
&+ \left[\mathfrak{D}^2\bar{\phi}\left(\frac{15}{2}D - \frac{9}{8}\mathfrak{D}^m\omega_{2m} - \frac{3}{4}\tau_1\right) + \bar{D}\left(\frac{3}{2}\mathfrak{D}^m\omega_{2m} + 9\tau_1 - 12\phi\right) + \text{h.c.}\right] \\
&- \frac{1}{8}\left(\mathfrak{D}^m\bar{\omega}_{1m}\mathfrak{D}^n\omega_{1n} + \mathfrak{D}^m\bar{\omega}_{2m}\mathfrak{D}^n\omega_{2n}\right) + \tau_{2mn}\bar{\tau}_2^{mn} - \tau_1\bar{\tau}_1 \\
&- \frac{1}{8}\left(\bar{\omega}_1^m\mathfrak{D}^2\omega_{1m} - \bar{\omega}_2^m\mathfrak{D}^2\omega_{2m}\right) + \left(\omega_1^m\mathfrak{D}^n\bar{\tau}_{2mn} - \frac{1}{4}\tau_1\mathfrak{D}^m\bar{\omega}_{2m} - \frac{1}{4}\tau_2\mathfrak{D}^m\bar{\omega}_{1m} + \text{h.c.}\right) \\
&- \frac{1}{8}\mathrm{i}\epsilon^{mn}\left(3\bar{\omega}_{1m}\omega_{1n} + \bar{\omega}_{2m}\omega_{2n}\right) + \left[\mathfrak{D}^2\bar{\varphi}\left(-\frac{1}{4}\tau_2 + \frac{3}{4}\mathfrak{D}^m a_m - \frac{1}{8}\mathfrak{D}^m\omega_{1m} - \frac{1}{2}G\right) + \text{h.c.}\right] \\
&- 2\bar{G}G + \left[\bar{G}\left(4\varphi + 3\mathfrak{D}^m a_m - \frac{1}{2}\mathfrak{D}^m\omega_{1m} - \tau_2\right) + \mathfrak{D}^m\bar{a}_m\left(\frac{9}{4}\mathfrak{D}^n\omega_{1n} + \frac{3}{2}\tau_2\right) + \text{h.c.}\right] \\
&+ \left[\bar{\omega}_{1m}\left(\frac{3}{2}\mathfrak{D}^2 a^m + 3\mathrm{i}\epsilon^{mn}a_n + \frac{3}{2}\mathrm{i}\bar{\epsilon}^{mn}\mathfrak{D}_n\phi\right) - 6\bar{a}_m\mathfrak{D}_n\tau_2^{mn} - \mathrm{i}\bar{\varphi}\epsilon^{mn}\tau_{2mn} + \text{h.c.}\right] \\
&+ \left[\frac{1}{2}\mathrm{i}\bar{c}_m\left(\bar{\epsilon}^{mn}\omega_{1n} - \epsilon^{mn}\omega_{2n}\right) - \frac{3}{2}\mathrm{i}\epsilon^{mn}\bar{\omega}_{2m}\mathfrak{D}_n\phi - \frac{1}{8}\mathrm{i}\bar{\epsilon}^{mn}\omega_{2m}\bar{\omega}_{1n} + \text{h.c.}\right]
\end{aligned}
$$

Bosons...

$$
\begin{aligned}
\mathcal{L}_F =& -\frac{\mathrm{i}}{8}\left[4\left(\lambda_1^m\sigma^n\mathfrak{D}_n\bar{\boldsymbol{\lambda}}_{1m}\right) + 4\left(\bar{\lambda}_2^m\bar{\sigma}^n\mathfrak{D}_n\boldsymbol{\lambda}_{2m}\right) - \left(\bar{\chi}_1^m\bar{\sigma}^n\sigma^k\bar{\sigma}^l\mathfrak{D}_n\mathfrak{D}_k\mathfrak{D}_l\boldsymbol{\chi}_{1m}\right) - \left(\chi_2^m\sigma^n\bar{\sigma}^k\sigma^l\mathfrak{D}_n\mathfrak{D}_k\mathfrak{D}_l\bar{\boldsymbol{\chi}}_{2m}\right)\right] \\
&- \frac{1}{4}\left[\left(\bar{\chi}_1^m\bar{\sigma}^n\sigma^k\mathfrak{D}_n\mathfrak{D}_k\bar{\boldsymbol{\lambda}}_{1m}\right) + \left(\lambda_1^m\sigma^n\bar{\sigma}^k\mathfrak{D}_n\mathfrak{D}_k\boldsymbol{\chi}_{1m}\right) + \left(\chi_2^m\sigma^n\bar{\sigma}^k\mathfrak{D}_n\mathfrak{D}_k\boldsymbol{\lambda}_{2m}\right) + \left(\bar{\lambda}_2^m\bar{\sigma}^n\sigma^k\mathfrak{D}_n\mathfrak{D}_k\bar{\boldsymbol{\chi}}_{2m}\right)\right] \\
&- \left[\left(\boldsymbol{\lambda}_1^m\boldsymbol{\chi}_{1m}\right) + (1\leftrightarrow 2) + \text{h.c.}\right] \\
&- \frac{33}{8}\mathrm{i}\left[\left(\bar{\xi}_1\bar{\sigma}^m\sigma^n\bar{\sigma}^k\mathfrak{D}_m\mathfrak{D}_n\mathfrak{D}_k\xi_1\right) + \left(\xi_2\sigma^m\bar{\sigma}^n\sigma^k\mathfrak{D}_m\mathfrak{D}_n\mathfrak{D}_k\bar{\xi}_2\right) - 4\left(\psi_1\sigma^m\mathfrak{D}_m\bar{\psi}_1\right) - 4\left(\bar{\psi}_2\bar{\sigma}^m\mathfrak{D}_m\psi_2\right)\right] \\
&+ \left[6\left(\psi_1\xi_1\right) + \frac{15}{4}\left(\psi_1\sigma^m\bar{\sigma}^n\mathfrak{D}_m\mathfrak{D}_n\xi_1\right) + (1\leftrightarrow 2) + \text{h.c.}\right] \\
&+ \frac{3}{2}\left[\mathrm{i}\left(\bar{\boldsymbol{\chi}}_1^m\mathfrak{D}_m\bar{\psi}_1\right) - \mathrm{i}\left(\boldsymbol{\chi}_2^m\mathfrak{D}_m\psi_2\right) + \mathrm{i}\left(\boldsymbol{\lambda}_1^m\mathfrak{D}_m\xi_1\right) - \mathrm{i}\left(\bar{\boldsymbol{\lambda}}_2^m\mathfrak{D}_m\bar{\xi}_2\right) - 2\left(\boldsymbol{\lambda}_1^m\sigma_m\bar{\psi}_1\right) + 2\left(\bar{\boldsymbol{\lambda}}_2^m\bar{\sigma}_m\psi_2\right)\right. \\
&\quad + \frac{1}{2}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}_m\mathfrak{D}^2\xi_1\right) - \frac{1}{2}\left(\boldsymbol{\chi}_2^m\sigma_m\mathfrak{D}^2\bar{\xi}_2\right) - 2\mathrm{i}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}_{mn}\mathfrak{D}^n\bar{\psi}_1\right) + 2\mathrm{i}\left(\boldsymbol{\chi}_2^m\sigma_{mn}\mathfrak{D}^n\psi_2\right) \\
&\quad - \left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}^n\mathfrak{D}_m\mathfrak{D}_n\xi_1\right) + \left(\boldsymbol{\chi}_2^m\sigma^n\mathfrak{D}_m\mathfrak{D}_n\bar{\xi}_2\right) + 2\mathrm{i}\left(\boldsymbol{\lambda}_1^m\sigma_{mn}\mathfrak{D}^n\xi_1\right) - 2\mathrm{i}\left(\bar{\boldsymbol{\lambda}}_2^m\bar{\sigma}_{mn}\mathfrak{D}^n\bar{\xi}_2\right) \\
&\quad \left. + \frac{1}{2}\mathrm{i}\bar{\boldsymbol{\chi}}_1^m\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}_m\xi_1 - \frac{1}{2}\mathrm{i}\boldsymbol{\chi}_2^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\xi}_2 + \text{h.c.}\right] \\
&- \frac{1}{4}\left[\left(\boldsymbol{\lambda}_1^m\mathfrak{D}_m v_1\right) + \left(\bar{\boldsymbol{\lambda}}_2^m\mathfrak{D}_m\bar{v}_2\right) + 2\left(\boldsymbol{\lambda}_1^m\sigma_{mn}\mathfrak{D}^n v_1\right) + 2\left(\bar{\boldsymbol{\lambda}}_2^m\bar{\sigma}_{mn}\mathfrak{D}^n\bar{v}_2\right)\right. \\
&\quad - 2\mathrm{i}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}_m\mu_1\right) - 2\mathrm{i}\left(\boldsymbol{\chi}_2^m\sigma_m\bar{\mu}_2\right) + \left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}^n\sigma_m\mathfrak{D}_n\bar{\zeta}_2\right) + \left(\boldsymbol{\chi}_2^m\sigma^n\bar{\sigma}_m\mathfrak{D}_n\zeta_1\right) \\
&\quad + 2\mathrm{i}\left(\boldsymbol{\lambda}_1^m\sigma_m\bar{\zeta}_2\right) + 2\mathrm{i}\left(\bar{\boldsymbol{\lambda}}_2^m\bar{\sigma}_m\zeta_1\right) - \mathrm{i}\left(\boldsymbol{\lambda}_{1m}\sigma^n\bar{\sigma}^m r_{1n}\right) + \mathrm{i}\left(\bar{\boldsymbol{\lambda}}_{2m}\bar{\sigma}^n\sigma^m\bar{r}_{2n}\right) \\
&\quad - \frac{1}{2}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}^n\mathfrak{D}_n r_{1m}\right) + \frac{1}{2}\left(\boldsymbol{\chi}_2^m\sigma^n\mathfrak{D}_n\bar{r}_{2m}\right) + \frac{1}{2}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}^n\mathfrak{D}_m r_{1n}\right) - \frac{1}{2}\left(\boldsymbol{\chi}_2^m\sigma^n\mathfrak{D}_m\bar{r}_{2n}\right) \\
&\quad + \frac{1}{2}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}_m\mathfrak{D}_n r_1^n\right) - \frac{1}{2}\left(\boldsymbol{\chi}_2^m\sigma_m\mathfrak{D}_n\bar{r}_2^n\right) - \frac{\mathrm{i}}{2}\varepsilon_{mnpq}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}^n\mathfrak{D}^p r_1^q\right) - \frac{\mathrm{i}}{2}\varepsilon_{mnpq}\left(\boldsymbol{\chi}_2^m\sigma^n\mathfrak{D}^p\bar{r}_2^q\right) \\
&\quad \left. + \frac{1}{2}\bar{\boldsymbol{\chi}}_1^m\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}_m v_1 + \frac{1}{2}\boldsymbol{\chi}_2^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{v}_2 + \text{h.c.}\right] \\
&+ \frac{1}{4}\left[\mathrm{i}\left(v_1\sigma^m\mathfrak{D}_m\bar{\mu}_1\right) - \frac{1}{4}\left(r_{1m}\sigma^m\mathfrak{D}^2\bar{v}_1\right) + \frac{1}{2}\left(r_{1m}\sigma^n\mathfrak{D}^m\mathfrak{D}_n\bar{v}_1\right) + \left(r_{1m}\sigma^m\bar{\mu}_1\right) + (1\leftrightarrow 2) + \text{h.c.}\right] \\
&+ \frac{\mathrm{i}}{8}\bar{\epsilon}_{mn}\left[\left(v_1\sigma^m\mathfrak{D}^n\bar{v}_1\right) - \left(v_2\sigma^m\mathfrak{D}^n\bar{v}_2\right)\right] + \left[\frac{\mathrm{i}}{8}r_{1m}\left(\epsilon\cdot\sigma\right)\sigma_m\bar{v}_1 - \frac{\mathrm{i}}{8}r_{2m}\left(\epsilon\cdot\sigma\right)\sigma_m\bar{v}_2 + \text{h.c.}\right] \\
&+ \left[\frac{\mathrm{i}}{2}\left(r_2^m\mathfrak{D}_m\zeta_1\right) - \left(\mu_2\zeta_1\right) - \frac{1}{4}\left(v_2\mathfrak{D}^2\zeta_1\right) + (1\leftrightarrow 2) + \text{h.c.}\right] \\
&+ \frac{\mathrm{i}}{2}\left(\rho_1 + \frac{1}{2}\mathrm{i}\bar{\gamma}_1\bar{\sigma}^m\mathfrak{D}_m\right)\sigma^n\mathfrak{D}_n\left(\bar{\rho}_1 + \frac{1}{2}\mathrm{i}\bar{\sigma}^k\mathfrak{D}_k\gamma_1\right) + \frac{\mathrm{i}}{2}\left(\bar{\rho}_2 + \frac{1}{2}\mathrm{i}\gamma_2\sigma^m\mathfrak{D}_m\right)\bar{\sigma}^n\mathfrak{D}_n\left(\rho_2 + \frac{1}{2}\mathrm{i}\sigma^k\mathfrak{D}_k\bar{\gamma}_2\right) \\
&- 2\left[\left(\rho_1\gamma_1\right) + (1\leftrightarrow 2) + \text{h.c.}\right] + \frac{3}{4}\left[\left(\rho_1 + \frac{1}{2}\mathrm{i}\bar{\gamma}_1\bar{\sigma}^m\mathfrak{D}_m\right)\left(\mathrm{i}\sigma^m\bar{\sigma}^n\mathfrak{D}_m\mathfrak{D}_n\xi_1 + 2\sigma^n\mathfrak{D}_n\bar{\psi}_1\right)\right. \\
&\quad \left. - \left(\bar{\rho}_2 + \frac{1}{2}\mathrm{i}\gamma_2\sigma^m\mathfrak{D}_m\right)\left(\mathrm{i}\bar{\sigma}^m\sigma^n\mathfrak{D}_m\mathfrak{D}_n\bar{\xi}_2 + 2\bar{\sigma}^n\mathfrak{D}_n\psi_2\right) + \text{h.c.}\right] \\
&- \left[\bar{\boldsymbol{\chi}}_1^m\left(\mathfrak{D}_m\bar{\rho}_1 - \frac{\mathrm{i}}{2}\bar{\epsilon}_{mn}\bar{\sigma}^n\gamma_1\right) + \boldsymbol{\chi}_2^m\left(\mathfrak{D}_m\rho_2 + \frac{\mathrm{i}}{2}\bar{\epsilon}_{mn}\sigma^n\bar{\gamma}_2\right) + \boldsymbol{\lambda}_1^m\mathfrak{D}_m\gamma_1 + \bar{\boldsymbol{\lambda}}_2^m\mathfrak{D}_m\bar{\gamma}_2 + \text{h.c.}\right] \\
&+ \frac{1}{8}\left[\left(\mathfrak{D}^2\bar{v}_1 - 2\mathrm{i}\bar{v}_1(\epsilon\cdot\bar{\sigma}) - 4\bar{\mu}_1 + 4\mathrm{i}\bar{r}_{1m}\bar{\sigma}^{mn}\mathfrak{D}_n\right)\left(\bar{\rho}_1 + \frac{1}{2}\mathrm{i}\bar{\sigma}^k\mathfrak{D}_k\gamma_1\right) + \text{h.c.}\right] \\
&+ \frac{1}{8}\left[\left(\mathfrak{D}^2 v_2 - 2\mathrm{i}v_2(\epsilon\cdot\sigma) - 4\mu_2 - 4\mathrm{i}r_{2m}\sigma^{mn}\mathfrak{D}_n\right)\left(\rho_2 + \frac{1}{2}\mathrm{i}\sigma^k\mathfrak{D}_k\bar{\gamma}_2\right) + \text{h.c.}\right] \\
&+ \left[\frac{\mathrm{i}}{2}\bar{v}_1(\epsilon\cdot\bar{\sigma})\bar{\rho}_1 - \frac{\mathrm{i}}{2}\bar{\zeta}_1(\epsilon\cdot\bar{\sigma})\bar{\gamma}_2 + \frac{\mathrm{i}}{4}\bar{r}_1^m\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}_m\gamma_1 + \text{h.c.}\right] \\
&+ \left[\frac{\mathrm{i}}{2}v_2(\epsilon\cdot\sigma)\rho_2 - \frac{\mathrm{i}}{2}\zeta_2(\epsilon\cdot\sigma)\gamma_1 - \frac{\mathrm{i}}{4}r_2^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\gamma}_2 + \text{h.c.}\right] \\
&- \frac{9}{8}\left[\left(\mathfrak{D}^2\bar{v}_1 - 2\mathrm{i}\bar{v}_1(\epsilon\cdot\bar{\sigma}) - 4\bar{\mu}_1 + 4\mathrm{i}\bar{r}_{1m}\bar{\sigma}^{mn}\mathfrak{D}_n\right)\left(\mathrm{i}\bar{\psi}_1 - \frac{1}{2}\bar{\sigma}^k\mathfrak{D}_k\xi_1\right) + \text{h.c.}\right] \\
&+ \frac{9}{8}\left[\left(\mathfrak{D}^2 v_2 - 2\mathrm{i}v_2(\epsilon\cdot\sigma) - 4\mu_2 - 4\mathrm{i}r_{2m}\sigma^{mn}\mathfrak{D}_n\right)\left(\mathrm{i}\psi_2 - \frac{1}{2}\sigma^k\mathfrak{D}_k\bar{\xi}_2\right) + \text{h.c.}\right] \\
&+ \left[3\bar{v}_1(\epsilon\cdot\bar{\sigma})\bar{\psi}_1 - \frac{3}{2}\mathrm{i}\bar{\epsilon}^{mn}\left(\bar{r}_{1m}\bar{\sigma}_n\xi_1\right) + \frac{3\mathrm{i}}{4}\bar{v}_1\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}^m\mathfrak{D}_m\xi_1 + \frac{3}{2}\left(\bar{r}_1^m\bar{\sigma}^n\sigma_m\mathfrak{D}_n\bar{\psi}_1\right)\right. \\
&\quad \left. + \frac{3\mathrm{i}}{4}\left(\bar{r}_1^m\bar{\sigma}_m\mathfrak{D}^2\xi_1\right) + 3\left(\bar{\mu}_1\bar{\sigma}^m\mathfrak{D}_m\xi_1\right) - 3\left(\bar{\zeta}_1\bar{\sigma}^m\mathfrak{D}_m\psi_2\right) + \frac{3\mathrm{i}}{2}\left(\bar{v}_1\mathfrak{D}^2\bar{\psi}_1\right) - \frac{3\mathrm{i}}{2}\left(\bar{\zeta}_1\mathfrak{D}^2\bar{\xi}_2\right) + \text{h.c.}\right] \\
&+ \left[-3v_2(\epsilon\cdot\sigma)\psi_2 + \frac{3}{2}\mathrm{i}\bar{\epsilon}^{mn}\left(r_{2m}\sigma_n\bar{\xi}_2\right) - \frac{3\mathrm{i}}{4}v_2\left(\epsilon\cdot\sigma\right)\sigma^m\mathfrak{D}_m\bar{\xi}_2 + \frac{3}{2}\left(r_2^m\sigma^n\bar{\sigma}_m\mathfrak{D}_n\psi_2\right)\right. \\
&\quad \left. + \frac{3\mathrm{i}}{4}\left(r_2^m\sigma_m\mathfrak{D}^2\bar{\xi}_2\right) - 3\left(\mu_2\sigma^m\mathfrak{D}_m\bar{\xi}_2\right) + 3\left(\zeta_2\sigma^m\mathfrak{D}_m\bar{\psi}_1\right) - \frac{3\mathrm{i}}{2}\left(v_2\mathfrak{D}^2\psi_2\right) + \frac{3\mathrm{i}}{2}\left(\zeta_2\mathfrak{D}^2\xi_1\right) + \text{h.c.}\right]
\end{aligned}
$$

Fermions...

Simplifying the Lagrangian

The unphysical DoF are:

- Auxiliary fields → *Integrate out*
- Lagrange multipliers → *Apply the constraints*
- Gauge DoF (including Stückelberg) → *Choose unitary gauge*
- **Transverse** components of some vector fields

Simplifying the Lagrangian

The unphysical DoF are:

- ♦ Auxiliary fields → *Integrate out*
- ♦ Lagrange multipliers → *Apply the constraints*
- ♦ Gauge DoF (including Stückelberg) → *Choose unitary gauge*
- ♦ **Transverse** components of some vector fields

Curtright, Freund (1980)
Deser, Townsend, Siegel (1980)

$$\mathcal{L} = \left(\partial^m a_m\right)^2 + M^2 a_m a^m \quad \xleftrightarrow{\text{is dual to}} \quad \mathcal{L} = \frac{1}{2} A \left(\partial^2 - M^2\right) A$$

EoM: $M^2 a_m = \partial_m \left(\partial_n a^n\right)$ ➡

- ♦ a_m is the **gradient** of a scalar (denoted by A)
- ♦ Only the **longitudinal** component of a_m is physical

Bosonic Lagrangian

We write it as a sum of two pieces: $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$

with

$$\mathcal{L}_1 = \bar{\mathcal{C}}^m \left(\mathfrak{D}^2 - M^2\right) \mathcal{C}_m + \mathfrak{D}^m \bar{\mathcal{C}}_m \mathfrak{D}^n \mathcal{C}_n + 2\mathrm{i}\epsilon_{mn} \bar{\mathcal{C}}^m \mathcal{C}^n + \bar{\mathcal{M}}_1 \left(\mathfrak{D}^2 - M^2\right) \mathcal{M}_1 + \bar{\mathcal{N}}_1 \left(\mathfrak{D}^2 - M^2\right) \mathcal{N}_1$$

	Cplx on-shell DoF
One massive vector	3
Two massive cplx scalars	2

As for $\mathcal{L}_2$, we present two equivalent forms, each with its own advantages:

I) Compact form

II) Deformation of Fierz-Pauli

Bosonic Lagrangian

I) Compact form

$$\begin{aligned}\mathcal{L}_2 =& \bar{a}^m\left(M^2\eta_{mn} - \mathrm{i}\epsilon_{mn}\right)a^n + \mathfrak{D}^m\bar{a}_m\mathfrak{D}^n a_n - M^2\bar{c}^m c_m - \frac{2}{5}\mathfrak{D}^m\bar{c}_m\mathfrak{D}^n c_n \\ &+ \frac{1}{\sqrt{2}}\left[M\bar{c}^m\left(-\frac{2}{5}\mathfrak{D}_m\mathcal{H} + \mathfrak{D}^n\mathcal{H}_{nm}\right) + \bar{\tilde{F}}^{mn}(a)\left(F_{mn}(c) - \frac{M}{\sqrt{2}}\mathcal{H}_{[mn]}\right) + \mathrm{h.c.}\right] \\ &+ \frac{1}{2}\bar{\mathcal{H}}_{mn}\mathfrak{D}^2 h^{mn} + \frac{1}{2}\mathfrak{D}^n\bar{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk} - \frac{M^2}{2}\bar{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \frac{M^2}{20}\bar{\mathcal{H}}\mathcal{H} + \mathrm{i}\epsilon^{nk}\bar{\mathcal{H}}_{mn}h_k{}^m\end{aligned}$$

(dual) field strength of a vector: $F_{mn}(a) \equiv \mathfrak{D}_m a_n - \mathfrak{D}_n a_m, \quad \tilde{F}_{mn}(a) \equiv \frac{1}{2}\varepsilon_{mnpq}F^{pq}(a)$

rescaled spin-2 field: $\mathcal{H}_{mn} \equiv \left(\eta_{mk} - \mathrm{i}\frac{2}{M^2}\epsilon_{mk}\right)h^k{}_n$

spin-2 on shell →

$$\begin{cases}\left(\mathfrak{D}^2 - M^2\right)h_{mn} - 2\mathrm{i}\left(\epsilon_{km}h^k{}_n + \epsilon_{kn}h^k{}_m\right) = 0 \\ Mh = -4\sqrt{2}\mathfrak{D}^m c_m \\ \mathfrak{D}^n h_{mn} + \sqrt{2}Mc_m = 0\end{cases}$$

$Mh = -4\sqrt{2}\mathfrak{D}^m c_m$ → **Trace**

$\mathfrak{D}^n h_{mn} + \sqrt{2}Mc_m = 0$ → **Divergence**

- ♦ Correct EoM
- ♦ Correct DoF
- ♦ But coupled to c_m

Bosonic Lagrangian

I) Compact form

$$\mathcal{L}_2 = \bar{a}^m \left(M^2\eta_{mn} - \mathrm{i}\epsilon_{mn}\right) a^n + \mathfrak{D}^m \bar{a}_m \mathfrak{D}^n a_n - M^2 \bar{c}^m c_m - \frac{2}{5}\mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n$$
$$+ \frac{1}{\sqrt{2}}\left[M\bar{c}^m\left(-\frac{2}{5}\mathfrak{D}_m \mathcal{H} + \mathfrak{D}^n \mathcal{H}_{nm}\right) + \bar{\tilde{F}}^{mn}(a)\left(F_{mn}(c) - \frac{M}{\sqrt{2}}\mathcal{H}_{[mn]}\right) + \text{h.c.}\right]$$
$$+ \frac{1}{2}\bar{\mathcal{H}}_{mn}\mathfrak{D}^2 h^{mn} + \frac{1}{2}\mathfrak{D}^n \bar{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk} - \frac{M^2}{2}\bar{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \frac{M^2}{20}\bar{\mathcal{H}}\mathcal{H} + \mathrm{i}\epsilon^{nk}\bar{\mathcal{H}}_{mn} h_k{}^m$$

(dual) field strength of a vector: $F_{mn}(a) \equiv \mathfrak{D}_m a_n - \mathfrak{D}_n a_m, \quad \tilde{F}_{mn}(a) \equiv \frac{1}{2}\varepsilon_{mnpq}F^{pq}(a)$

rescaled spin-2 field: $\mathcal{H}_{mn} \equiv \left(\eta_{mk} - \mathrm{i}\frac{2}{M^2}\epsilon_{mk}\right) h^k{}_n$

vectors on shell

$$\begin{cases} M^2 a_m = \mathfrak{D}_m\left(\mathfrak{D}_n a^n\right) + \mathcal{O}(\epsilon) \\ M^2 c_m = \mathfrak{D}_m\left(\mathfrak{D}_n c^n\right) + \mathcal{O}(\epsilon) \end{cases}$$

The EoM of a_m and c_m are **coupled** to other fields through background dependent terms

Bosonic Lagrangian

I) Compact form

These EoM and constraints can be decoupled on shell with the redefinitions:

$$a'_m \equiv a_m - \frac{\mathrm{i}}{M^2}\epsilon_{mn}a^n - \cdots \qquad c'_m \equiv c_m - \frac{\sqrt{2}\mathrm{i}}{2M^2}\tilde{\epsilon}_{mn}a^n + \cdots$$

$$\mathfrak{h}_{mn} \equiv \frac{2}{3}h_{mn} - \frac{1}{6}\eta_{mn}h - \frac{\mathrm{i}}{M^2}\epsilon_m{}^k h_{kn} + \frac{\sqrt{2}}{3M}\mathfrak{D}_m c_n - \cdots$$

Check out the complete expressions:

2211.13689

New EoM& constraints

$$\left(\mathfrak{D}^2 - M^2\right)\mathfrak{h}_{mn} = 2\mathrm{i}\left(\epsilon_{km}\mathfrak{h}^k{}_n + \epsilon_{kn}\mathfrak{h}^k{}_m\right)$$

$$\mathfrak{D}^n\mathfrak{h}_{mn} = 0, \quad \mathfrak{h} = 0$$

$$\mathfrak{D}_m\mathfrak{D}_n a'^n = M^2 a'_m, \quad \mathfrak{D}_m\mathfrak{D}_n c'^n = M^2 c'_m$$

Bosonic Lagrangian

I) Compact form

These EoM and constraints ca… …ll with the redefinitions:

$$a'_m \equiv a_m - \frac{\mathrm{i}}{M^2}\epsilon \ldots$$

$$\mathfrak{h}_{mn} \equiv \frac{2}{3} h_{mn} - \frac{1}{6}\eta \ldots$$

Check out the complete expressions:

2211.13689

Same as the equations of the **Argyres-Nappi** Lagrangian!

New EoM& constraints

$$\left(\mathfrak{D}^2 - M^2\right)\mathfrak{h}_{mn} = 2\mathrm{i}\left(\epsilon_{km}\mathfrak{h}^k{}_n + \epsilon_{kn}\mathfrak{h}^k{}_m\right)$$

$$\mathfrak{D}^n\mathfrak{h}_{mn} = 0, \quad \mathfrak{h} = 0$$

$$\mathfrak{D}_m\mathfrak{D}_n a'^n = M^2 a'_m, \quad \mathfrak{D}_m\mathfrak{D}_n c'^n = M^2 c'_m$$

They are dual to **scalars**

Bosonic Lagrangian

II) Deformation of Fierz-Pauli

We can eliminate unphysical (transverse) DoF in a_m and c_m , leaving the physical scalars A and B.

$$
\begin{aligned}
\mathcal{L}_2 = &\left[\bar{A}\mathfrak{D}_m - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{mb}\bar{B}\mathfrak{D}^b + \frac{1}{2M^4}\left(\epsilon\tilde{\epsilon}\right)\bar{B}\mathfrak{D}_m - \frac{1}{2}\varepsilon_{mabc}\bar{\mathcal{H}}^{bc}\mathfrak{D}^a - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{ma}\bar{\mathcal{H}}^{ba}\mathfrak{D}_b + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{mb}\bar{\mathcal{H}}\mathfrak{D}^b\right] \\
&\times\left(\eta^{mn} - \frac{\mathrm{i}}{M^2}\epsilon^{mn} - \frac{2}{M^4}\tilde{\epsilon}^{mk}\tilde{\epsilon}_k{}^n\right)^{-1} \\
&\times\left[\mathfrak{D}_n A + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}^l B + \frac{1}{2M^4}\left(\epsilon\tilde{\epsilon}\right)\mathfrak{D}_n B - \frac{1}{2}\varepsilon_{nlpq}\mathfrak{D}^l\mathcal{H}^{pq} + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}_p\mathcal{H}^{pl} - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}^l\mathcal{H}\right] \\
&- M^2\bar{A}A + \bar{B}\left(\mathfrak{D}^2 - M^2\right)B - \frac{2}{M^4}\epsilon_{mn}\epsilon^{mk}\bar{B}\mathfrak{D}^n\mathfrak{D}_k B + \frac{1}{2}\bar{\mathcal{H}}_{(mn)}\mathfrak{D}^2 h^{mn} + \frac{1}{2}\mathfrak{D}^n\bar{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk} \\
&- M^2\bar{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \frac{M^2}{2}\bar{\mathcal{H}}^{(mn)}h_{mn} - \frac{1}{2}\bar{\mathcal{H}}\left(\mathfrak{D}^2 - M^2\right)h + \frac{1}{2}\mathfrak{D}^n\bar{\mathcal{H}}_{nm}\mathfrak{D}_k\mathcal{H}^{km} \\
&+ \frac{1}{2}\left(\bar{\mathcal{H}}^{mn}\mathfrak{D}_m\mathfrak{D}_n h + \text{h.c.}\right) + \frac{1}{2M^2}\left[2\mathrm{i}\left(\mathfrak{D}^n\bar{\mathcal{H}}_{nm}\epsilon^{mk}\mathfrak{D}_k B\right) - \left(\epsilon\epsilon\right)\bar{\mathcal{H}}B + \text{h.c.}\right] \\
&+ \frac{M^2}{2}\left(\bar{\mathcal{H}}^{[mn]} + \frac{1}{M^2}\mathrm{i}\epsilon^{mn}\bar{B}\right)\left(\mathcal{H}_{[mn]} - \frac{1}{M^2}\mathrm{i}\epsilon_{mn}B\right)
\end{aligned}
$$

dual field strength

Bosonic Lagrangian

II) Deformation of Fierz-Pauli

$$\mathcal{L}_2 = \left[\bar{A}\mathfrak{D}_m - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{mb}\bar{B}\mathfrak{D}^b + \frac{1}{2M^4}(\epsilon\tilde{\epsilon})\bar{B}\mathfrak{D}_m - \frac{1}{2}\varepsilon_{mabc}\bar{\mathcal{H}}^{bc}\mathfrak{D}^a - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{ma}\bar{\mathcal{H}}^{ba}\mathfrak{D}_b + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{mb}\bar{\mathcal{H}}\mathfrak{D}^b\right]$$
$$\times\left(\eta^{mn} - \frac{\mathrm{i}}{M^2}\epsilon^{mn} - \frac{2}{M^4}\tilde{\epsilon}^{mk}\tilde{\epsilon}_k{}^n\right)^{-1}$$
$$\times\left[\mathfrak{D}_n A + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}^l B + \frac{1}{2M^4}(\epsilon\tilde{\epsilon})\mathfrak{D}_n B - \frac{1}{2}\varepsilon_{nlpq}\mathfrak{D}^l\mathcal{H}^{pq} + \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}_p\mathcal{H}^{pl} - \frac{\mathrm{i}}{M^2}\tilde{\epsilon}_{nl}\mathfrak{D}^l\mathcal{H}\right]$$
$$- M^2\bar{A}A + \bar{B}\left(\mathfrak{D}^2 - M^2\right)B - \frac{2}{M^4}\epsilon_{mn}\epsilon^{mk}\bar{B}\mathfrak{D}^n\mathfrak{D}_k B + \frac{1}{2}\bar{\mathcal{H}}_{(mn)}\mathfrak{D}^2 h^{mn} + \frac{1}{2}\mathfrak{D}^n\bar{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk}$$
$$- M^2\bar{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \frac{M^2}{2}\bar{\mathcal{H}}^{(mn)}h_{mn} - \frac{1}{2}\bar{\mathcal{H}}\left(\mathfrak{D}^2 - M^2\right)h + \frac{1}{2}\mathfrak{D}^n\bar{\mathcal{H}}_{nm}\mathfrak{D}_k\mathcal{H}^{km}$$
$$+ \frac{1}{2}\left(\bar{\mathcal{H}}^{mn}\mathfrak{D}_m\mathfrak{D}_n h + \mathrm{h.c.}\right) + \frac{1}{2M^2}\left[2\mathrm{i}\left(\mathfrak{D}^n\bar{\mathcal{H}}_{nm}\epsilon^{mk}\mathfrak{D}_k B\right) - (\epsilon\epsilon)\bar{\mathcal{H}}B + \mathrm{h.c.}\right]$$
$$+ \frac{M^2}{2}\left(\bar{\mathcal{H}}^{[mn]} + \frac{1}{M^2}\mathrm{i}\epsilon^{mn}\bar{B}\right)\left(\mathcal{H}_{[mn]} - \frac{1}{M^2}\mathrm{i}\epsilon_{mn}B\right)$$

One spin-2

Two cplx scalars

- ♦ Can also be **decoupled** on shell
- ♦ It has **physical** DoF only
- ♦ **In absence of EM background, we recover the Fierz-Pauli Lagrangian+decoupled scalars**

Fermionic Lagrangian (2-component notation)

$$\mathcal{L}_F = -\frac{\mathrm{i}}{\sqrt{2}}\left[2\left(\lambda_1^m\sigma^n\mathfrak{D}_n\bar{\lambda}_{1m}\right) + \left(\bar{\chi}_{1m}\bar{\sigma}^n\sigma^k\bar{\sigma}^m\mathfrak{D}_k\chi_{1n}\right)\right]$$
$$- \sqrt{2}M\left[\left(\lambda_1^m\chi_{1m}\right) + \text{h.c.}\right]$$
$$+ \sqrt{2}\left[-\frac{\mathrm{i}}{4}\left(\psi_1\sigma^m\mathfrak{D}_m\bar{\psi}_1\right) + 2\mathrm{i}\left(\gamma_1\sigma^m\mathfrak{D}_m\bar{\gamma}_1\right)\right]$$
$$+ \left[3\left(\chi_1^m\sigma_{mn}\mathfrak{D}^n\psi_1\right) - \frac{1}{2}\left(\chi_1^m\mathfrak{D}_m\psi_1\right) - 2\left(\lambda_1^m\mathfrak{D}_m\gamma_1\right)\right.$$
$$\left.-\frac{\mathrm{i}}{2}M\left(\lambda_1^m\sigma_m\bar{\psi}_1\right) - 2\mathrm{i}M\left(\bar{\chi}_1^m\bar{\sigma}_m\gamma_1\right) + \text{h.c.}\right]$$
$$+ M\left[\frac{1}{\sqrt{2}}\left(\psi_1\gamma_1\right) + \text{h.c.}\right] + (1 \leftrightarrow 2)$$
$$- \frac{1}{M}\left[\bar{\chi}_1^m\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}_m\gamma_1 + \chi_2^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\gamma}_2 + \text{h.c.}\right]$$

- ♦ The physical states are the **spin-3/2** λ_{mj}, χ_{mj} coupled to **spin-1/2** ψ_j, γ_j with j=1, 2.

- ♦ Spin-3/2 constraints:

$$\bar{\sigma}^m\chi_{1m} = 0$$
$$\mathfrak{D}^m\chi_{1m} = -\frac{\mathrm{i}}{\sqrt{2}M}\left(\epsilon\cdot\sigma\right)\gamma_1$$
$$\sigma^m\bar{\lambda}_{1m} = \frac{3}{\sqrt{2}}\mathrm{i}\gamma_1 - \frac{\sqrt{2}}{M^2}\left(\epsilon\cdot\sigma\right)\gamma_1$$
$$\mathfrak{D}^m\bar{\lambda}_{1m} = \frac{3M}{2\sqrt{2}}\bar{\psi}_1 - \frac{1}{2M}\bar{\sigma}^m\left(\epsilon\cdot\sigma\right)\chi_{1m}$$

Coupling to spin-1/2!

Fermionic Lagrangian (2-component notation)

$$
\begin{aligned}
\mathcal{L}_F = & -\frac{\mathrm{i}}{\sqrt{2}}\left[2\left(\lambda_1^m\sigma^n\mathfrak{D}_n\bar{\lambda}_{1m}\right)+\left(\bar{\chi}_{1m}\bar{\sigma}^n\sigma^k\bar{\sigma}^m\mathfrak{D}_k\chi_{1n}\right)\right] \\
& -\sqrt{2}M\left[(\lambda_1^m\chi_{1m})+\text{h.c.}\right] \\
& +\sqrt{2}\left[-\frac{\mathrm{i}}{4}(\psi_1\sigma^m\mathfrak{D}_m\bar{\psi}_1)+2\mathrm{i}\,(\gamma_1\sigma^m\mathfrak{D}_m\bar{\gamma}_1)\right] \\
& +\left[3\left(\chi_1^m\sigma_{mn}\mathfrak{D}^n\psi_1\right)-\frac{1}{2}\left(\chi_1^m\mathfrak{D}_m\psi_1\right)-2\,(\lambda_1^m\mathfrak{D}_m\gamma_1)\right. \\
& \left.-\frac{\mathrm{i}}{2}M\,(\lambda_1^m\sigma_m\bar{\psi}_1)-2\mathrm{i}M\left(\bar{\chi}_1^m\bar{\sigma}_m\gamma_1\right)+\text{h.c.}\right] \\
& +M\left[\frac{1}{\sqrt{2}}(\psi_1\gamma_1)+\text{h.c.}\right]+(1\leftrightarrow 2) \\
& -\frac{1}{M}\left[\bar{\chi}_1^m(\epsilon\cdot\bar{\sigma})\,\bar{\sigma}_m\gamma_1+\chi_2^m(\epsilon\cdot\sigma)\,\sigma_m\bar{\gamma}_2+\text{h.c.}\right]
\end{aligned}
$$

- We can **decouple** spin-3/2 and spin-1/2 on shell by the redefinitions:

$$
\begin{aligned}
\bar{\lambda}'_{1m} \equiv\ & \bar{\lambda}_{1m}+\frac{\mathrm{i}}{2\sqrt{2}}\left[1-\mathrm{i}\frac{2}{M^2}(\epsilon\cdot\bar{\sigma})\right]\bar{\sigma}_m\gamma_1 \\
& -\frac{1}{\sqrt{2}M}\left[\eta_{mn}-\mathrm{i}\frac{2}{M^2}(\epsilon_{mn}+\mathrm{i}\tilde{\epsilon}_{mn})\right]\mathfrak{D}^n\bar{\psi}_1
\end{aligned}
$$

$$
\chi'_{1m} \equiv \chi_{1m}+\frac{1}{\sqrt{2}M^2}(\epsilon\cdot\sigma)\,\sigma_m\bar{\psi}_1
$$

Fermionic Lagrangian (2-component notation)

$$\mathcal{L}_F = -\frac{\mathrm{i}}{\sqrt{2}}\Big[2\big(\lambda^{m} \ldots$$

$$-\ \sqrt{2}M \ldots$$

$$+\ \sqrt{2}\Big[- \ldots$$

$$+\Big[3\big(\chi_1^m \sigma_{mn}\mathfrak{D} \ldots \quad 2\,(\bar{\lambda}_1 \ldots \mathfrak{D}_n \ldots$$

$$-\frac{\mathrm{i}}{2}M\left(\lambda_1^m \sigma_m \bar{\psi}_1\right) - 2\mathrm{i}M\left(\bar{\chi}_1^m \bar{\sigma}_m \gamma_1\right) + \text{h.c.}\Big]$$

$$+\,M\left[\frac{1}{\sqrt{2}}\left(\psi_1\gamma_1\right) + \text{h.c.}\right] + (1 \leftrightarrow 2)$$

$$-\frac{1}{M}\left[\bar{\chi}_1^m\left(\epsilon\cdot\bar{\sigma}\right)\bar{\sigma}_m\gamma_1 + \chi_2^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\gamma}_2 + \text{h.c.}\right]$$

Spin-1/2 decoupled with **Dirac EoM** in QED

- We can **decouple** spin-3/2 and spin-1/2 on shell by the redefinitions:

$$\bar{\lambda}'_{1m} \equiv \bar{\lambda}_{1m} + \frac{\mathrm{i}}{2\sqrt{2}}\left[1 - \mathrm{i}\frac{2}{M^2}\left(\epsilon\cdot\bar{\sigma}\right)\right]\bar{\sigma}_m\gamma_1 - \frac{1}{\sqrt{2}M}\left[\eta_{mn} - \mathrm{i}\frac{2}{M^2}\left(\epsilon_{mn} + \mathrm{i}\tilde{\epsilon}_{mn}\right)\right]\mathfrak{D}^n\bar{\psi}_1$$

$$\chi'_{1m} \equiv \chi_{1m} + \frac{1}{\sqrt{2}M^2}\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\psi}_1$$

- New equations:

$$\mathrm{i}\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = -M\bar{\psi}_1, \quad \mathrm{i}\sigma^m\mathfrak{D}_m\bar{\psi}_1 = -M\gamma_1$$

$$\mathrm{i}\bar{\sigma}^n\mathfrak{D}_n\chi'_{1m} = -M\bar{\lambda}'_{1m},$$

$$\mathrm{i}\sigma^n\mathfrak{D}_n\bar{\lambda}'_{1m} = -M\left(\eta_{mn} - \mathrm{i}\frac{2}{M^2}\epsilon_{mn}\right)\chi_1'^{n}$$

$$\mathfrak{D}^m\chi'_{1m} = 0, \quad \bar{\sigma}^m\chi'_{1m} = 0$$

$$\mathfrak{D}^m\bar{\lambda}'_{1m} = -\frac{1}{2M}\bar{\sigma}^m\left(\epsilon\cdot\sigma\right)\chi'_{1m}, \quad \sigma^m\bar{\lambda}'_{1m} = 0$$

spin-3/2 equations (4-component notation)

$$\mathbf{\Psi}_{1m} \equiv \begin{pmatrix} \chi'_{1m\alpha} \\ \bar{\lambda}'^{\dot\alpha}_{1m} \end{pmatrix}$$

charged spin-3/2

EoM:

$$(\mathrm{i}\not{\mathfrak{D}} + M)\,\mathbf{\Psi}_{1m} = \mathrm{i}\frac{2}{M}\left(\epsilon_{mn}\mathbf{\Psi}^{n}_{1L}\right)$$

Divergence constraint:

$$\left[\mathfrak{D}^m - \frac{1}{2M}\left(\epsilon^{mn} + \mathrm{i}\tilde{\epsilon}^{mn}\right)\gamma_n\right]\mathbf{\Psi}_{1m} = 0,$$

γ-trace constraint:

$$\gamma^m \mathbf{\Psi}_{1m} = 0$$

spin-3/2 equations (4-component notation)

$$\boldsymbol{\Psi}_{1m} \equiv \begin{pmatrix} \chi'_{1m\alpha} \\ \bar{\lambda}'^{\dot{\alpha}}_{1m} \end{pmatrix}$$

charged spin-3/2

Without background:

EoM:

$$(\mathrm{i}\not{\partial} + M)\boldsymbol{\Psi}_{1m} = \cancel{\mathrm{i}\frac{2}{M}\left(\epsilon_{mn}\boldsymbol{\Psi}^{n}_{1L}\right)}\ 0$$

Divergence constraint:

$$\left[\partial^{m} - \cancel{\frac{1}{2M}(\epsilon^{mn} + \mathrm{i}\tilde{\epsilon}^{mn})\gamma_{n}}\right]\boldsymbol{\Psi}_{1m} = 0,$$

γ-trace constraint:

$$\gamma^{m}\boldsymbol{\Psi}_{1m} = 0$$

→ ***Rarita-Schwinger***

Side remark:
Other forms of the spin-3/2 Lagrangian

With additional redefinitions, we can write the Lagrangian in alternative (more complicated) forms:

1) One that gives directly decoupled EoM and constraints
2) One that reduces trivially to RS Lagrangian in absence of background

Check out 👉 2211.13691

Conclusions

- ♦ A Lagrangian formulation is important to describe **composite states** in the point particle approximation

- ♦ We found **consistent 4D** Lagrangians including charged massive spin-3/2 and spin-2

Bosons: **12** complex DoF on shell ; Fermions: **12** complex DoF on shell

- ♦ On-shell equations can be decoupled !

Bosons: Same as Argyres-Nappi ; Fermions: (NEW) explicit EoM and constraints

- ♦ Another paper with all calculation details in preparation…

Thank you!