Gravitational radiation from a binary black hole coalescence in Einstein-scalar-Gauss-Bonnet gravity

Félix-Louis Julié

Max Planck Institute For Gravitational Physics
(Albert Einstein Institute), Potsdam

Mathématique-Physique group seminar
Institut de Mathématiques de Bourgogne

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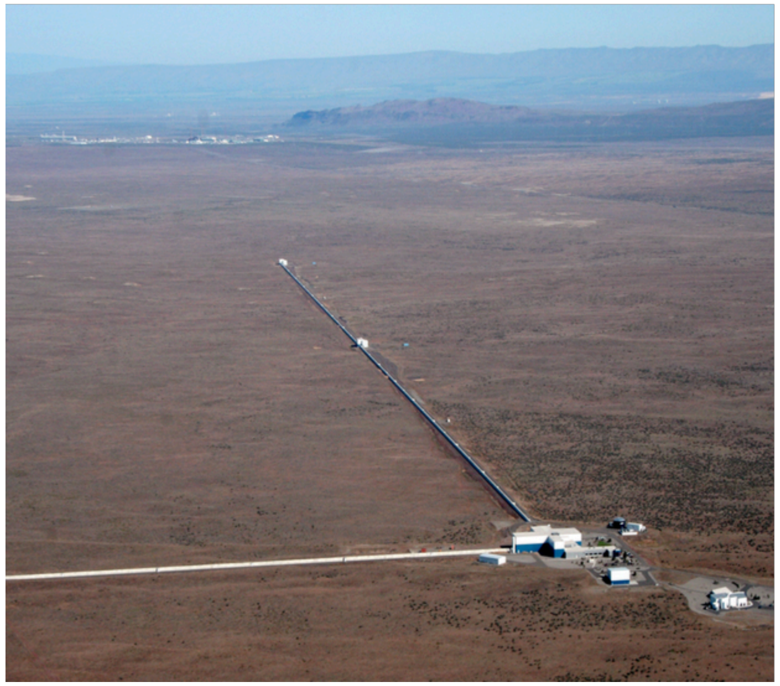
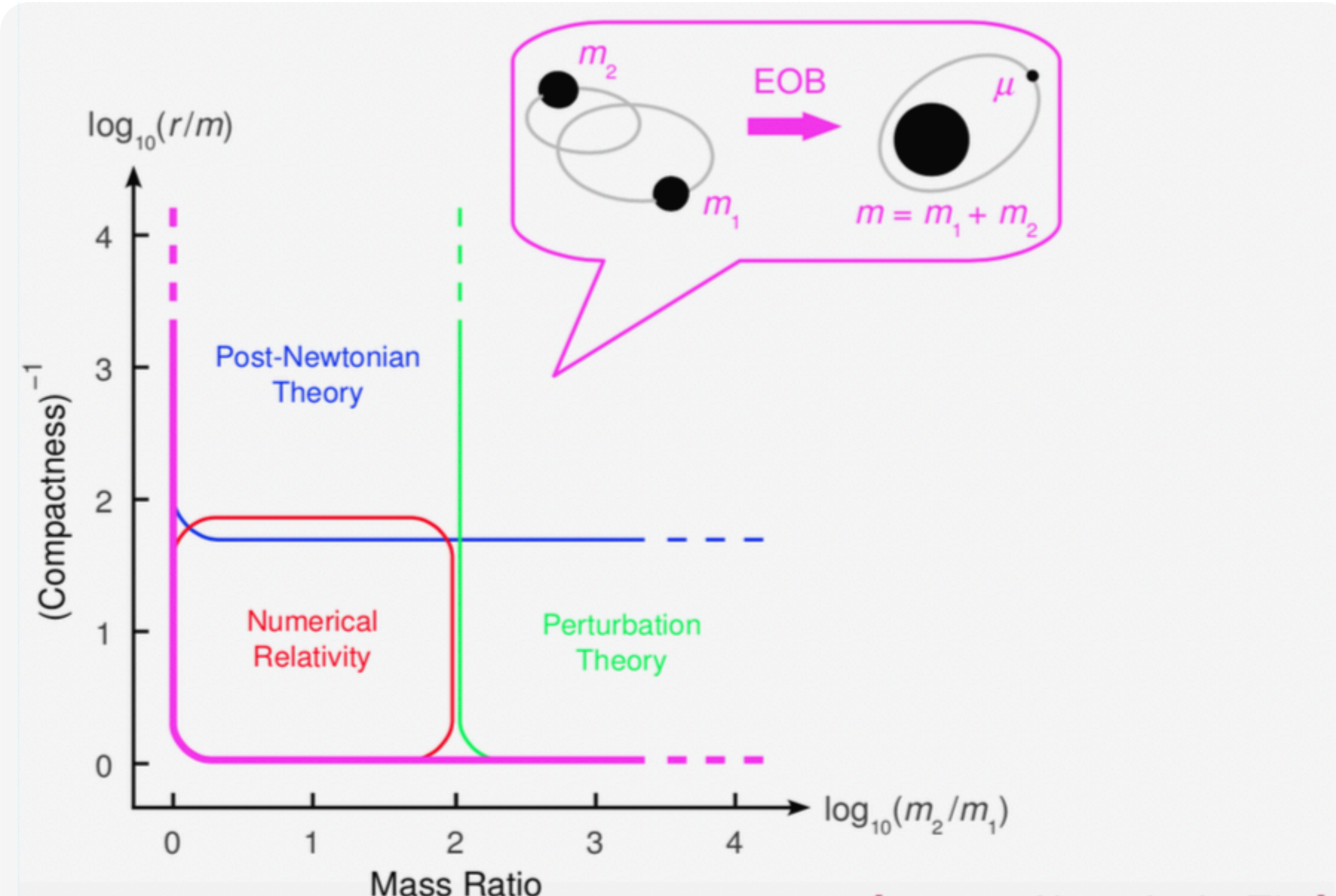
The era of gravitational wave astronomy

- GW150914: first observation of a binary black hole coalescence by LIGO-Virgo
- GW170817: first binary neutron star with electromagnetic counterparts
- Third observation run (O3): 90 candidate gravitational-wave event detections
- Since March 2020: O4 in preparation, possibly with the Japanese interferometer KAGRA...



Opportunity for **new tests of general relativity and modified gravities**, in the strong-field regime of a compact binary coalescence.

"Knowing the chirp to hear it"...



[courtesy Alexandre Le Tiec]

In general relativity: PN theory, self-force calculations, EOB framework, numerical relativity...

How to adapt these tools to derive analytic waveforms in modified gravities ?

Consider the example of **Einstein-scalar-Gauss-Bonnet** (ESGB) theories.

- **Félix-Louis Julié, Hector O. Silva, Emanuele Berti, Nicolás Yunes**, "Black hole sensitivities in scalar-Gauss-Bonnet gravity," arXiv:2202.01329, 2022
- **Félix-Louis Julié, Emanuele Berti**, "Post-Newtonian dynamics and black hole thermodynamics in Einstein-scalar-Gauss-Bonnet gravity," Phys.Rev. D100 (2019) no.10, 104061
- **Marcela Cardenas, Félix-Louis Julié, Nathalie Deruelle**, "Thermodynamics sheds light on black hole dynamics," Phys. Rev. D97, 12, 124021, 2018.
- **Félix-Louis Julié**, "Gravitational radiation from compact binary systems in Einstein-Maxwell-dilaton theories," JCAP 1810, 10, 033, 2018.
- **Félix-Louis Julié**, "Reducing the two-body problem in scalar-tensor theories to the motion of a test particle: a scalar-tensor effective-one-body approach," Phys. Rev. D97, 2, 024047, 2018.
- **Félix-Louis Julié, Nathalie Deruelle**, "Two-body problem in scalar-tensor theories as a deformation of general relativity: an effective-one-body approach," Phys. Rev. D95, 12, 124054, 2017.

Einstein-Scalar-Gauss-Bonnet gravity

ESGB vacuum action ($G = c = 1$)

$$I_{\rm ESGB} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)\mathscr{R}^2_{\rm GB}\right)$$

- Massless scalar field φ
- Gauss-Bonnet scalar $\mathscr{R}^2_{\rm GB} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$
- Fundamental coupling α with dimensions L^2 and $f(\varphi)$ defines the ESGB theory
- $\int d^Dx\sqrt{-g}\mathscr{R}^2_{\rm GB}$ is a boundary term in $D \leqslant 4$ [see e.g. Myers 87]
- Subclass of Horndeski theories *[Kobayashi 19]*, compactified 5D Lovelock theories *[Charmousis 15]*, low-energy limit of string theory *[Gross-Sloan 87]*,...

Second order field equations

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - 4\alpha\left(P_{\mu\alpha\nu\beta} - \frac{g_{\mu\nu}}{2}P_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta f(\varphi)$$

$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)\mathscr{R}^2_{\rm GB}$$

with $P_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2g_{\mu[\rho}R_{\sigma]\nu} + 2g_{\nu[\rho}R_{\sigma]\mu} + g_{\mu[\rho}g_{\sigma]\nu}R$

Hairy black holes in ESGB gravity

Analytical solutions in the small Gauss-Bonnet coupling ℓ limit

- **Einstein-dilaton-Gauss-Bonnet,** $f(\varphi) = e^{\varphi}$

  Mignemi-Stewart 93 at $\mathcal{O}(\ell^4)$, *Maeda at al. 97* at $\mathcal{O}(\ell^2)$, *Yunes-Stein 11* at $\mathcal{O}(\ell^2)$

  Ayzenberg-Yunes 14 at $\mathcal{O}(\ell^4, S^2)$, *Pani et al. 11* at $\mathcal{O}(\ell^4, S^2)$, *Maselli et al. 15* at $\mathcal{O}(\ell^{14}, S^5)$

- **Shift-symmetric theories,** $f(\varphi) = \varphi$

  Sotiriou-Zhou 14 at $\mathcal{O}(\ell^4)$

- **Generic ESGB theories**

  FLJ-Berti 19 at $\mathcal{O}(\ell^8)$

Numerical solutions

- **Einstein-dilaton-Gauss-Bonnet,** $f(\varphi) = e^{\varphi}$

  Kanti et al. 95, *Pani-Cardoso 09*, *Kleihaus 15* (includes spins), ***FLJ-Silva-Berti-Yunes 22***

- **Shift-symmetric theories,** $f(\varphi) = \varphi$

  Delgado et al. 20 (includes spin), ***FLJ-Silva-Berti-Yunes 22***

- **Generic ESGB theories**

  Antoniou et al. 18

- **Quadratic couplings,** $f(\varphi) = \varphi^2(1 + \lambda\varphi^2)$ and $f(\varphi) = -\, e^{-\lambda\varphi^2}$

  Doneva-Yazadjiev 17, Silva et al. 17, Minamitsuji-Ikeda 18, Macedo et al. 19,

  Dima-Barausse et al. 20, ***FLJ-Silva-Berti-Yunes 22***

How to address analytically the motion and gravitational radiation of two coalescing ESGB black holes?

See also *Witek et al. 19, Okounkova 20, East-Ripley 20* for numerical waveforms in the small α limit;

***East-Ripley 21* for numerical head-on collisions with large α.**

1. **ESGB black holes and their thermodynamics**

2. The post-newtonian (PN) dynamics of an ESGB black hole binary

3. Beyond the PN approximation: "EOBization" of an ESGB black hole binary

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Static, spherically symmetric ESGB black holes

Just coordinate system

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + B(r)\,r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

Solve iteratively the field equations around a Schwarzschild spacetime with $\boxed{\epsilon = \frac{\alpha f'(\varphi_\infty)}{m^2} \ll 1}$

$$A(r) = 1 - \frac{2m}{r} + \sum_i \epsilon^i A_i(r)\,, \qquad B(r) = 1 + \sum_i \epsilon^i B_i(r)\,, \qquad \varphi(r) = \varphi_\infty + \sum_i \epsilon^i \varphi_i(r)$$

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - 4\alpha\left(P_{\mu\alpha\nu\beta} - \frac{g_{\mu\nu}}{2}P_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta f(\varphi)$$

$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)\mathscr{R}^2_{\rm GB}$$

with $\mathscr{R}^2_{\rm GB} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$

ESGB black hole, at leading order for simplicity:

$$A = 1 - \frac{2m}{r} + \mathcal{O}\left(\frac{\alpha f'_\infty}{m^2}\right)^2, \qquad B = 1 + \mathcal{O}\left(\frac{\alpha f'_\infty}{m^2}\right)^2, \qquad \varphi = \varphi_\infty + \frac{\alpha f'(\varphi_\infty)}{m^2}\left(\frac{m}{2r} + \frac{m^2}{2r^2} + \frac{2m^3}{r^3}\right) + \mathcal{O}\left(\frac{\alpha f'_\infty}{m^2}\right)^2$$

Two integration constants: m and φ_∞, at all orders in the Gauss-Bonnet coupling.

ESGB black hole thermodynamics

- **Temperature:**

$$\boxed{T = \frac{\kappa}{4\pi}} \quad \text{where } \kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu)_{r_H} \quad \text{is the surface gravity}$$

- **Wald entropy:**

$$S_w = -8\pi \int_{r_H} d\theta d\phi \sqrt{\sigma} \frac{\partial \mathscr{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \qquad \text{with } \epsilon_{\mu\nu} = n_{[\mu} l_{\nu]}$$

$$\boxed{S_w = \frac{\mathscr{A}_H}{4} + 4\alpha\pi f(\varphi_H)} \qquad \text{in ESGB gravity.}$$

- **Mass as a global charge:**

$$\boxed{M = m + \int D \, d\varphi_\infty}$$

D is the scalar "charge" defined as $\varphi = \varphi_\infty + \frac{D}{r} + \mathscr{O}\left(\frac{1}{r^2}\right)$

[Henneaux et al. 02, Cardenas et al. 16, Anabalon-Deruelle-FLJ 16,...]

The quantities above are calculated in terms of m and φ_∞. At leading order for simplicity:

$$T = 8\pi m \left[1 + \mathscr{O}\left(\frac{\alpha f'_\infty}{m^2}\right)^2\right], \quad S_w = 4\pi m^2 \left[1 + \frac{\alpha f(\varphi_\infty)}{m^2} + \mathscr{O}\left(\frac{\alpha f'_\infty}{m^2}\right)^2\right], \quad D = \frac{\alpha f'(\varphi_\infty)}{2m}\left[1 + \mathscr{O}\left(\frac{\alpha f'_\infty}{m^2}\right)\right]$$

The variations of S_w and M with respect to m and φ_∞ are such that:

$$T\delta S_w = \delta M$$

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“Skeletonizing” an ESGB black hole

[in GR: Mathisson 1931, Infeld 1950,...]

$$I_{\text{ESGB}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)\mathscr{R}^2_{\text{GB}}\right) + I^A_{\text{pp}}$$

Generic ansatz for compact bodies

$$I^A_{\text{pp}}[g_{\mu\nu}, \varphi, x^\mu_A] = -\int m_A(\varphi)\, ds_A$$

with $ds_A = \sqrt{-g_{\mu\nu}dx^\mu_A dx^\nu_A}$.

- $m_A(\varphi)$ is a function of the local value of φ to encompass the effect of the background scalar field on the equilibrium of a body [Eardley 75, Damour-Esposito-Farèse 92].
- Strong equivalence principle violation

Question: How to derive $m_A(\varphi)$ for an ESGB black hole?

Answer: by identifying the BH's fields to those sourced by the particle.

Comparing the asymptotic expansions of the fields

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - 4\alpha\left(P_{\mu\alpha\nu\beta} - \frac{1}{2}g_{\mu\nu}P_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta f(\varphi) + 8\pi\left(T^A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^A\right)$$

$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)\mathscr{R}^2_{\rm GB} + 4\pi\frac{ds_A}{dt}\frac{dm_A}{d\varphi}\frac{\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))}{\sqrt{-g}}$$

with $$T^{\mu\nu}_A = m_A(\varphi)\frac{\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))}{\sqrt{g g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}\frac{dx^\mu_A}{dt}\frac{dx^\nu_A}{dt}$$

Fields of particle A in its rest frame, $x^i_A = 0$

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{2m_A(\varphi_\infty)}{\tilde{r}}\right) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)$$

$$\varphi = \varphi_\infty - \frac{1}{\tilde{r}}\frac{dm_A}{d\varphi}(\varphi_\infty) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)$$

Fields of the ESGB black hole

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta_{\mu\nu}\left(\frac{2m}{\tilde{r}}\right) + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)$$

$$\varphi = \varphi_\infty + \frac{D}{\tilde{r}} + \mathcal{O}\left(\frac{1}{\tilde{r}^2}\right)$$

Matching

- the **identification** yields

Matching conditions

$$m_A(\varphi_\infty) = m$$
$$m'_A(\varphi_\infty) = -D$$

- **For an ESGB black hole** with **"secondary hair"**, $D = D(m, \varphi_\infty)$ yields a first order differential equation.

At leading order, for simplicity:

$$\frac{dm_A}{d\varphi} + \frac{\alpha f'(\varphi)}{2m_A(\varphi)}\left[1 + \mathcal{O}\left(\frac{\alpha f'}{m_A^2}\right)\right] = 0$$

- Its resolution involves a **unique integration constant** μ_A.

The sensitivity of a hairy ESGB black hole

$$I^A_{\rm pp}[g_{\mu\nu}, \varphi, x^\mu_A] = -\int m_A(\varphi)\, ds_A$$

- In an **arbitrary ESGB theory**, BHs are described by a **unique constant parameter:**

$$\boxed{m_A(\varphi) = \mu_A\left(1 - \frac{\alpha f(\varphi)}{2\mu_A^2} + \cdots\right)}$$

where $\mu_A = M_{\rm irr} = \sqrt{\frac{S_{\rm w}}{4\pi}}$

- Recall: ESGB first law of thermodynamics: $\boxed{T\delta S_{\rm w} = \delta M}$

where $\delta M = \delta m + D\delta\varphi_\infty.$

Matching conditions

(a) $m_A(\varphi_\infty) = m$

(b) $m'_A(\varphi_\infty) = -D$

(a) and (b) $\Rightarrow$ $\delta M = 0$

As a consequence, $\delta S_{\rm w} = 0$

When φ_∞ varies slowly, the black hole readjusts its equilibrium configuration, i.e. m and D, in keeping its Wald entropy fixed.

$$I_{\rm ESGB} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)\mathscr{R}^2_{\rm GB}\right) - \sum_A \int m_A(\varphi) ds_A$$

ESGB two-body Lagrangian at 1PN order

- Harmonic gauge $\partial_\mu(\sqrt{-g}\, g^{\mu\nu}) = 0$
- Conservative 1PN dynamics: $\mathcal{O}\left(\frac{v}{c}\right)^2 \sim \mathcal{O}\left(\frac{GM}{r}\right)$ corrections to Newtonian dynamics
- Solve iteratively the field equations with point particle sources around

$$g_{00} = -e^{-2U} + \mathcal{O}(v^6) \qquad \varphi = \varphi_0 + \delta\varphi$$
$$g_{0i} = -4g_i + \mathcal{O}(v^5)$$
$$g_{ij} = \delta_{ij}e^{2U} + \mathcal{O}(v^4)$$

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - 4\alpha\left(P_{\mu\alpha\nu\beta} - \frac{1}{2}g_{\mu\nu}P_{\alpha\beta}\right)\nabla^\alpha\nabla^\beta f(\varphi) + 8\pi\sum_A\left(T^A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^A\right)$$
$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)\mathscr{R}^2_{\rm GB} + 4\pi\sum_A \frac{ds_A}{dt}\frac{dm_A}{d\varphi}\frac{\delta^{(3)}(\mathbf{x} - \mathbf{x}_A(t))}{\sqrt{-g}}$$

- The sensitivities $m_A(\varphi)$ and $m_B(\varphi)$ are expanded around φ_0

$$\ln m_A(\varphi) = \ln m_A^0 + \alpha_A^0(\varphi - \varphi_0) + \frac{1}{2}\beta_A^0(\varphi - \varphi_0)^2 + \cdots$$
$$\ln m_B(\varphi) = \ln m_B^0 + \alpha_B^0(\varphi - \varphi_0) + \frac{1}{2}\beta_B^0(\varphi - \varphi_0)^2 + \cdots$$

ESGB two-body Lagrangian *[FLJ-Berti 2019]*

$$
\begin{aligned}
L_{AB} = & - m_A^0 - m_B^0 + \frac{1}{2} m_A^0 \mathbf{v}_A^2 + \frac{1}{2} m_B^0 \mathbf{v}_B^2 + \frac{G_{AB} m_A^0 m_B^0}{r} \\
& + \frac{1}{8} m_A^0 \mathbf{v}_A^4 + \frac{1}{8} m_B^0 \mathbf{v}_B^4 + \frac{G_{AB} m_A^0 m_B^0}{r} \left[\frac{3}{2} (\mathbf{v}_A^2 + \mathbf{v}_B^2) - \frac{7}{2} (\mathbf{v}_A \cdot \mathbf{v}_B) - \frac{1}{2} (\mathbf{n} \cdot \mathbf{v}_A)(\mathbf{n} \cdot \mathbf{v}_B) + \bar{\gamma}_{AB} (\mathbf{v}_A - \mathbf{v}_B)^2 \right] \\
& - \frac{G_{AB}^2 m_A^0 m_B^0}{2r^2} \left[m_A^0 (1 + 2\bar{\beta}_B) + m_B^0 (1 + 2\bar{\beta}_A) \right] + L_{AB}^{ST,2PN} + L_{AB}^{ST,3PN} + \Delta L_{AB}^{\mathrm{GB}} + \mathcal{O}(v^{10})
\end{aligned}
$$

$$
\begin{aligned}
G_{AB} &= G(1 + \alpha_A^0 \alpha_B^0) \\
\bar{\gamma}_{AB} &= -2 \frac{\alpha_A^0 \alpha_B^0}{1 + \alpha_A^0 \alpha_B^0} \\
\bar{\beta}_A &= \frac{1}{2} \frac{\beta_A^0 {\alpha_B^0}^2}{(1 + \alpha_A^0 \alpha_B^0)^2} \quad \text{and} \quad (A \leftrightarrow B).
\end{aligned}
$$

where $\alpha_A^0 = (d \ln m_A / d\varphi)(\varphi_0)$, $\beta_A^0 = (d\alpha_A^0 / d\varphi)(\varphi_0)$

- L_{AB} has the same structure as that of scalar-tensor theories $(\alpha = 0)$ up to 3PN, modulo a finite Gauss-Bonnet contribution:

$$
\Delta L_{AB}^{\mathrm{GB}} = \frac{\alpha f'(\varphi_0)}{(GM)^2} \left(\frac{GM}{r} \right)^2 \frac{G^2 m_A^0 m_B^0}{r^2} \left[m_A^0 (\alpha_B^0 + 2\alpha_A^0) + m_B^0 (\alpha_A^0 + 2\alpha_B^0) \right]
$$

- In scalar-tensor theories, L_{AB} is known at 2PN *[Mirshekari-Will 13]* and 3PN *[Bernard 19]*
- Leap forward in our knowledge of the **conservative ESGB** dynamics, from 0PN *[Yagi et al. 12]* to **3PN**.

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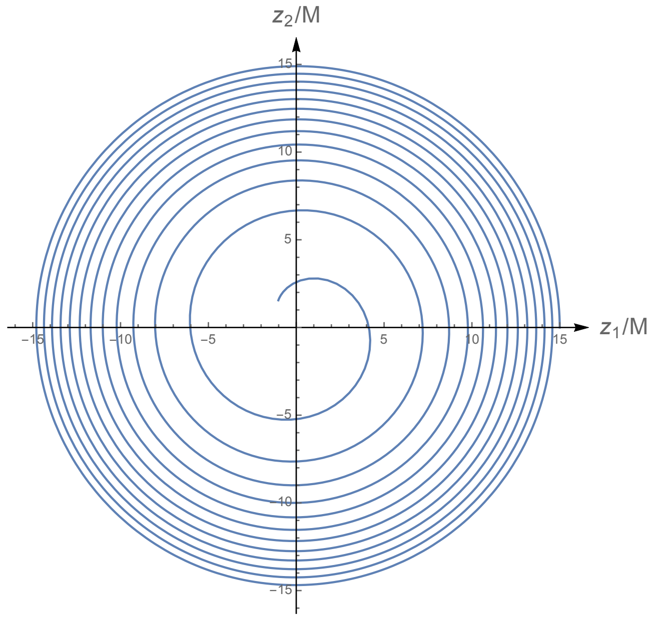
In general relativity, "effective-one-body" (EOB) :

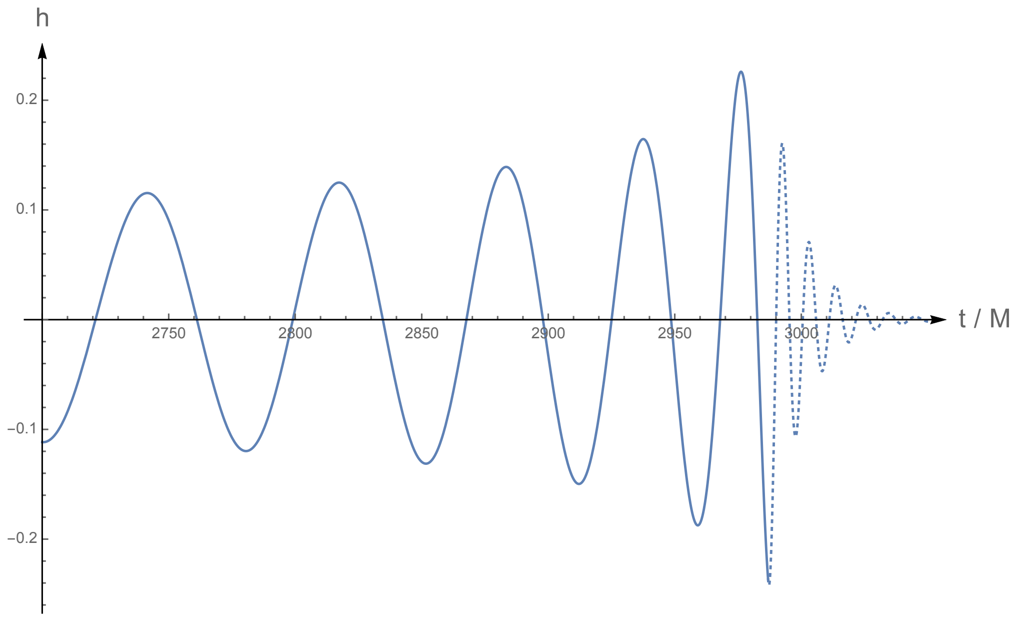
- Map the two-body PN dynamics to the motion of a **test particle** in an **effective static, spherically symmetric metric** [Buonanno-Damour 98]

$$H(Q,P)\ , \quad \epsilon = \left(\frac{v}{c}\right)^2 \qquad \longrightarrow \qquad H_e(q,p)\ , \quad ds_e^2 = g^e_{\mu\nu} dx^\mu dx^\nu$$

$$\boxed{H_e = f_{\mathrm{EOB}}(H)}$$

- Defines a resummation of the PN dynamics, hence describes **analytically** the coalescence of 2 compact objects in **general relativity**, from inspiral **to merger**.





- Instrumental to build libraries of waveform templates for LIGO-Virgo

In practice, on the simple example of ESGB at 1PN:

Compute the two-body hamiltonian $H(Q,P) = P_R \dot{R} + P_\phi \dot{\phi} - L_{AB}$.

In the center-of-mass frame :

$$\boxed{\overrightarrow{P}_A + \overrightarrow{P}_B = \overrightarrow{0}}$$

7 coefficients (polar coordinates)

$$H = M + \left(\frac{P^2}{2\mu} - \mu \frac{G_{AB} M}{R} \right) + H^{1\text{PN}} + \cdots$$

with $$\frac{H^{1\text{PN}}}{\mu} = \left(h_1^{1\text{PN}} \hat{P}^4 + h_2^{1\text{PN}} \hat{P}^2 \hat{P}_R^2 + h_3^{1\text{PN}} \hat{P}_R^4 \right) + \frac{1}{\hat{R}} \left(h_4^{1\text{PN}} \hat{P}^2 + h_5^{1\text{PN}} \hat{P}_R^2 \right) + \frac{h_6^{1\text{PN}}}{\hat{R}^2}$$

The 7 $h_i^{N\text{PN}}$ coefficients are computed explicitly and depend on the 6 parameters $(m_A^0, \alpha_A^0, \beta_A^0)$ and $(m_B^0, \alpha_B^0, \beta_B^0)$ built from $m_A(\varphi)$ and $m_B(\varphi)$

The effective Hamiltonian H_e

Geodesic motion in a static, spherically symmetric metric

In Schwarzschild-Droste coordinates (equatorial plane $\theta = \pi/2$) :

$$ds_e^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2$$

$A(r)$ and $B(r)$ are arbitrary

Effective Hamiltonian $H_e(q, p)$:

$$H_e(q,p) = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{\hat{r}^2}\right)} \quad \text{with} \quad p_r = \frac{\partial L_e}{\partial \dot{r}} \; , \quad p_\phi = \frac{\partial L_e}{\partial \dot{\phi}}$$

Can be expanded :

$$A(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots$$

$$B(r) = 1 + \frac{b_1}{r} + \cdots$$

and depends on 3 effective parameters at 1PN order, to be determined.

EOB mapping [Buonanno-Damour 98]

(i) Canonically transform H :

$$H(Q,P) \to H(q,p)$$

Generic ansatz $G(Q,p)$ that depends on **3 parameters** at 1PN order :

$$G(Q,p) = R\, p_r \left(\alpha_1 \mathscr{P}^2 + \beta_1 \hat{p}_r^2 + \frac{\gamma_1}{\hat{R}} + \cdots \right)$$

(ii) Relate H to H_e through the quadratic relation [Damour 2016]

$$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p) - M}{\mu} \right) \left[1 + \frac{\nu}{2} \left(\frac{H(q,p) - M}{\mu} \right) \right]$$

where $\nu = \dfrac{m_A^0 m_B^0}{(m_A^0 + m_B^0)^2}$, $M = m_A^0 + m_B^0$, $\mu = \dfrac{m_A^0 m_B^0}{M}$

3. Beyond the PN approximation: "EOBization" of an ESGB black hole binary

$$ds_e^2 = -A(r)dt + B(r)dr^2 + r^2 d\phi^2$$

It works, i.e., it yields a **unique** solution in ESGB theories:

FLJ, N. Deruelle [PRD 95, 12, 124054, 2017]

$$A(r) = 1 - 2\left(\frac{G_{AB}M}{r}\right) + 2\left[\langle\bar{\beta}\rangle - \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right)^2 + \cdots$$

$$B(r) = 1 + 2\left[1 + \bar{\gamma}_{AB}\right]\left(\frac{G_{AB}M}{r}\right) + \cdots$$

we recognize the PPN Eddington metric written in Droste coordinates, with :

$$\boxed{\beta^{\text{Edd}} = 1 + \langle\bar{\beta}\rangle\ , \quad \gamma^{\text{Edd}} = 1 + \bar{\gamma}_{AB}}$$

where

$$\langle\bar{\beta}\rangle \equiv \frac{m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A}{m_A^0 + m_B^0} \qquad \bar{\gamma}_{AB} \equiv -\frac{2\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0} \qquad \bar{\beta}_A \equiv \frac{1}{2}\frac{\beta_A^0(\alpha_B^0)^2}{(1+\alpha_A^0\alpha_B^0)^2} \qquad G_{AB} \equiv G(1+\alpha_A^0\alpha_B^0)$$

(See also [Damour, Jaranowski, Schaefer 15] at 4PN in GR; and [FLJ, N.Deruelle 17] at 2PN in scalar-tensor theories.)

A resummed dynamics

- The inversion of $$\frac{H_e(q,p)}{\mu} - 1 = \left(\frac{H(q,p)-M}{\mu}\right)\left[1 + \frac{\nu}{2}\left(\frac{H(q,p)-M}{\mu}\right)\right]$$

defines a "resummed" EOB Hamiltonian :

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}\,, \quad \text{where} \quad H_e = \sqrt{A\left(\mu^2 + \frac{p_r^2}{B} + \frac{p_\phi^2}{r^2}\right)}$$

- HEOB hence defines a resummed dynamics, e.g., up to the innermost stable circular orbit (ISCO) or light-ring (LR).

$$\dot{r} = \frac{\partial H_{\text{EOB}}}{\partial p_r}\,, \qquad \dot{p}_r = -\frac{\partial H_{\text{EOB}}}{\partial r}\,, \qquad \dot{\phi} = \frac{\partial H_{\text{EOB}}}{\partial p_\phi}\,, \qquad \dot{p}_\phi = -\frac{\partial H_{\text{EOB}}}{\partial \phi}$$

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Radiated energy fluxes at infinity

$$-\frac{d\mathscr{E}}{dt} = \mathscr{F}_g + \mathscr{F}_\varphi \qquad \text{with} \qquad \mathscr{E} = \int d^3x\,|g|\left(t^{00} + T^{00}_{(\varphi)} + T^{00}_{(\mathrm{m})}\right)$$

$$\mathscr{F}_g = \int_{x\to\infty} t^{0i}\, n_i x^2 d\Omega^2\,, \qquad \mathscr{F}_\varphi = \int_{x\to\infty} T^{0i}_{(\varphi)}\, n_i x^2 d\Omega^2\,,$$

- $\mathscr{F}_g$ is given by Einstein's 2nd quadrupole formula at leading order, $t^{\mu\nu}$ reduces to the Landau-Lifshitz pseudo-tensor
- $\mathscr{F}_\varphi$ is the extra scalar flux.

In the center-of-mass frame ($P^i_A + P^i_B = 0$) and for circular orbits:

[Yagi et al. 12, FLJ 18,
see also Shiralilou et al. 20-21]

- **Metric flux** ("dressed up" quadrupole formula)

Kepler: $(G_{AB}M\dot\phi)^{2/3} = \mathcal{O}(v^2)$

$$\mathscr{F}_g = \frac{32}{5}\frac{\nu^2\left(G_{AB}M\dot\phi\right)^{10/3}}{G\left(1+\alpha^0_A\alpha^0_B\right)^2} + \cdots$$

- **Scalar flux** (with dipolar contribution if $\alpha^0_A \neq \alpha^0_B$)

$$\mathscr{F}_\varphi = \frac{\nu^2\left(G_{AB}M\dot\phi\right)^{8/3}}{G_*\left(1+\alpha^0_A\alpha^0_B\right)^2}\left\{\frac{1}{3}(\alpha^0_A-\alpha^0_B)^2 + \left(G_{AB}M\dot\phi\right)^{2/3}\left[\frac{16}{15}\left(\frac{m^0_A\alpha^0_B + m^0_B\alpha^0_A}{M}\right)^2 + \frac{2}{9}(\alpha^0_A-\alpha^0_B)^2\left(\nu - 3 - \bar\gamma_{AB} - 2\langle\bar\beta\rangle\right)\right.\right.$$
$$\left.\left. + 2(\alpha^0_A-\alpha^0_B)\left(\frac{(m^0_A)^2\alpha^0_B - (m^0_B)^2\alpha^0_A}{5M^2} + \frac{m^0_A\left[\alpha^0_B + \alpha^0_A(\alpha^0_B)^2 + \beta^0_B\alpha^0_A\right] - (A\leftrightarrow B)}{3M(1+\alpha^0_A\alpha^0_B)}\right)\right] + \cdots\right\}$$

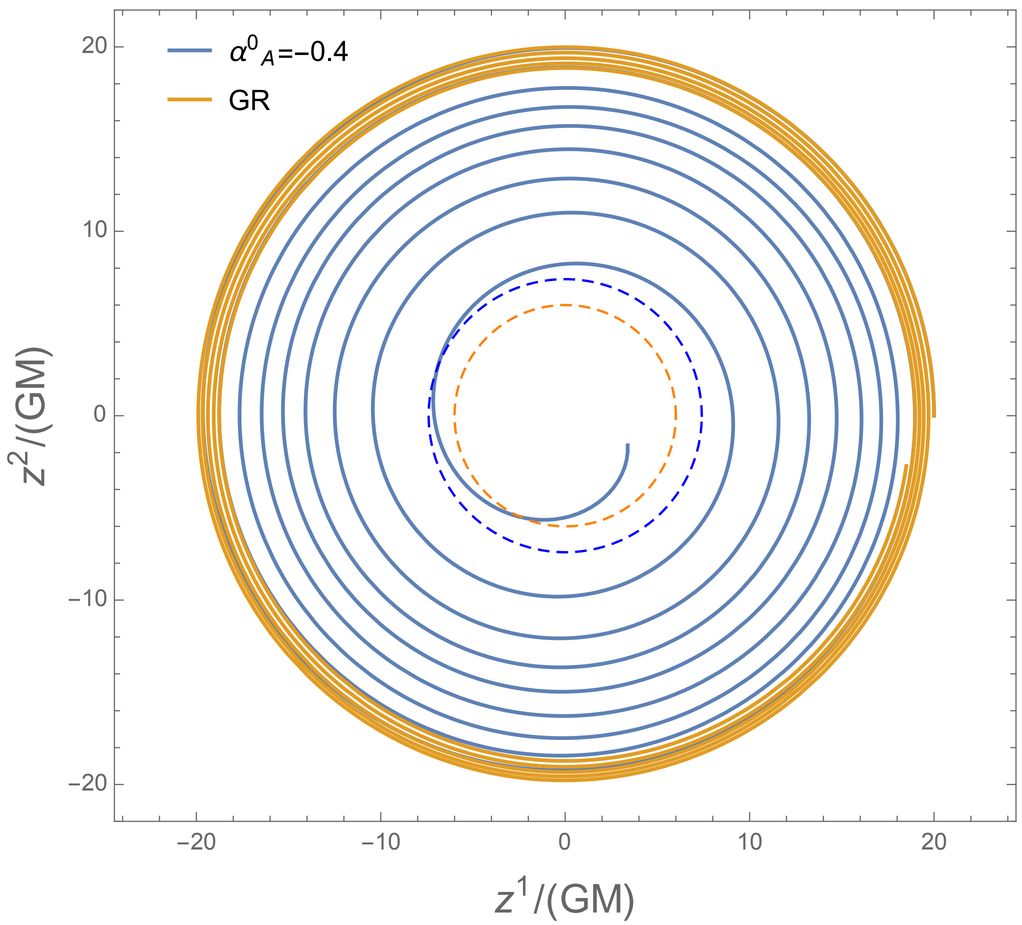
EOB dynamics and waveform

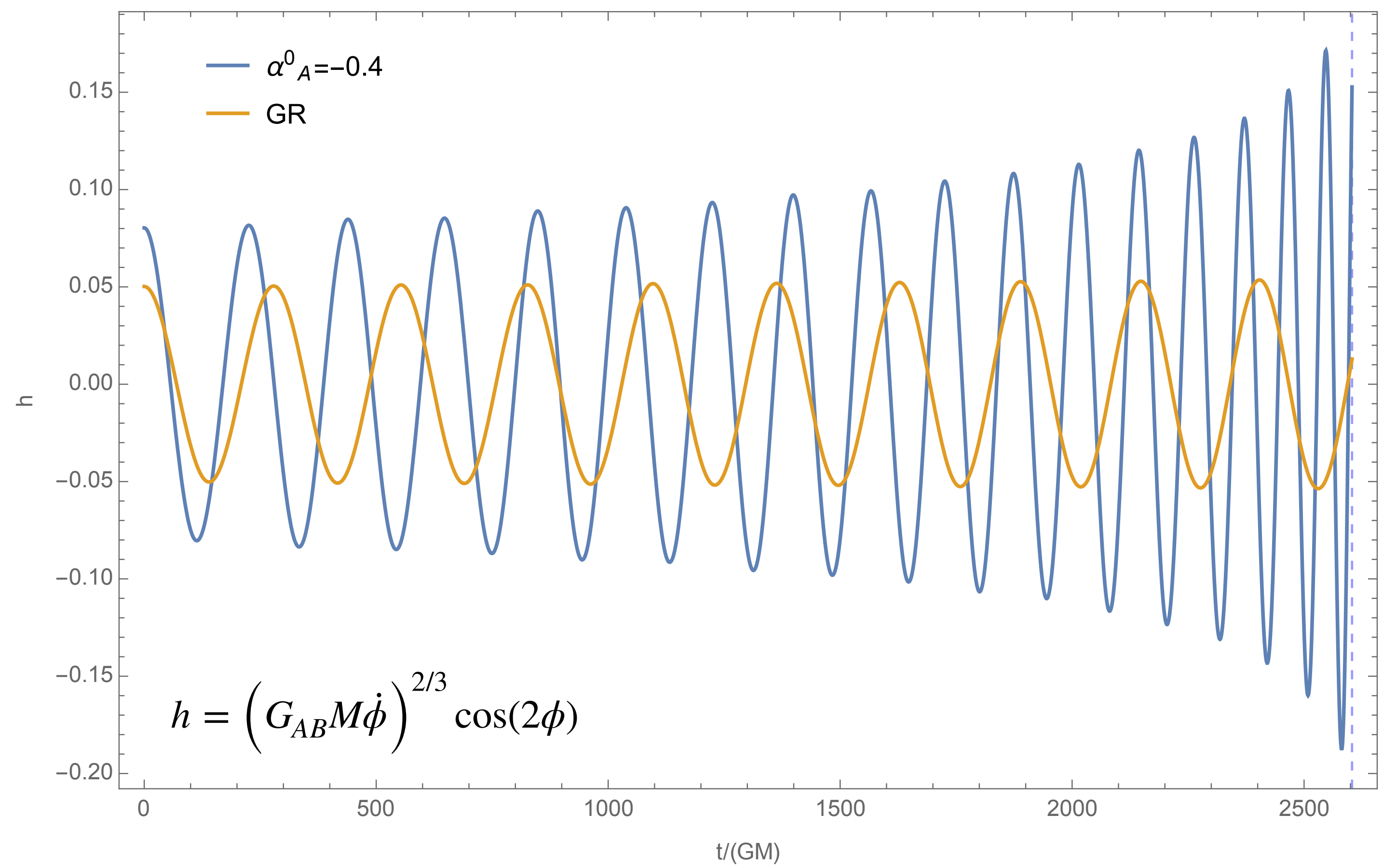
- On quasi-circular orbits : tangential force $\boxed{F_\phi = -(\mathscr{F}_g + \mathscr{F}_\varphi)/\dot{\phi}}$

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r}\,, \qquad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r}\,, \qquad \dot{\phi} = \frac{\partial H_{\rm EOB}}{\partial p_\phi}\,, \qquad \dot{p}_\phi = -\frac{\partial H_{\rm EOB}}{\partial \phi} + F_\phi$$

where $H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}$ and $H_e = \mu\sqrt{A\left(1 + \frac{p_r^2}{\mu^2 B} + \frac{p_\phi^2}{\mu^2 r^2}\right)}$

Example: analytic EOB trajectory and waveform for two EdGB black holes ($f(\varphi) = e^{2\varphi}/4$):





$m_A^0/m_B^0 = 2$

$\alpha_A^0 = -0.4$

$\alpha_B^0 = -1.6$

Recap

- Remarkably, the EOB approach can be extended beyond general relativity. In **ESGB** and **scalar-tensor gravity:**

$$A^{2\mathrm{PN}}(u) = \mathscr{P}^1_5[A^{Taylor}_{5\mathrm{PN}} + 2\epsilon_{1\mathrm{PN}}u^2 + (\epsilon^0_{2\mathrm{PN}} + \nu\,\epsilon^\nu_{2\mathrm{PN}})u^3]$$

- Also works in **Einstein-Maxwell-dilaton** (EMD) theories at 1PN [FLJ 18]

$$I_{\mathrm{EMD}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi}F^{\mu\nu}F_{\mu\nu}\right)$$

Future developments

- Refine our waveforms using **higher PN order** Lagrangians and fluxes; e.g., ST-ESGB at 3PN [Bernard 18]
- Out-of-equilibrium black holes: **finite-size effects**?

$$I_{\mathrm{pp}} = -\int ds_A\left[m_A(\varphi) + n_A(\varphi)(\partial\varphi)^2 + \cdots\right]$$

[Damour-Esposito-Farèse 98, Bini-Geralico-Steinhoff 20, Zimmerman-Lewis-Pfeiffer 16, Borhanian et al. 19, etc.]

- Match our waveforms to the **quasi-normal modes** of the final black hole [Blázquez-Salcedo et al. 16, Brito-Pacilio 18, Bryant et al. 21]

Thank you for your attention.