

# Quantum measurements, probabilities and reversibility: some naïve remarks and questions

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Trimestre IHES : Le Monde Quantique

F. David, Importance of reversibility in the quantum formalism. Phys. Rev. Lett. 107, 180401 (2011)

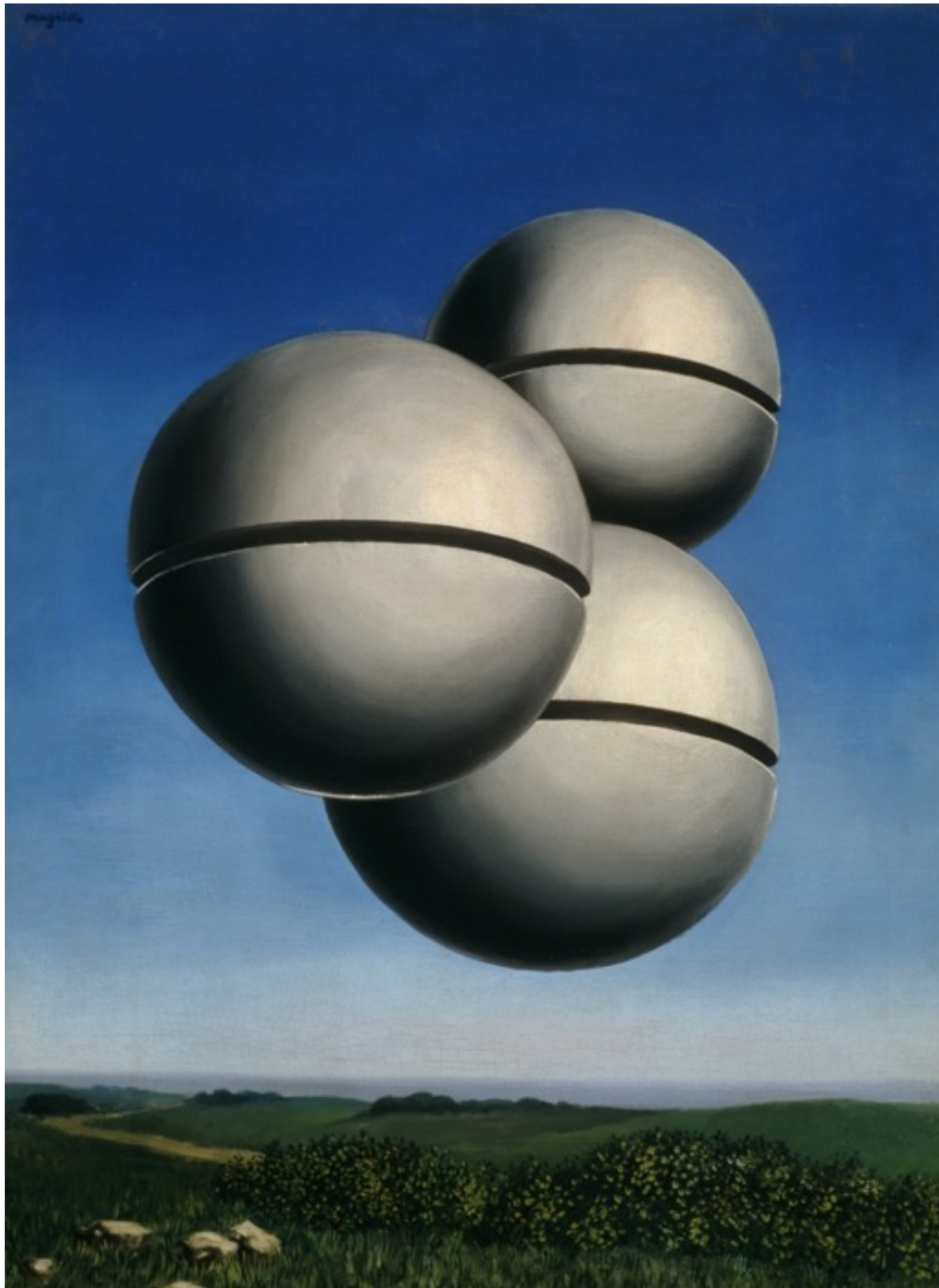
F. David, The Formalisms of Quantum Mechanics: An Introduction, Lecture Notes in Physics 893 (2015)

- Introduction: some questions?
- Time symmetry and quantum probabilities
- Time symmetry in the quantum logic formalism
- Time symmetry in the algebraic formalism
- What about space-time and causal separability
- Quantum informational formulations of quantum mechanics
- And what about Quantum Gravity?



Is Quantum  
Mechanics  
an unfinished  
construction  
which has left its  
builders in a state  
of mental  
confusion, mutual  
misunderstanding  
and discord ?





... or is Quantum Mechanics a closed and consistent physical theory (or at least a mathematical framework for physical theories) which rules over and beyond our familiar classical world, and is there «as it is» (whether we like it or not) ?

## **In this trimester...**

Many beautiful talks about how Quantum Mechanics works and how to probe and use its non-classical features in physics and as a resource in quantum information science.

But this trimester is (according to Juerg) also about discussing «foundational questions»...

### **What about Time in quantum physics?**

- What is Time? *Too difficult a question, at least for me...*
- Does Time plays a similar role in quantum physics than in classical physics (special and general relativity) ?
- ... or does it plays a very different role?
- Why irreversibility ?
- Why reversibility ?

I think that (most) physicists agree on the statement that quantum Mechanics is about

- Observables (what we can measure/action on the system)
- States (preparation of the system/ information we have on the system)
- Probabilities\*\* (of outcome of an observation on a state)
- Causality (relations between actions and outcomes)

... but they may and (often do) disagree on

- Importance of «locality»
- Concept and meaning of «physical reality»
- Origin and significance of chance and indeterminism

*This is not so different from Classical Physics ...*

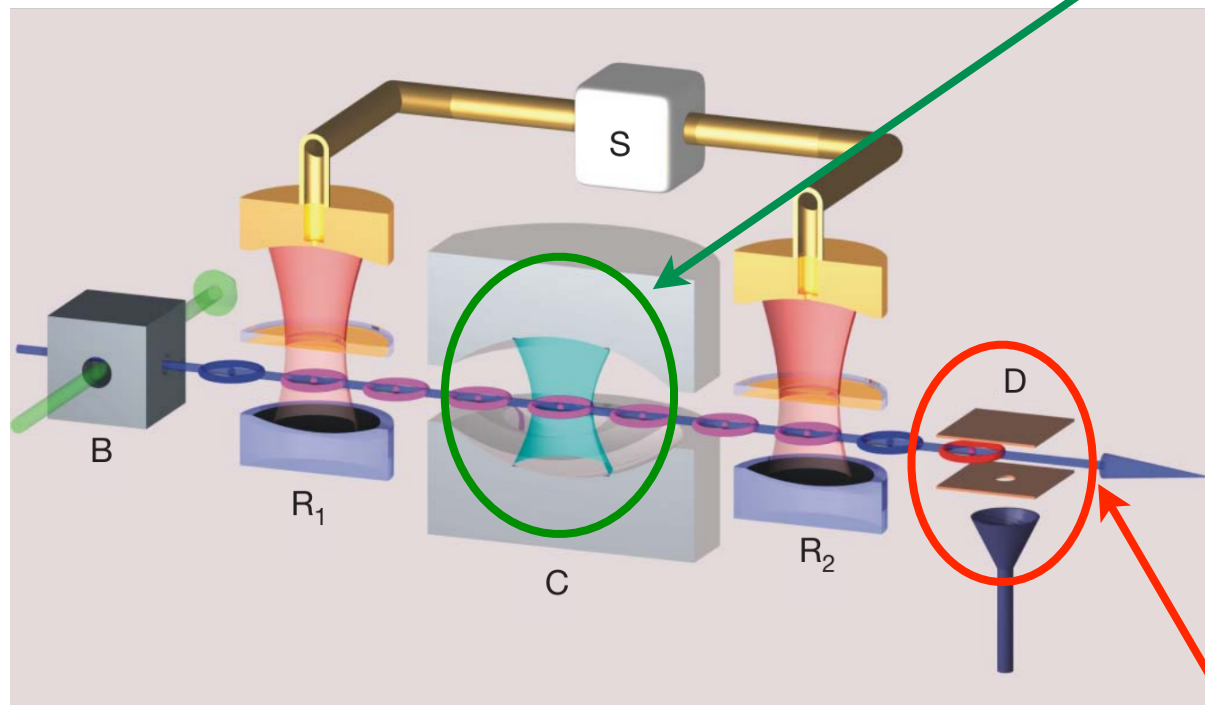
*Here probabilities follow the same rules as classical probabilities, but applies to different objects, since quantum processes  $\neq$  classical ones*

# Back to ideal measurements

Ideal projective measurements have been crucial in the construction and in the presentation of QM.

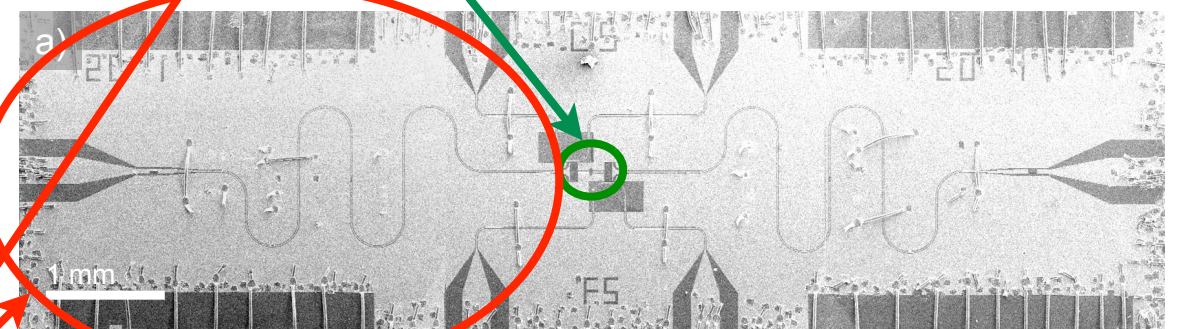
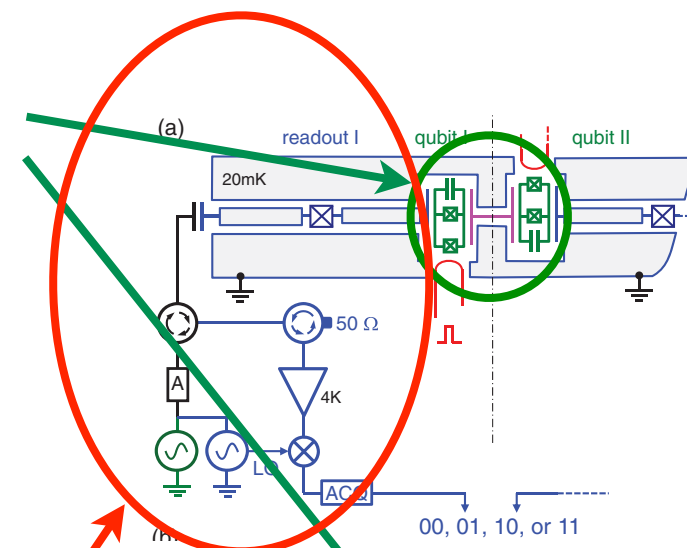
Although physicists are now able to perform indirect measurements and weak measurements in the lab ...

... there is a projective or destructive measurement on the probe at the end of the process.



cavity QED, LKB, ENS

quantum  
system



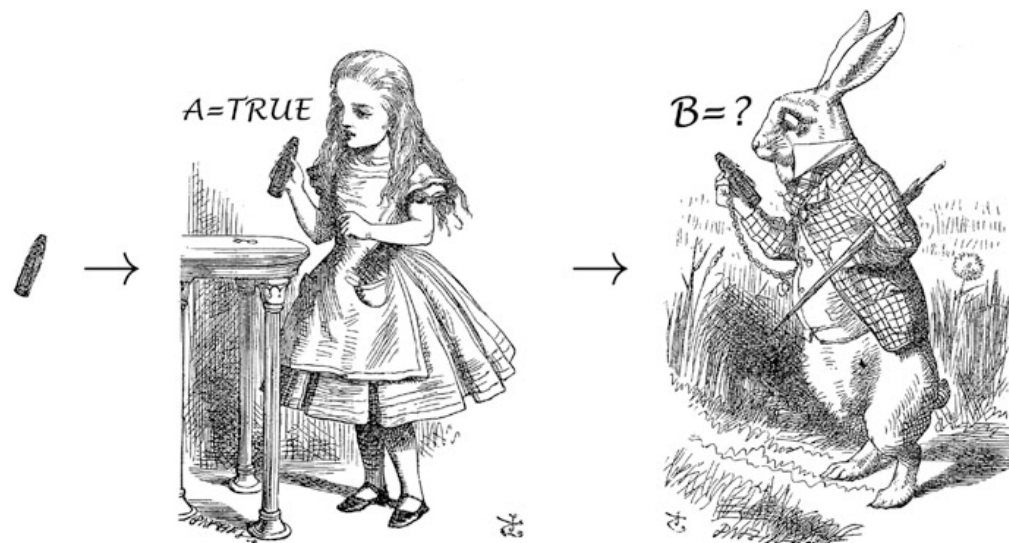
circuit QED, Quantronic Group, Saclay



# Time symmetry of quantum probabilities\*

Alice and Bob performs successive projective measurements on a system

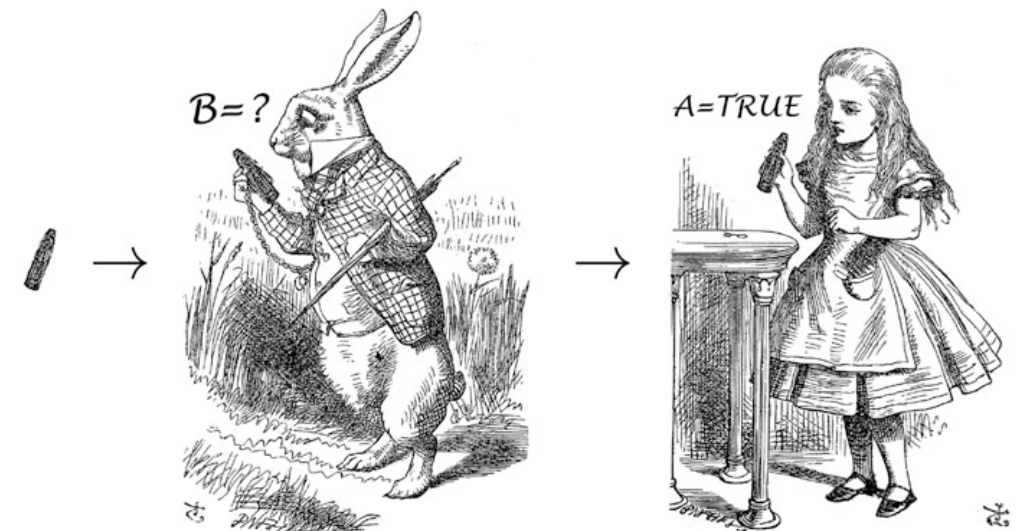
$\mathbf{P}_A$  &  $\mathbf{P}_B$  non commuting projectors (non compatible observables)



Alice measures  $A$  and gets  $A=1$ .

She asks Bob to measure  $B$ .

What is the probability (for Alice) that Bob will get  $B=1$ ?



Alice measures  $A$  and gets  $A=1$ .

She knows that Bob has measured  $B$ .

What is the probability (for Alice) that Bob had gotten  $B=1$ ?

$$P(B \leftarrow A) = P(B \rightarrow A) = \text{Tr}[\mathbf{P}_A \mathbf{P}_B] / \text{Tr}[\mathbf{P}_A]$$



\* Aharonov, Bergmann & Lebowitz '61 for dummies

Conditional probabilities have to be taken in a Bayesian sense (but still objective probabilities).



Can we view this feature as one of the basis of quantum physics?  
... at least from a pedagogical point of view...

I find this useful...

This can be done in the framework of **Algebraic Quantum Theory**

- Observables generate a  $C^*$ -algebra (complex)
- States are linear  $>0$  functionals on this algebra
- Time acts as automorphism (Poincaré for AQFT)

But also in the **Quantum Logic** framework

- Geometry of propositions associated to projective Y/N ideal measurements
- Initiated by G. Birkhoff & J. von Neumann (1936)
- Motivation for Gleason's theorem (1957, *its consequences for contextuality predate Bell-Kochen-Specker*)
- One of the very few derivations (if not a full proof) of the Hilbert space structure of quantum states and of the  $C^*$ -algebra structure of observables. Addition law + of observables, superposition of states and Born rule are not postulates.
- But very un-operational for doing physics...

# Time symmetry in the Quantum Logic Formulation(s)

*Birkhoff & v. Neumann '36, G. Mackey '63, J.M. Jauch '68, C. Piron '64, ...*

Projectors corresponds to «quantum propositions» on quantum systems (YES-NO measurements similar to TRUE-FALSE propositions).

But they do not generate a Boolean algebra (distributivity fails).

Which axioms should they satisfy? Can they be only represented as the projectors on closed subspaces of a complex Hilbert space?

Starting point: the propositions must form an orthocomplemented lattice. Plus some additional axioms (orthomodularity, etc..) that I won't discuss here.

## **POSET structure:**

Order relation (a IMPLIES b)  $a \preceq b$  defined as

«for any state, if a is found TRUE, then b will be found TRUE»

with standard logical relations:  $a \preceq b$  and  $b \preceq a$  then  $b = a$

$a \preceq b$  and  $b \preceq c$  then  $a \preceq c$

$a \preceq a$  **important!**

## Complete lattice

Cunjunction (AND)       $a \wedge b$       (= intersection of subspaces)  
Meet (OR)       $a \vee b$       (= direct sum of subspaces)  
with a largest and a smallest element       $I$        $\emptyset$

## Orthocomplementation

Negation or complement (NOT)       $\neg a$       (=  $\perp$  complement of subspace)

Its crucial property is: if  $a \preceq b$  then  $\neg b \preceq \neg a$

In a causal framework, this is equivalent to reversibility:

If  $a \preceq b$  is taken to mean: «if **a** is found true, then **b** will be found true»

then it means also: «if **b** is found false, then **a** was found false»

while  $\neg b \preceq \neg a$  means: «if **b** is found false, then **a** will be found false»

The two propositions are equivalent only if the causal structure represented by the order relation is symmetric (time symmetry)!

## Other axioms

weak-modularity (replaces distributivity), atomicity, covering (there are some physical and logical justifications...)

{Propositions} = Orthomodular AC Lattice

## Geometrization theorem:

Any OM AC lattice of length  $\geq 3$  (analog to  $\dim H \geq 3$  condition for Gleason) can be represented as the lattice of orthogonal projections on closed subspaces of a (left) module  $V$  (analog of a vector space) on a division ring  $K$  (analog of  $\mathbb{C}$ ) with an Hermitian form  $f$  (analog of  $\langle \cdot | \cdot \rangle$ ), with some additional nice properties (closure, unity, ...).

NB: This is an analog for orthomodular geometry of the geometrization theorem of abstract projective geometry.

Under some «regularity assumptions» the division ring  $K$  must contain the real numbers  $\mathbb{R}$ . This implies that  $V$  is a Hilbert space over

$$K = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}$$

Therefore, the Hilbert space geometry of projective measurements (and states, see later) is not so mysterious, but comes from the possible symmetries of the projective measurements (tests) on quantum systems. It is a geometrical representation.

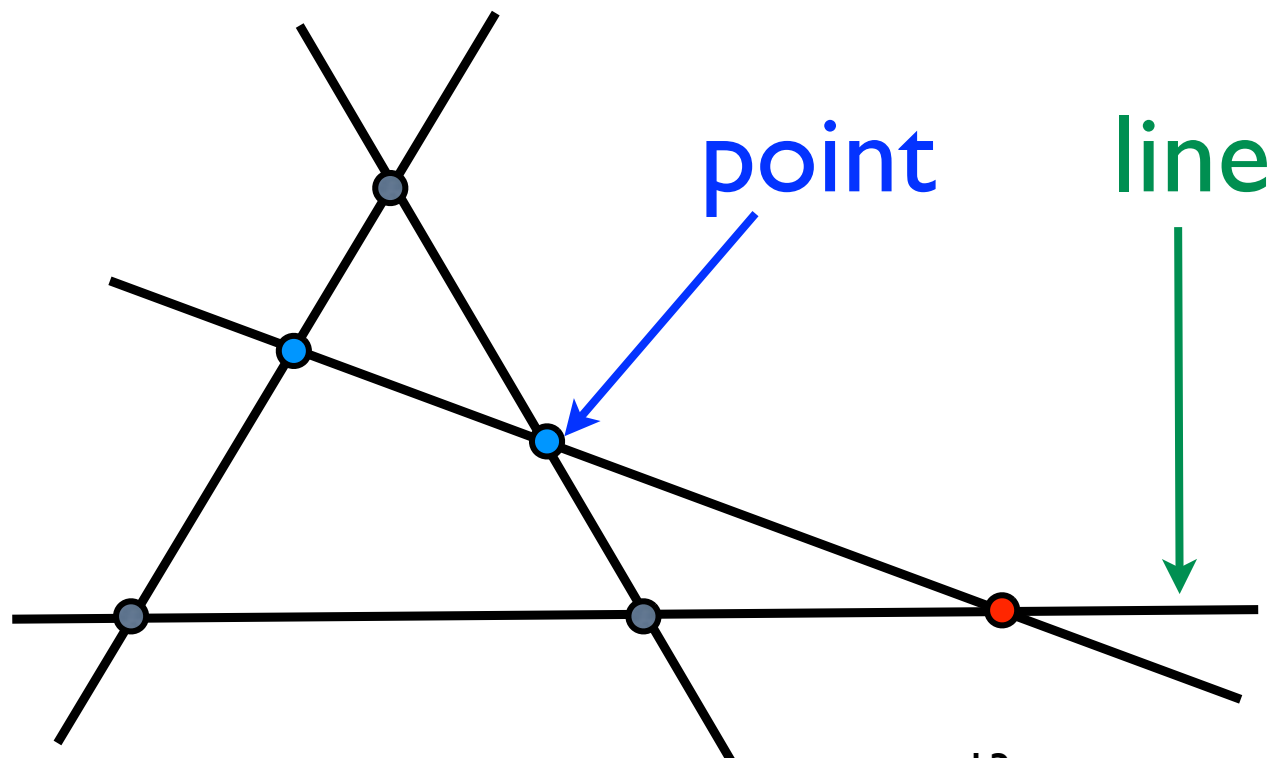


# Geometrization theorem of projective geometry (Veblen)

Theorem: If the geometry (points, lines) satisfies the axioms:

1. Any line contains at least three points,
2. Two points lie in a unique line,
3. A line meeting two sides of triangle, not at a vertex of the triangle, meets the third side also (Veblen's axiom),
4. There are at least four points non coplanar (a plane is defined in the usual way from lines),

then the corresponding geometry is the geometry of the affine subspaces of a left module  $M$  on a division ring  $K$  (a division ring is a general non-commutative field).



## Quantum probabilities and Born Rule:

Then Gleason's theorem states that probabilities are given by Born rule, i.e. any state  $\omega$  is described by a density matrix  $\rho_\omega$  so that the probability that **a** is found true is of the form

$$P(a|\omega) = \text{tr}[\Pi_a \rho_\omega] \qquad \Pi_a = \text{projector for } \mathbf{a}$$

This is Born rule!

Moreover, if the outcome of the measurement of **a** is 1 (true) then the conditional state is

$$\rho_{\omega_a} = \frac{\Pi_a \rho_\omega \Pi_a}{\text{tr}(\Pi_a \rho_\omega)}$$

This is the projection postulate (here consequence of repeatability of propositions, i.e. of  $\mathbf{a} \preceq \mathbf{a}$  )

# Algebraic Quantum Formalism

Standard formulation:

**Observables:** operators elements of a complex  $C^*$ -algebra  $\mathfrak{A}_{\mathbb{C}}$ , i.e.

1. addition law (non trivial)  $\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$
2. associative product  $(\mathbf{a}\mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b}\mathbf{c}) = \mathbf{a}\mathbf{b}\mathbf{c}$
3. involution  $(\mathbf{a}\mathbf{b})^* = \mathbf{b}^* \mathbf{a}^* \quad (\lambda \mathbf{a})^* = \bar{\lambda} \mathbf{a}^*$
4. closed  $*$ -norm  $\|\mathbf{a}\|^2 = \|\mathbf{a}^* \mathbf{a}\| = \|\mathbf{a}^*\|^2$
5. physical observables  $\mathbf{a} = \mathbf{a}^*$

**States:** fully characterised by the expectation value of observables, hence states are positive linear forms on  $\mathfrak{A}_{\mathbb{C}}$

$$\langle \mathbf{a} \rangle_{\omega} = \omega(\mathbf{a}) \quad \omega(\mathbf{a}^*) = \overline{\omega(\mathbf{a})} \quad \omega(\mathbf{a}\mathbf{a}^*) \geq 0 \quad \omega(\mathbf{1}) = 1$$

**Pure states** are extremal elements of the convex set of states  $\mathcal{E}_{\mathbb{C}}$

**GNS construction:**  $\mathfrak{A}_{\mathbb{C}} \subset \mathcal{B}(\mathcal{H}_{\mathbb{C}})$  operator algebra over some Hilbert space, reconstructed from its representations as acting on states.

# Reconsider this, starting from real algebras

**Observables** (operators): elements of a real  $*$ -algebra  $a \in \mathfrak{A}_{\mathbb{R}}$  ,

1. addition law (non trivial!)  $c = \lambda a + \mu b$

2. associative product  $(ab)c = a(bc) = abc$

associativity may be related to causality (causal ordering when combining «operations» represented by «observables»)

3. real involution  $(ab)^* = b^* a^* \quad (\lambda a)^* = \lambda a^*$

this involution represents time symmetry (causal description by  $a$  and anti-causal description by  $a^*$  of the system are equivalent)

4. physical observables (invariant under time symmetry)

$$a = a^*$$



# Real algebras, continued

**States:** positive symmetric linear forms on  $\mathfrak{A}_{\mathbb{R}}$

$\langle \mathbf{a} \rangle_{\omega} = \omega(\mathbf{a})$       expectation value

$\omega(\mathbf{a}^*) = \omega(\mathbf{a})$       symmetry: no observable distinguishes causal from anti-causal description

$\omega(\mathbf{a}\mathbf{a}^*) \geq 0$       positivity (or rather  $\omega(\mathbf{a}^2) \geq 0$  when  $\mathbf{a} = \mathbf{a}^*$ )

$\omega(\mathbf{1}) = 1$       normalisation of probabilities

**Norm and real C\*-algebra** structure follow! Just define the norm as

$$\|\mathbf{a}\|^2 = \sup_{\omega \in \mathcal{E}_{\mathbb{R}}} \omega(\mathbf{a}^* \mathbf{a})$$

This makes  $\mathfrak{A}_{\mathbb{R}}$  a real C\*-algebra (assuming closure)

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad \|\mathbf{a}\mathbf{b}\| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

$$\|\mathbf{a}\|^2 = \|\mathbf{a}^* \mathbf{a}\| = \|\mathbf{a}^*\|^2$$

$\mathbf{1} + \mathbf{a}\mathbf{a}^*$  invertible

# GNS theorems for real $C^*$ -algebras

Abstract real  $C^*$ -algebras are similar to complex ones.

See e.g. *K. R. Goodearl, Notes on real and complex  $C^*$ -algebras. Shiva Mathematics Series. Shiva Publishing Ltd., 1982.*

## Finite dimensional case

Algebraic problem. Artin-Wedderburn theorem implies that  $\mathfrak{A}_{\mathbb{R}}$  is (a direct sum of...) matrix algebras over the reals, the complex or the quaternions

$$M_n(\mathbb{R})$$

$$M_n(\mathbb{C})$$

$$M_n(\mathbb{H})$$

## Infinite dimensional case

Analysis more involved, but similar conclusion: the algebra of observables is a closed subalgebra of the algebra of operators over some real Hilbert space

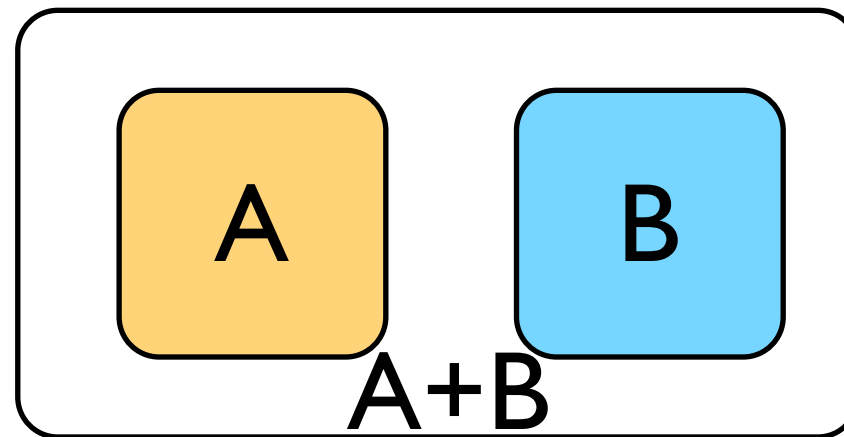
$$\mathfrak{A}_{\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_{\mathbb{R}})$$

# Why $\mathbb{C}$ , not $\mathbb{R}$ ? Has this to do with locality?

This is an old question! ..., *Stueckelberg circa 1960*, ...

- Standard argument, because it works! (Hamiltonian dynamics, etc.)
- More refined argument, because of locality and separability.

One wants to be able to construct all the physical observables of a system out of the observables of its (causally) independent subparts (... , *Araki 1980*, *Wooters 1990*,...)



This is possible only for complex algebras, since:

Complex case:  $(nm)^2 = n^2 m^2$

Real case 
$$\frac{(nm)(nm+1)}{2} \neq \frac{n(n+1)}{2} \frac{m(m+1)}{2}$$

Note that in these two frameworks (Algebraic and Quantum Logic) all observables are born equal (at least in a local sense)!

This is already a feature of Classical Hamiltonian Mechanics, where canonical transformations and Poisson brackets are the fundamental objects.

This is an important difference with de Broglie - Bohm formulations. Although dBB looks equivalent to standard QM formulations, it make reference to a particular set of commuting observables that have a special ontological status (be-ables).

For instance the positions of the particle(s) in the dBB theory of a non-relativistic particle (with or without spin!).

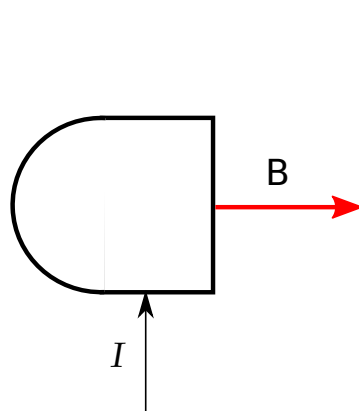


# Quantum informational formulations of quantum mechanics

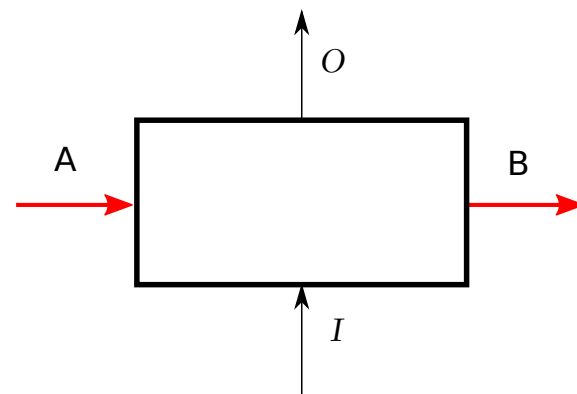
An old idea going back to J. Wheeler «It from Bit»

Recent developements with quantum information tools: preparations, effects, quantum channels, etc.

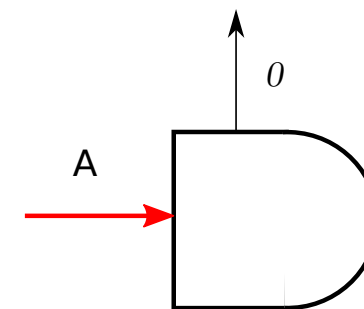
See in particular *L. Hardy '01 & '11*, *G. Chiribella, G.M. D'Ariano, P. Perinotti '11*, *L. Masanes, M. Muller '11*.



preparation of a **out** state  $B$  according to some input  $I$

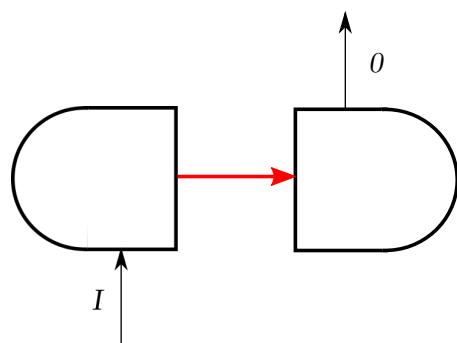


general device acting on a state  $A$  with some input  $I$  and output  $O$



measurement (effect) on a **in** state  $B$  with some output  $O$

Build out «circuits» of operations on multipartite systems, and look for «informational axioms» for these operations.



probabilities are associated to (state / effect)

# Here are the axioms of Chiribella et al.

**Principle 1 (Causality)** The probability of an outcome at a certain step does not depend on the choice of experiments performed at later steps.

**Principle 2 (Fine-Grained Composition)** The sequence of two fine-grained processes is a fine-grained process.  
or “maximal knowledge of the episodes implies maximal knowledge of the history”

**Principle 3 (Perfect Distinguishability)** If a state is not compatible with some preparation, then it is perfectly distinguishable from some other state.

**Principle 4 (Ideal Compression)** Information can be compressed in a lossless and maximally efficient fashion.

**Principle 5 (Local tomography)** The state of a composite system is determined by the statistics of local measurements on the components

**Principle 6 (Purity and Reversibility of Physical Processes)** Every random process can be simulated in an essentially unique way as a reversible interaction of the system with a pure environment.

Note that the ingredients of more physical approaches are here:

- States and Probabilities
- Causality (principle 1)
- Reversibility (principle 6)
- Locality (principle 5)

but some concepts are different and more information theoretical  
(projective measurements or unitary transformations are replaced by  
general POVM-like operations acting on states)

So is there a way to unify these different formulations?

## And what about Quantum Gravity?

It is well known that General Relativity is at clash with Quantum Mechanics. Quantization is problematic, despite the efforts of some of the best brains of Theoretical Physics of the last 50 years...

I have no claim to bring something fresh in this debate...

However... Causality seems to precede Time (continuous time evolution) in the formulations of Quantum Mechanics

This is fortunate since in SR and GR there is no absolute time, and since there is probably no time in quantum gravity (Wheeler-deWitt equation)

But in Quantum Gravity, Locality\* is expected to disappear too (Holography, String dualities, etc.).

Is it possible that Causality and Reversibility still make sense, in a weaker sense? If not, what replaces quantum mechanics?

\* in the Einstein 1905/15 sense, not in the EPR-Bell 1935/64 sense

# A list of questions

- What do we understand and what do we really not understand in quantum mechanics?
- What are the basic physical principles we should insist on?  
axioms/principles versus features/properties
- Can we agree on a preferred formulation?
- Or are different complementary formulations better?
- Are we still missing some important features of QM?
- Where should we expect quantum mechanics to fail?
- Why one theory and so many «interpretations»?
- Why are we so interested in the historical aspects of quantum mechanics?
- Where is the boundary between physics and philosophy in these discussions?



# Thank you