Analogue Gravity

A playground to develop and test mathematical ideas.

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March 16, 2022

Overview

A brief history of black holes

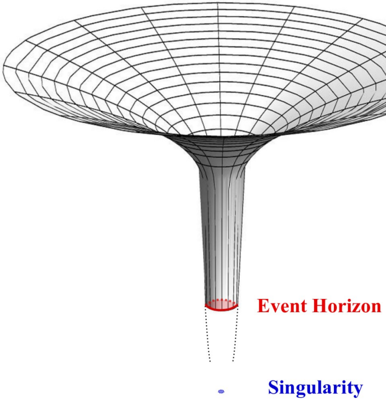
Black holes are regions of space-time from which nothing can escape.

Mathematically, they are solutions to Einstein's field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

with $G_{\mu\nu}$ the Einstein tensor constructed from the metric $g_{\mu\nu}$, Λ the cosmological constant, $T_{\mu\nu}$ the stress-energy tensor and κ is Einstein's gravitational constant.

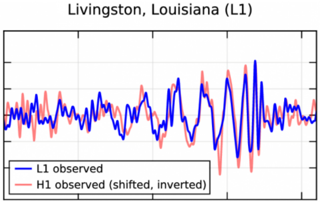
Black holes come with their own set of mathematical tools and astonishing effects.



A brief history of black holes

Black holes are the perfect systems to look for new physics.

- Gravitational wave astronomy.



The LISA mission will be able to probe Extreme-Mass Ratio Inspirals (EMRIs), which will probe the vicinity of the black hole geometry.
Theoretical challenge: Self-Force calculation.

- Effect of the drastic geometry on field propagation.

Hawking Radiation: Hawking predicted that quantum effects will lead black holes to emit particles and evaporate
Superradiance: Extraction of energy from a rotating black hole.
Challenges: Experimental detection.
Hawking Radiation for astrophysical black hole $\approx 10^{-8}$K. Difficult to test the theoretical predictions.

Analogue Gravity - Introduction

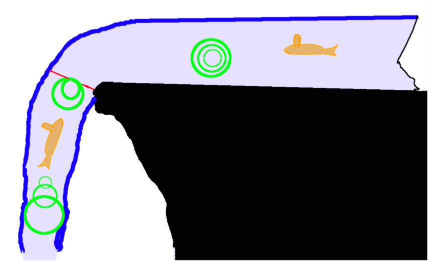


Figure: Illustration of the analogy. Credits: W.G. Unruh.

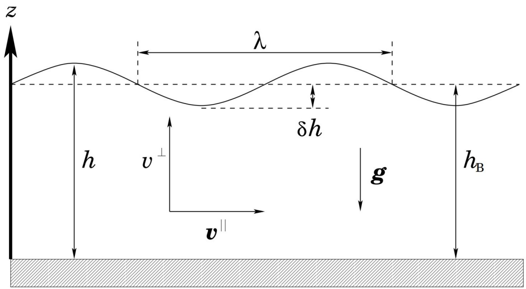
Fish swimming in a waterfall and talking to each other via sound waves, which propagate with speed c_{sound}.

- When $c_{waterfall} < c_{sound}$, waves can propagate in all directions.
- When $c_{waterfall} > c_{sound}$, waves are dragged down the waterfall

$c_{waterfall} = c_{sound}$ defines the analogue horizon.

Derivation of the analogy

We consider here an irrotational and inviscid fluid with velocity $\vec{v}$, pressure p and density ρ. We define the material derivative operator $\mathcal{D}_t = \partial_t + \vec{v}.\nabla$.



Fluid equations

$$\begin{cases} \nabla.\vec{v} = 0 \\ \mathcal{D}_t \vec{v} = -\frac{\nabla p}{\rho} + \vec{g} + \vec{F} \end{cases}$$

Irrotational

$$\vec{v} = \nabla \phi$$

Boundary conditions

$$\begin{cases} p\big|_{z=h} = 0 \\ v_z\big|_{z=0} = \frac{\partial \phi}{\partial_z}\Big|_{z=0} = 0 \\ v_z\big|_{z=h} = \frac{\partial \phi}{\partial_z}\Big|_{z=h} = \mathcal{D}_t h \end{cases}$$

Derivation of the analogy

Linearization: We then look at small displacements of the water surface $h = h_0 + \delta h$ which correspond to small changes in the velocity potential $\phi = \phi_0 + \delta\phi$.

Linearized governing equations

$$\Delta\delta\phi = 0, \tag{1}$$

$$\mathcal{D}_t\delta\phi = -g\delta h, \tag{2}$$

$$\frac{\partial\delta\phi}{\partial_z}\Big|_{z=h_0} = \mathcal{D}_t\delta h, \tag{3}$$

$$\frac{\partial\delta\phi}{\partial_z}\Big|_{z=0} = 0, \tag{4}$$

with $\mathcal{D}_t = \partial_t + \vec{v}_0.\nabla$.

We then integrate Eq.(1) through the bulk to relate the bottom of the tank to the free surface.

$$\Delta\delta\phi = \left(\Delta_{||} + \partial_z^2\right)\delta\phi = 0,$$

$$\begin{aligned}\delta\phi &= \sum_{n=0}^{\infty}\delta\phi_n(x,y)\frac{z^n}{n!}\\ &= \cosh(iz\nabla_{||})\delta\phi_0\end{aligned}$$

Derivation of the analogy

We finally obtain the wave equation for surface gravity waves propagating in a flowing fluid with velocity $\vec{v}_0$:

$$\mathcal{D}_t^2 \delta\phi - ig\nabla_{||} \tanh(ih_0\nabla_{||})\delta\phi = 0.$$

In the long-wavelength limit, it reduces to:

$$\mathcal{D}_t^2 \delta\phi - c^2 \Delta_{||}\delta\phi = 0, \quad \text{with} \quad c^2 = \sqrt{gh_0}.$$

Derivation of the analogy

Statement of the analogy

The wave equation describing surface gravity waves of small amplitudes in shallow water reduces to the Klein-Gordon equation for a massless scalar field on a curved spacetime with (inverse) metric $g^{\mu\nu}$:

$$\mathcal{D}_t^2\phi - c^2\Delta_{||}\phi = 0 \iff \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0,$$

where the effective metric is given by:

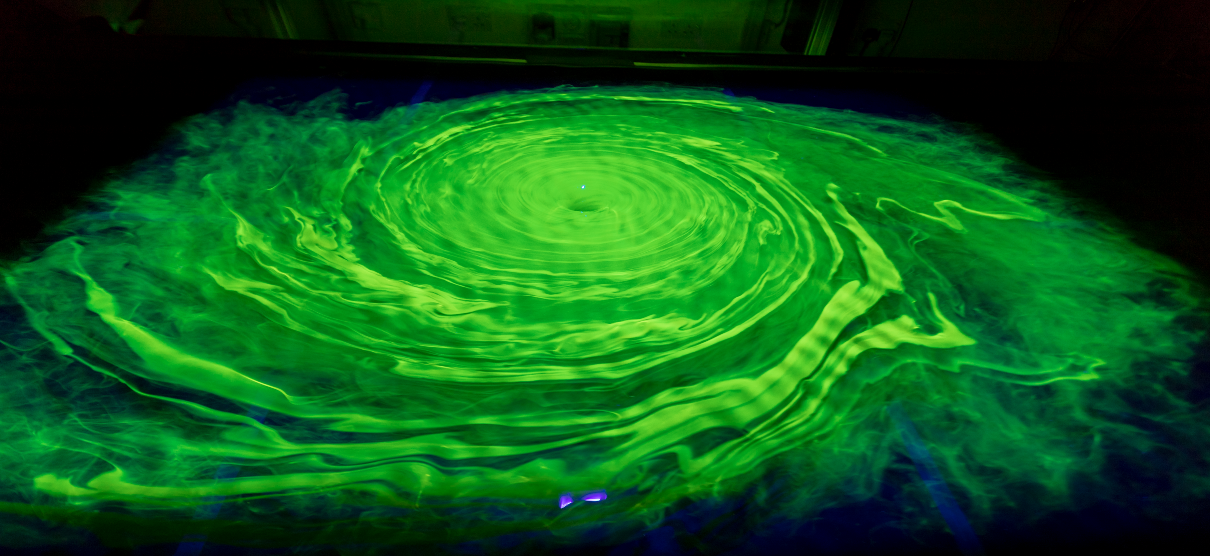
$$g_{\mu\nu} = \frac{1}{c^2}\begin{pmatrix} -(c^2 - \vec{v}_0^2) & -v_0^x & -v_0^y \\ -v_0^x & 1 & 0 \\ -v_0^y & 0 & 1 \end{pmatrix}.$$

Remarks on the analogy

- For superfluid/optics described by a wave function ψ obeying the Non-Linear Schrodinger Equation, one can obtain a similar derivation by performing the Madelung transform: $\psi = \sqrt{\rho}e^{iS}$ and linearizing around a background phase/density profile (S_0, ρ_0).
- We are not mimicking the dynamical aspects of gravity (Einstein equations), instead we are focusing on the kinematics.

Analogue gravity allows for the identification of universal and robust features of fundamental effects.

"It would seem that the physical intuition ought not only provide the mathematician with interesting and challenging conjectures, but also show him the way toward a proof and toward possible generalisation." Kac



A JOURNEY THROUGH AN ANALOGUE ROTATING BLACK HOLE.

From superradiance to quasi-normal modes.

Analogue rotating black hole

To mimick a rotating black hole, we study a rotating and draining fluid, i.e. a vortex. An irrotational vortex flow is described by the DBT model:

$\vec{v_0} = -\frac{D}{r}\vec{e_r} + \frac{C}{r}\vec{e_\theta}$. We define $r_e = \frac{\sqrt{C^2+D^2}}{c}$ and $r_h = \frac{D}{c}$.

<u>Line elements</u>:

$$ds^2_{DBT} = -\left(1 - \frac{r_e^2}{r^2}\right)dt^2 + \left(1 - \frac{r_h^2}{r^2}\right)^{-1} dr^2 - 2C\frac{r_e^2}{r^2}dtd\theta + \left(r^2 + \frac{C^2}{c^2} - \frac{C^2 r_e^2}{c^2 r^2}\right)d\theta^2$$

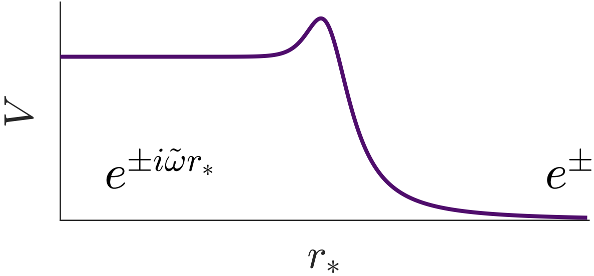
$$ds^2_{Kerr} = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r + a^2/r^2} - \frac{4Ma}{r}dtd\theta + \left[r^2 + a^2 + \frac{2Ma^2}{r}\right]d\theta^2$$

Scattering with a vortex flow

After separation of variables, $\phi(t,\theta,r) = e^{-i\omega t + im\theta} f(r)$, the scattering of waves with a DBT is governed by the wave equation:

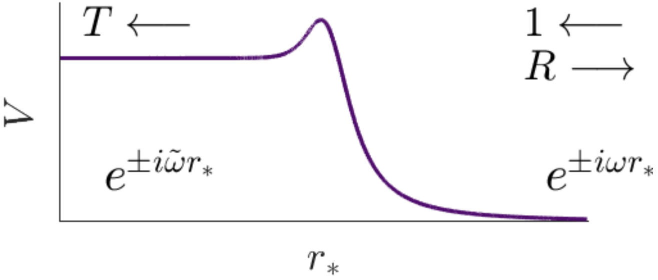
$$\frac{d^2R}{dr_*^2} - V(r)R = 0, \text{ with } V(r) = -\left[(\omega - \frac{Cm}{r^2})^2 - \left(c^2 - \frac{D^2}{r^2}\right)\left(\frac{m^2 - 1/4}{r^2} + \frac{5D^2}{4r^2}\right)\right].$$

Note that $r(r_*)$, where $dr_*/dr = \left(1 - \frac{D^2}{c^2 r^2}\right)^{-1}$ defines the tortoise coordinate.



Scattering with a vortex flow - superradiance

Physical boundary conditions: No outgoing mode at the horizon.



Conservation of the Wronskian $W = f\dot{f}^* - f^*\dot{f}$, implies the following energy conservation condition:

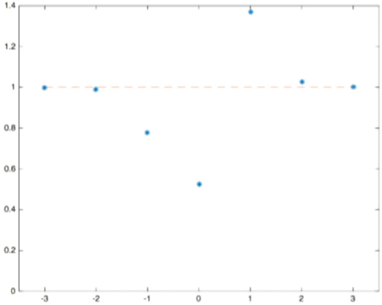
$$|R|^2 = 1 - \frac{\tilde{\omega}}{\omega}|T|^2$$

If $\tilde{\omega} = \omega - m\Omega_h < 0$, then $|R|^2 > 1$. This amplification process is known as *superradiance*.

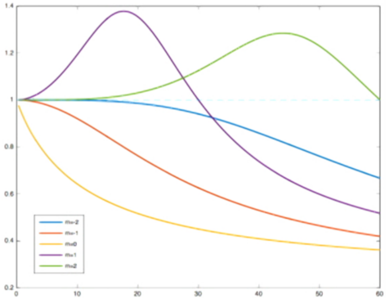
Scattering with a vortex flow - superradiance

Typical behaviour

- Counter rotating waves (m's negative) are absorbed
- Co-rotating waves (m's positive) are amplified



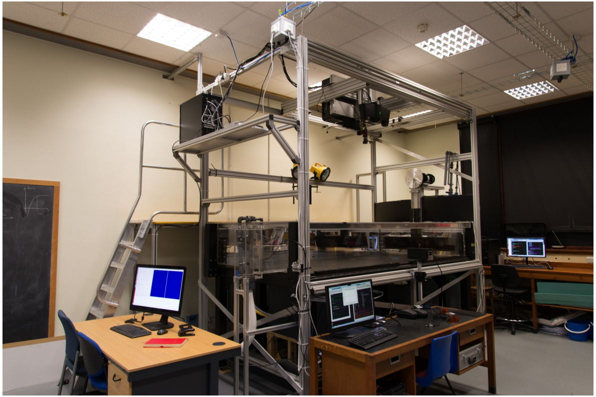
Fixed background and ω
Varying m.



Fixed background
Varying ω.

Observation of superradiance scattering

Superradiance was an effect known since the 70's but it had never been observed experimentally.
Analogue gravity offered a new platform to investigate this scattering effect.



Experimental and technical details can be found in:

nature physics | LETTERS

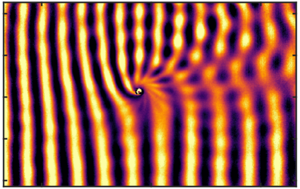
PUBLISHED ONLINE: 12 JUNE 2017 | DOI: 10.1038/NPHYS4151

Rotational superradiant scattering in a vortex flow

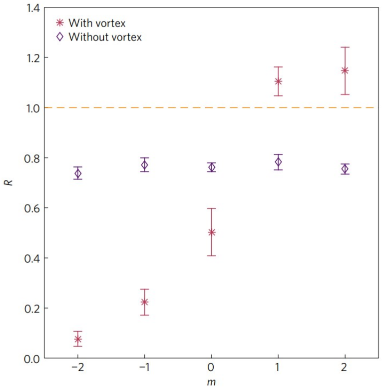
Theo Torres[1], Sam Patrick[1], Antonin Coutant[1], Maurício Richartz[2], Edmund W. Tedford[3] and Silke Weinfurtner[1,4,5]*

Observation of superradiance scattering

Experimental data from scattering of a plane wave incident on a vortex flow (rotating counterclockwise).

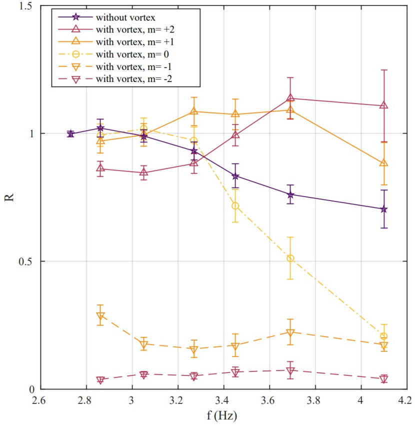


We extract the azimuthal modes by angular Fourier transform and then compute the ratio of the energy current going away/inside the vortex.



Observation of superradiance scattering

Repeat the experiment for various frequencies



We have a successful observation of superradiant scattering!
However we do not have a theoretical model because the experiment was conducted in a non-trivial regime:

- Dispersion
- Vorticity
- Water height non constant in the centre

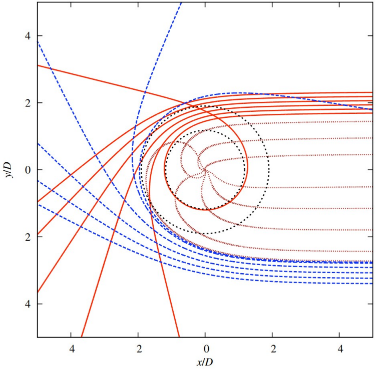
Question: How can we built a mathematical model capturing the key features of our observations?

Eikonal description

- Wave, $\phi = e^{iS}$, is considered as a collection of rays $(x^\mu(\tau), k^\mu(\tau))$.
- Rays are the characteristic curves of the Hamilton-Jacobi equation $(\partial_t S + \vec{v}_0.\nabla S)^2 - F(\nabla S) = 0$.
- The characteristics can be interpreted as the trajectories of fictitious particles. Their momentum is given as the gradient of the eikonal phase $\vec{k} = \nabla S$ and $\omega = -\partial_t S$.
- Their dynamics is governed by a Hamiltonian, $\mathcal{H} = -\frac{1}{2}(\omega - \vec{v}_0.\vec{k})^2 + \frac{1}{2}F(k)$.
- From the rays, we can reconstruct the eikonal wavefronts by looking for constant phase surfaces.

In the shallow water limit, $F(k) = c^2 k^2$, and Hamilton's equation reduces to the geodesic equation of massless particles, photons, in the effective geometry.

Eikonal description - Rays

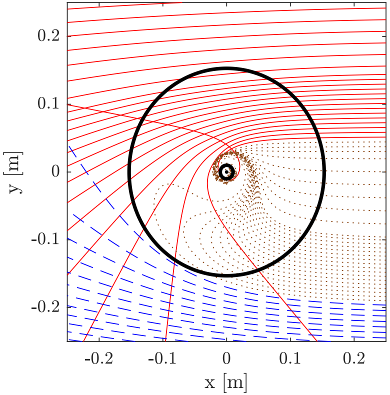


From Dolan *et al.* Phys. Rev. D 87, 124038 (2013).

Corotating escaping rays

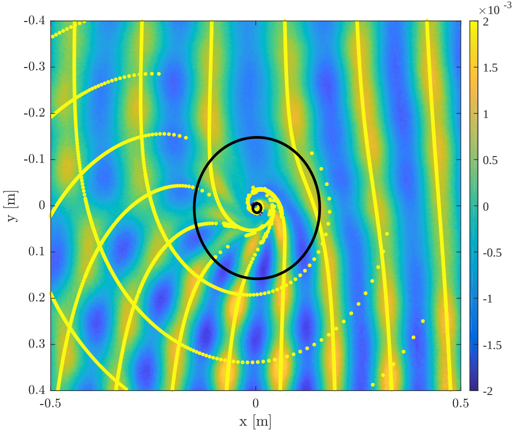
Falling rays

Counter-rotating escaping rays



From Torres *et al.* J. Fluid Mech. 857, 291–311 (2018).

Eikonal description - Waves



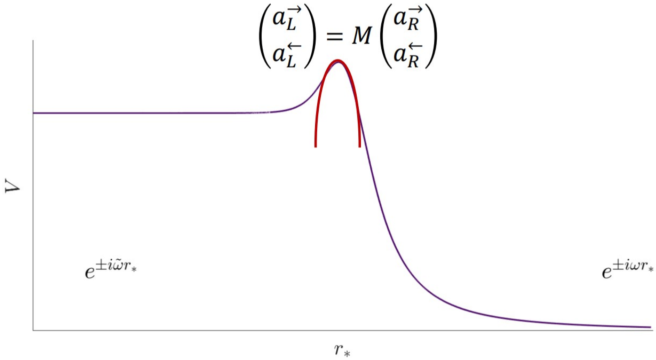
Good agreement between the eikonal wavefronts and the data.

Can we explain the observed reflection spectrum with this description?

WKB estimate

$\phi = Ae^{iS}$: S is found from the rays and A obeys a conservation equation $\mathrm{div}(\vec{J}A^2) = 0$.

Asymptotic behaviour: $a_L^{\leftarrow} e^{-i\tilde{\omega} r_*} + a_L^{\rightarrow} e^{+i\tilde{\omega} r_*} \longleftarrow \phi \longrightarrow a_R^{\leftarrow} e^{-i\omega r_*} + a_R^{\rightarrow} e^{+i\omega r_*}$



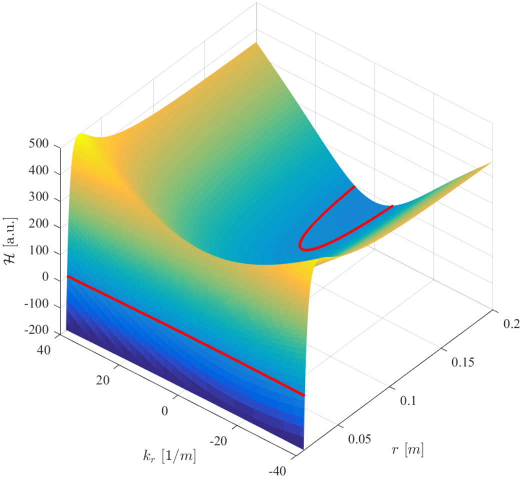
We solve the wave equation exactly in the vicinity of the maximum in terms of parabolic cylinder functions and match the asymptotics to connect the coefficients $(a_L^{\leftrightarrow})$ to $(a_R^{\leftrightarrow})$.

WKB estimate - Tunneling through a saddle point

We apply the same approach to the dispersive case.

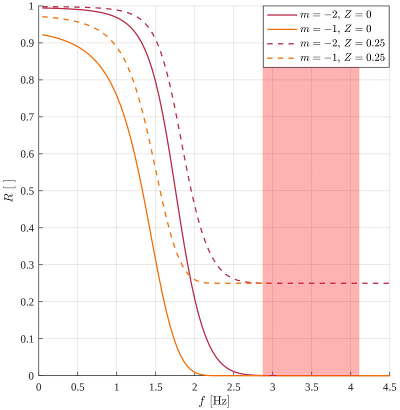
We expand the Hamiltonian in the vicinity of the saddle point and lift it to an operator to connect rays on each side of the saddle point.

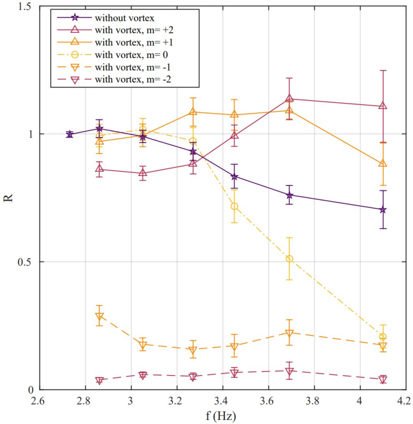
(Torres - Phil. Trans. R. Soc. A. 37820190236)



WKB estimate - Tunneling through a saddle point

Boundary condition: $\binom{a_L^{\rightarrow}}{a_L^{\leftarrow}} = \binom{Z}{T} = M\binom{R}{1}$



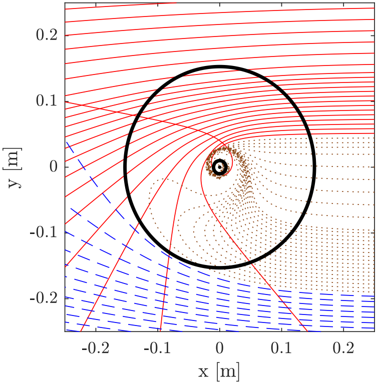


Summary on superradiance

- We have a successful observation of superradiance ($R > 1$).
- A geometrical description allows us to understand the wave pattern (interferences and AB effet)
- The WKB estimate indicates that the vortex is not purely absorbing.
- We are operating in a regime well oustide the analogy, however the superradiance effect persists.

Question: Are there other effects associated with the geometrical description?

Analogue light-rings



The light-rings are critical points of the Hamiltonian, $\dot{r} = \dot{k}_r = 0$.

They can be linked to the resonances of the systems, that is modes which are purely outgoing at the boundaries.

The resonances, quasinormal modes, have a complex frequency ω_{QNM} which can be approximated by the light-rings.

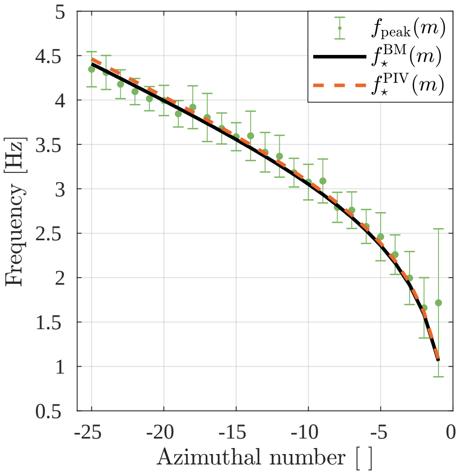
$$\omega_{QNM}(m) \approx \omega_{LR}(m) = \omega_\star(m) - i\Lambda(m)\left(n + \frac{1}{2}\right),$$

where $\omega_\star$ is the angular frequency of the orbit and Λ the Lyapunov exponent.

Analogue light-rings

Experimental realisation: We observe the emission of waves from a perturbed vortex flow.

- Oscillation spectrum agrees with the light rings prediction, where the flow parameters have been measured independently.
- The LR spectrum allows for the identification of the flow parameters: Analogue Black Hole Spectroscopy.



Carnet de voyage

- We started from the analogy which motivated us to observe the superradiance effect.
- Successful observation but in a regime where the analogy breaks down (dispersion, vorticity, etc...)
- We investigated the effect of dispersion in wave propagation over vortices $\rightarrow$ Robustness of the superradiance effect, identification of analogue light-rings and new predictions for QNMs frequencies in dispersive systems.
- We conducted an experiment to validate our prediction.
- Development of the Analogue Black Hole Spectroscopy method as a non-invasive flow measurement method.

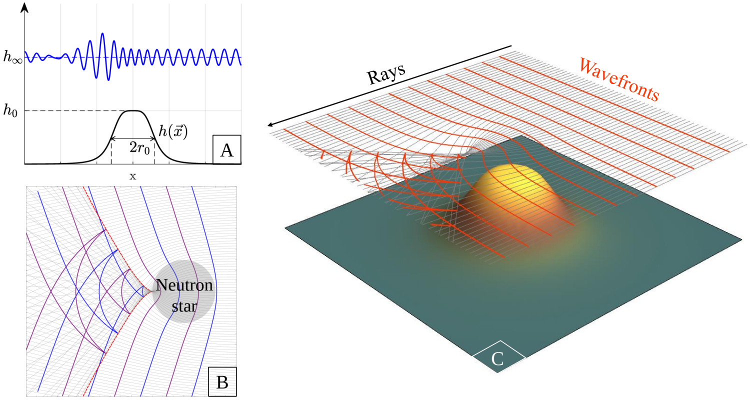
Caustics in an analogue neutron star

We mimic gravitational lensing using surface waves over an submerged island.

Our system reveals the presence of a caustic with a shape that depends on the island parameters.

This caustic will lead to amplification of the wave behind the obstacle.

(Torres *et al*, arXiv:2202.05926)



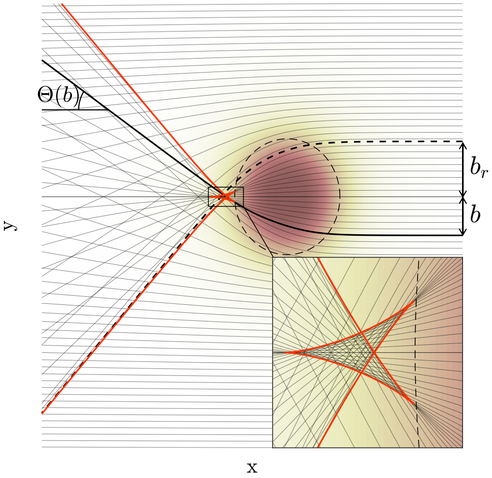
Caustics in an analogue neutron star

Using the analogy, we can use the Raychauduri equation to compute the shape of the caustic.

$$\frac{d\vartheta}{d\lambda} = -\vartheta^2 - R_{\mu\nu}k^\mu k^\nu,$$

ϑ is the expansion scalar, $R_{\mu\nu}$ the Ricci tensor and k^μ the tangent vector.

Intricate structure of the caustic.
How can we understand it?



Catastrophe theory in a nutshell

Definitions

Let C be an n-dimensional space, called the control space. Elements in C correspond to points which may be reached by rays, they are given by the control parameters $(c_1, c_2, ...c_n)$.
Let S be an m-dimensional space, called the state space. Elements in S characterise the various rays and are given by the state variables $(s_1, s_2, ..., s_m)$.

If different rays reach the same point, then the eikonal phase, $S : C \to \mathbb{R}$ is a multivalued function (value depends on the ray one follows to reach an element in C).

Catastrophe theory in a nutshell

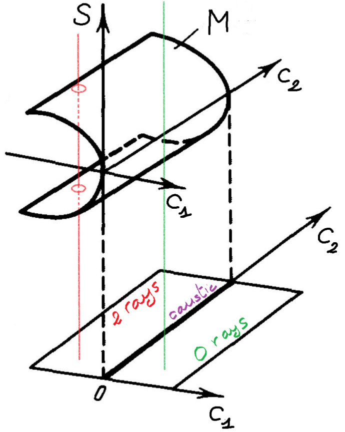
We define the *single-valued* function, called the generating function:

$$\begin{aligned} \Phi : S \times C &\to \mathbb{R} \\ (s, c) &\to S(c) \text{ from } s \text{ ray.} \end{aligned}$$

Physical rays minimize the "optical distance": $\frac{\partial \Phi}{\partial s_i} = 0$. These provides m relations which define a submanifold $M \subset S \times C$. The branches of S corresponds to the folding of M and caustics are the "folding lines" of M.

Adapted from Kravtsov's "Caustics, Catastrophes and Wave Fields".

Caustic are singularities of the gradient maps: $\partial^2\Phi/\partial s_i \partial s_j = 0$



Catastrophe theory in a nutshell

Thom's theorem

There exists a finite number of equivalence classes of structurally stable caustic for each codimension $K = \dim(C)$.

Standard polynomials Φ for the elementary catastrophes with codimension $K \leq 4$

Name	Symbol	K	$\Phi(s;C)$
fold	A_2	1	$s^3/3 + Cs$
cusp	A_3	2	$s^4/4 + C_2 s^2/2 + C_1 s$
swallowtail	A_4	3	$s^5/5 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
elliptic umbilic	D_4^-	3	$s_1^3 - 3 s_1 s_2^2 - C_3(s_1^2 + s_2^2) - C_2 s_2 - C_1 s_1$
hyperbolic umbilic	D_4^+	3	$s_1^3 + s_2^3 - C_3 s_1 s_2 - C_2 s_2 - C_1 s_1$
butterfly	A_5	4	$s^6/6 + C_4 s^4/4 + C_3 s^3/3 + C_2 s^2/2 + C_1 s_1$
parabolic umbilic	D_5	4	$s_1^4 + s_1 s_2^2 + C_4 s_2^2 + C_3 s_1^2 + C_2 s_2 + C_1 s_1$

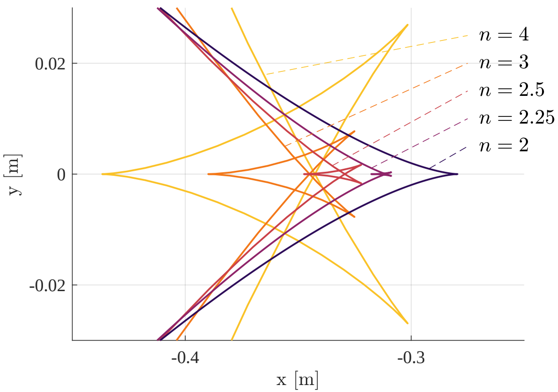
Caustics in an analogue neutron star

Butterfly caustic:

$$\Phi = \frac{s^6}{6} + C_4\frac{s^4}{4} + C_3\frac{s^3}{3} + C_2\frac{s^2}{2} + C_1 s$$

- The symmetry of the system sets $C_3 = 0$.
- (C_1, C_2) are related to the Cartesian coordinates
- C_4 depends on the island parameter n.

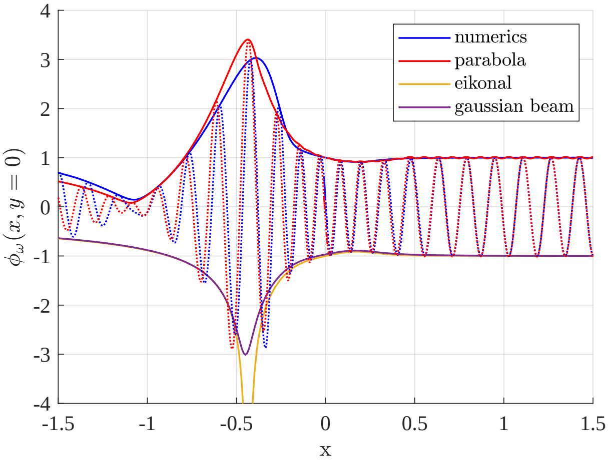
Varying n allows us to slice through the caustic surface.



Caustics in an analogue neutron star

Wave profile along $y = 0$ axis.

- Eikonal approximation: diverges at the caustic.
- Gaussian beam: Derives a local wave equation near along a ray $\iff$ complex rays. Allow to propagate rays through caustics.
- CT estimates the factor of focusing as $K_{foc} \approx (kF)^\sigma$ with $\sigma = 1/3$ for the butterfly caustic. We find $K_{foc} \approx 2.9$



Uniform approximation from CT

WKB gives an asymptotic solution to wave propagation.
On caustics, WKB fails and one usually turn to local solution (eg. Airy treatment of rainbows).

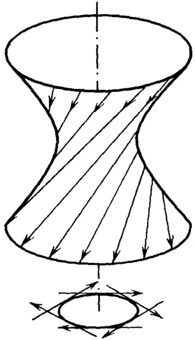
Problem: The local solution and WKB modes match only asymptotically.
Solution: Build a uniform solution which reduces to the local solution in the vicinity of the caustic and to WKB solutions far from the caustic.
Ingredients for the method:

- Geometrical quantities: phase S and amplitude A along rays.
- A local solution in the vicinity of the caustic: **given by catastrophe theory**

$$\phi(C) = \frac{1}{(2\pi)^{m/2}} \int \dots \int d^m s \; e^{i\Phi(s,C)}.$$

Example of uniform approximation

Cylindrical caustic



Exact solution: Bessel functions.
Fold caustic: $\Phi(s; C) = s^3/3 + Cs$.

$$\phi(C) = \frac{1}{\sqrt{2\pi}} \int ds\ e^{i\Phi(s;C)} = \sqrt{2\pi}\mathrm{Ai}(C).$$

A change of coordinate allows to extend the local solution far away from the caustic using geometrical quantities (recipe in eg: Kravtsov).

$$J_n(r) \approx \left(\frac{4\xi}{r^2 - n^2}\right)^{1/4} \mathrm{Ai}(-\xi)$$

with $\xi = \left((r^2 - n^2)^{1/2} - n \arccos(n/r)\right)^{2/3}$. (Langer)

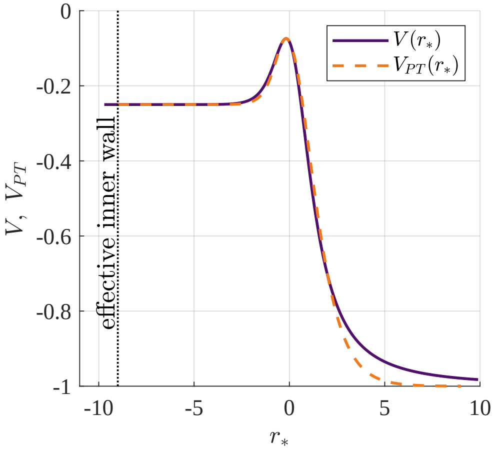
CT can do even better by including derivative of the Airy function (Berry, Chester)

Imperfect vortex

Motivated by the WKB analysis, we study a vortex with an extra structure located just outside the analogue horizon.
New boundary condition:

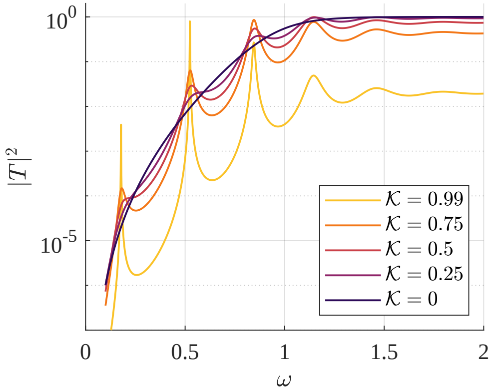
$$\phi_r(r_* \simeq r_{*0}) \sim A^{\mathrm{wall}} \left(e^{-i\tilde{\omega} r_*} + \mathcal{K} e^{-2i\tilde{\omega} r_{*0}} e^{i\tilde{\omega} r_*}\right).$$

The presence of the wall allows for the existence of resonances trapped between the wall and the light-ring.
This leads to Breit-Wigner type lines in absorption spectra.

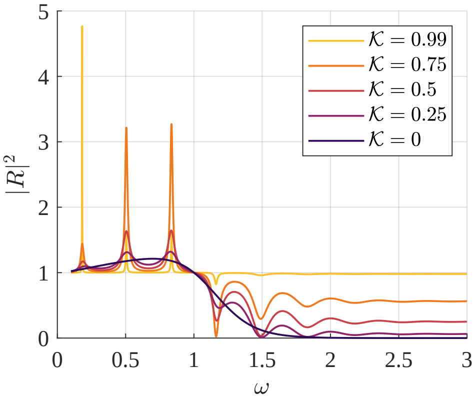


Imperfect vortex

Non-rotating vortex: Enhanced absorption.



Rotating vortex: Enhanced superradiance.

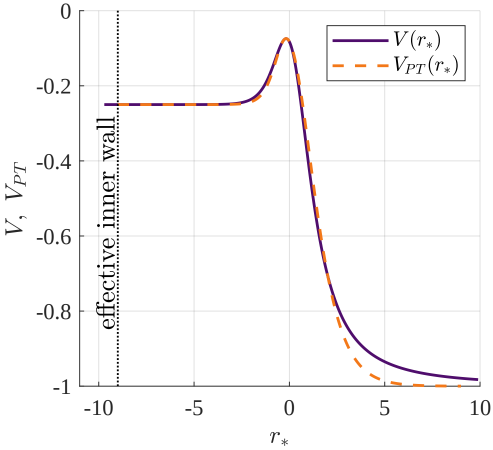


Imperfect vortex model

To describe the numerics we build two analytical descriptions.

- A WKB approach
- A toy model where we approximate the potential with 2 Poschl-Teller like potential. We can solve this problem exactly in terms of Legendre functions. This gives an analytic expression for the condition satisfied by the bound state. For low frequencies:

$$\omega_{mn} = \frac{\pi n}{|r_{*0}|} + \sigma + i\frac{\ln|\mathcal{K}|}{2|r_{*0}|}.$$



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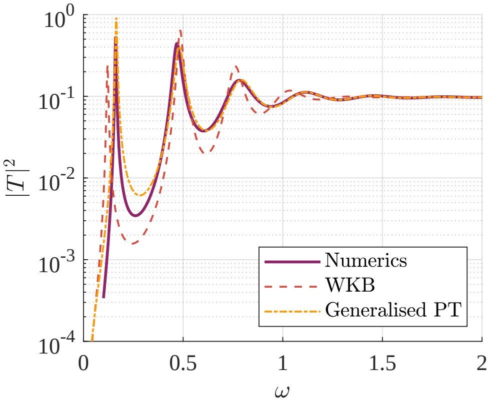
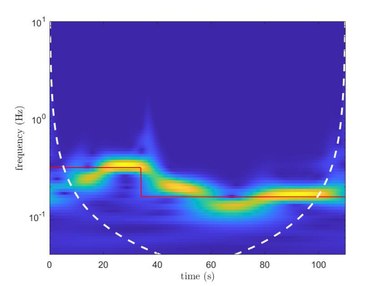


More on QNMs

QNMs of time-dependent systems

Time-dependent vortex flow:

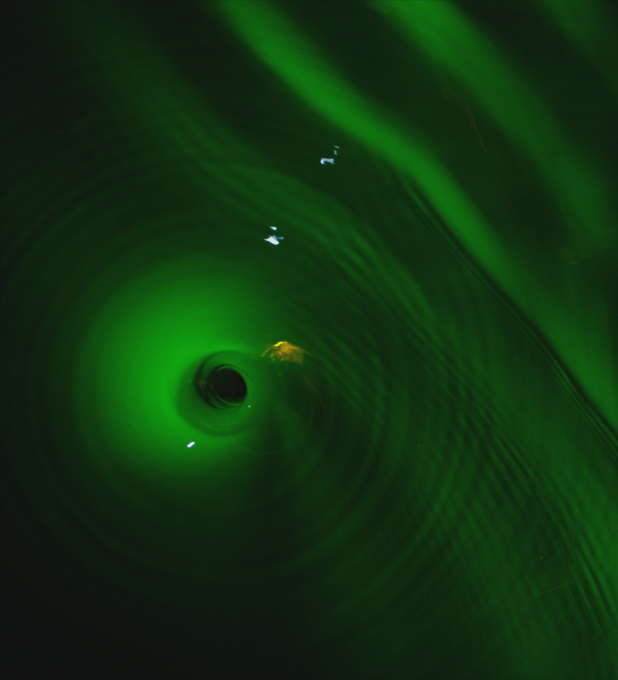


Black hole surrounded by a collapsing shell of matter.

QNMs emission in fibre optics

Light in fibre obeys the NLSE.
We can send a soliton in the fibre which creates a background geometry for small amplitude probe fields $\rightarrow$ Poschl-Teller potential.

- Stability of the QNM spectrum (pseudospectrum).
- QNMs excitation factors $B_n \int_{-\infty}^{+\infty} \frac{I(\omega, r_*)\hat{X}(r_*)}{A_{out}} dr_*$, with I a source and $\hat{X}$ the QNM radial function (divergent at ∞). Padé approximant?



"It is probably true quite generally that in the history of human thinking the most fruitful developments frequently take place at those points where two different lines of thought meet. [...] (I)f they are at least so much related to each other that a real interaction can take place, then one may hope that new and interesting developments will follow."

Heisenberg.