Stability analysis of high order discrete boundary condition

Pierre Le Barbenchon

PhD thesis supervised by
Benjamin Boutin and Nicolas Seguin

Université de Rennes 1, IRMAR

Coastal flow models and boundary conditions

27 octobre 2022

Contents

Advection equation

We want to approximate the solution $u : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$ of the advection equation ($a > 0$):

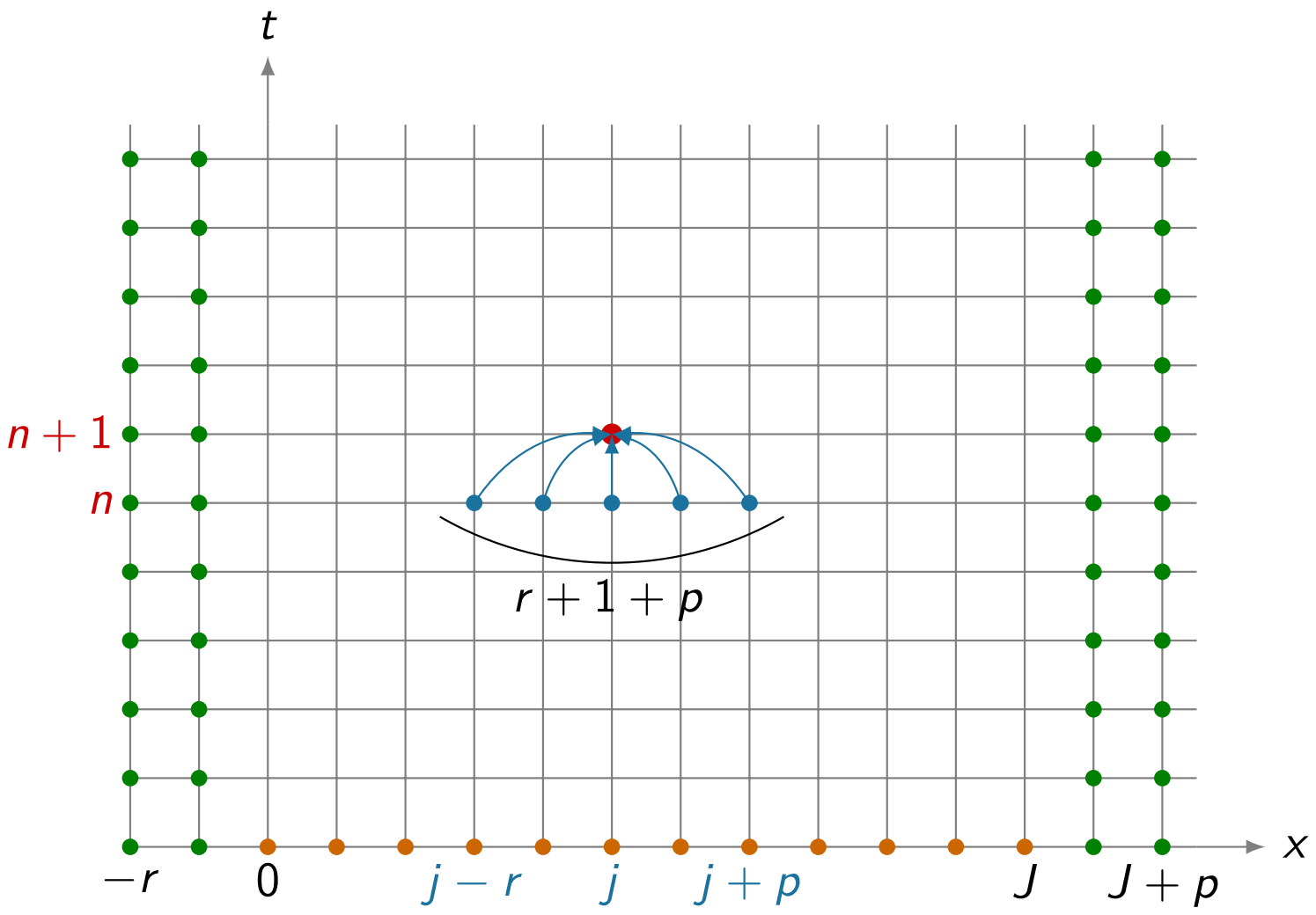
$$\begin{cases} \partial_t u + a\partial_x u = 0 & (t,x) \in \mathbb{R}^+ \times [0,1] \\ u(t,0) = g(t) & g : \mathbb{R}^+ \to \mathbb{R} \\ u(0,x) = f(x) & f : [0,1] \to \mathbb{R}. \end{cases} \tag{1}$$

The finite difference scheme we used are the following.

$$\begin{cases} U_j^{n+1} = \sum_{k=-r}^{p} a_k U_{j+k}^n & n \in \mathbb{N}, j \in [\![0; J]\!] \\ U_j^n + \sum_{i=0}^{m-1} b_{i,j} U_i^n = g_j^n & n \in \mathbb{N}, j \in [\![-r; -1]\!] \\ U_j^n + \sum_{i=0}^{m-1} c_{i,j} U_{J-i}^n = g_j^n & n \in \mathbb{N}, j \in [\![J+1; J+p]\!] \\ U_j^0 = f(x_j) & j \in [\![0; J]\!]. \end{cases} \tag{2}$$

with $J \in \mathbb{N}^*$, $\Delta x = \frac{1}{J}$ and $x_j = j\Delta x$ for $j \in [\![0; J]\!]$ and $\Delta t \in]0,1[$ and $t^n = n\Delta t$ for $n \in \mathbb{N}$ satisfying the Courant number $\lambda := \dfrac{a\Delta t}{\Delta x}$ fixed.

Discretization



Convergence

Consistency the exact solution of the PDE is almost a solution of the scheme

Stability the solution is continuous with respect to the initial data, the boundary data and a source term F

Theorem (Lax)

A linear scheme is convergent if and only if it is consistent and stable.

Consistency

Study $u(t^{n+1}, x_j) - \sum_{k=-r}^{p} a_k u(t^n, x_{j+k})$ for the interior and $u(t^n, x_j) + \sum_{i=0}^{m-1} b_{i,j} u(t^n, x_i) - g_j^n$ for the boundary

Stability

Find an inequality of the form $\|U\| \lesssim \|f\| + \|g\| + \|F\|$

Strategy to prove stability ?

GKS Theory

GKS theory (Gustafsson, Kreiss and Sundström) is introduced in the article [GKS72] and gives the following proposition.

Proposition

To have the stability of the problem with two boundaries, it is sufficient to prove :

(a) the Cauchy-stability of the problem without boundary (on $\mathbb{Z}$),

(b) the stability of the problem with only a left boundary (on $\mathbb{N}$),

(c) the stability of the problem with only a right boundary (on $-\mathbb{N}$).

Points (b) and (c) can be handled in the same way.

Cauchy-Stability

Definition (Symbol)

The symbol of the scheme is defined, for $\xi \in \mathbb{R}$, by

$$\gamma(\xi) = \sum_{j=-r}^{p} a_j e^{ij\xi}$$

We have $\widehat{U^{n+1}}(\xi) = \gamma(\xi)\widehat{U^n}(\xi)$ for all $\xi \in \mathbb{R}$.

Definition (Cauchy-stability)

The scheme is *Cauchy-stable* if

$$\forall \xi \in \mathbb{R}, |\gamma(\xi)| \leqslant 1$$

$$\|U^n\|_{\Delta x} \leqslant \sup_{\xi} |\gamma(\xi)|^n \|U^0\|_{\Delta x}$$

Cauchy-Stability

Beam-Warming

For example, the Beam-Warming scheme is given by

$$U_j^{n+1} = \frac{\lambda(\lambda-1)}{2} U_{j-2}^n + \lambda(2-\lambda) U_{j-1}^n + \frac{(\lambda-1)(\lambda-2)}{2} U_j^n$$

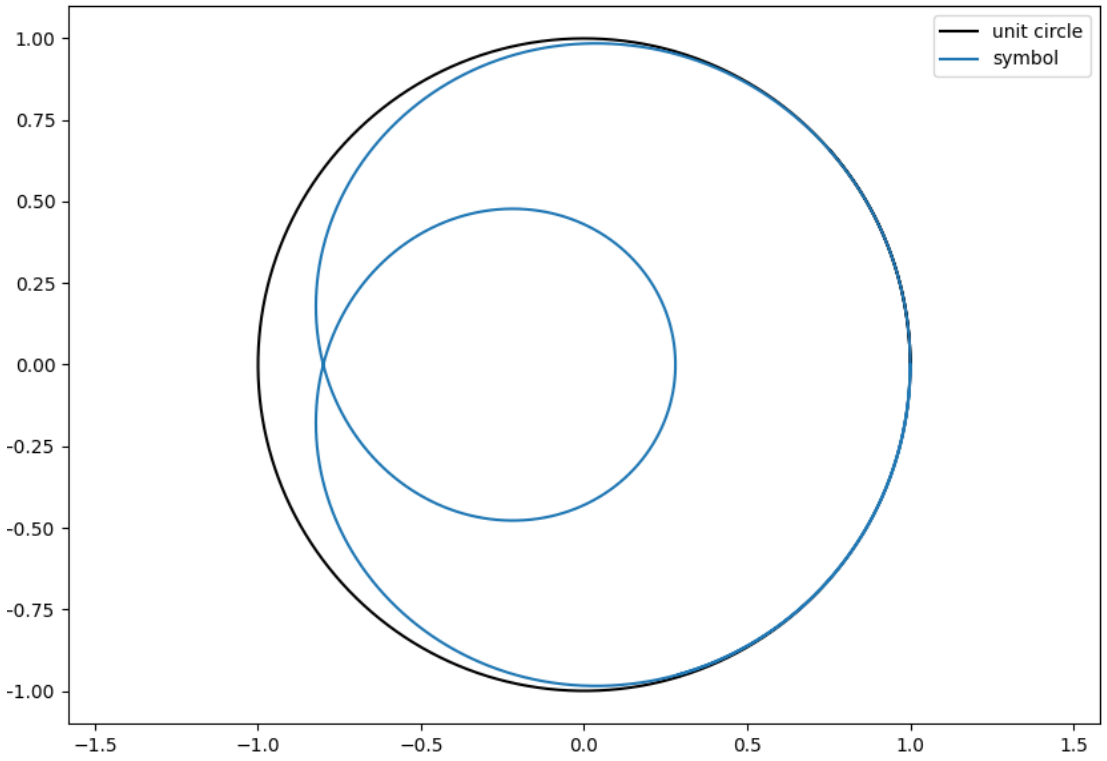


Figure: Symbol of Beam-Warming for $\lambda = 1.8$.

Cauchy-stable for the CFL condition given by: $0 < \lambda \leqslant 2$.

Cauchy-Stability

Third Order

For example, let us take the Third Order scheme (O3) given by

$$U_j^{n+1} = \left(\frac{\lambda^3}{6} - \frac{\lambda}{6}\right) U_{j-2}^n + \left(\lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{2}\right) U_{j-1}^n + \left(1 - \frac{\lambda}{2} - \lambda^2 + \frac{\lambda^3}{2}\right) U_j^n + \left(\frac{\lambda^2}{2} - \frac{\lambda^3}{6} - \frac{\lambda}{3}\right) U_{j+1}^n$$

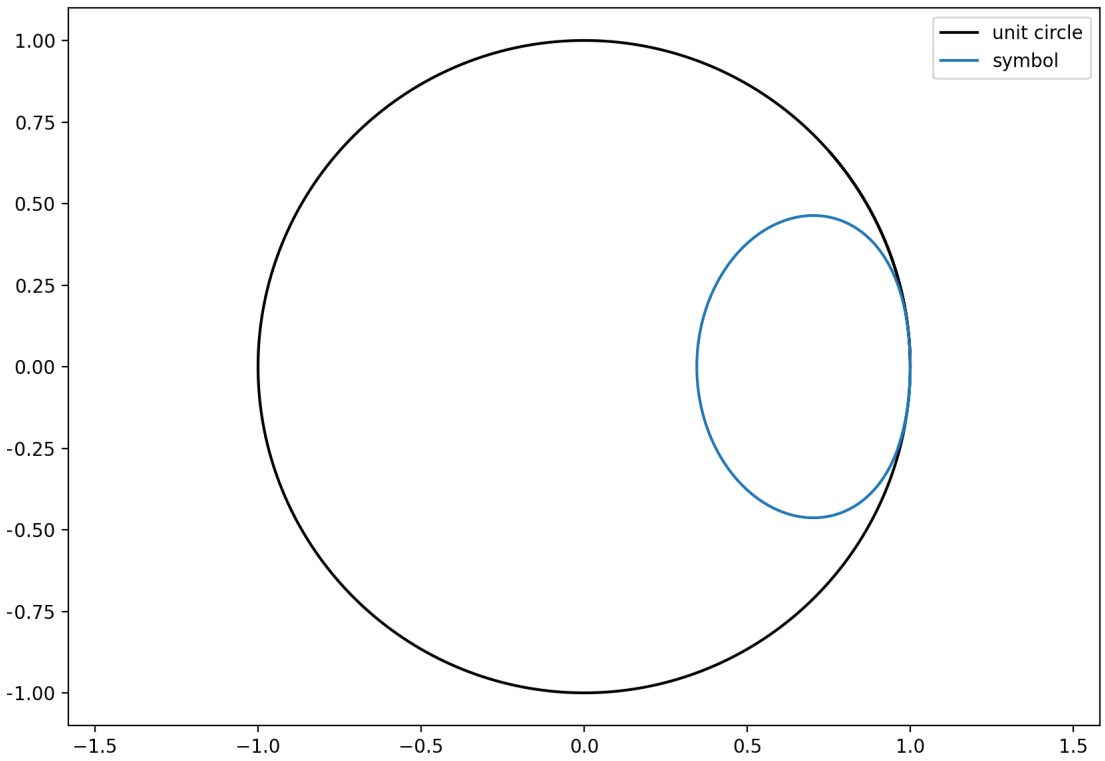


Figure: Symbol of Third Order for $\lambda = 0.35$.

Cauchy-stable for the CFL condition given by: $0 < \lambda \leqslant 1$.

Stablity of the scheme with only one boundary

We study the following problem:

$$\begin{cases} U_j^{n+1} = \sum_{k=-r}^{p} a_k U_{j+k}^n, & n \in \mathbb{N}, j \in \mathbb{N} \\ U_j^n + \sum_{i=0}^{m-1} b_{i,j} U_i^n = g_j^n, & n \in \mathbb{N}, j \in [\![-r; -1]\!] \\ (U_j^n)_j \in \ell^2(\mathbb{N}) \end{cases} \tag{3}$$

Stablity of the scheme with only one boundary

We study the following problem:

$$\begin{cases} U_j^{n+1} = \sum_{k=-r}^{p} a_k U_{j+k}^n, & n \in \mathbb{N}, j \in \mathbb{N} \\ U_j^n + \sum_{i=0}^{m-1} b_{i,j} U_i^n = g_j^n, & n \in \mathbb{N}, j \in [\![-r;-1]\!] \\ (U_j^n)_j \in \ell^2(\mathbb{N}) \end{cases} \tag{3}$$

GKS-Stability: for $f = 0$ and $F = 0$, there exist K, α_0 such that for all $\alpha > \alpha_0$, we have

$$\sum_{j=-r}^{-1} \|e^{-\alpha n \Delta t} U_j\|_{\Delta t}^2 + \left(\frac{\alpha - \alpha_0}{\alpha \Delta t + 1}\right) \|e^{-\alpha n \Delta t} U\|_{\Delta x, \Delta t}^2 \leqslant K^2 \sum_{j=-r}^{-1} \|e^{-\alpha n \Delta t} g_j\|_{\Delta t}^2$$

Stablity of the scheme with only one boundary

We study the following problem:

$$\begin{cases} U_j^{n+1} = \sum_{k=-r}^{p} a_k U_{j+k}^n, & n \in \mathbb{N}, j \in \mathbb{N} \\ U_j^n + \sum_{i=0}^{m-1} b_{i,j} U_i^n = g_j^n, & n \in \mathbb{N}, j \in [\![-r;-1]\!] \\ (U_j^n)_j \in \ell^2(\mathbb{N}) \end{cases} \tag{3}$$

GKS-Stability: for $f = 0$ and $F = 0$, there exist K, α_0 such that for all $\alpha > \alpha_0$, we have

$$\sum_{j=-r}^{-1} \|e^{-\alpha n \Delta t} U_j\|_{\Delta t}^2 + \left(\frac{\alpha - \alpha_0}{\alpha \Delta t + 1}\right) \|e^{-\alpha n \Delta t} U\|_{\Delta x, \Delta t}^2 \leqslant K^2 \sum_{j=-r}^{-1} \|e^{-\alpha n \Delta t} g_j\|_{\Delta t}^2$$

Theorem (Kreiss)

The following assertions are equivalent:

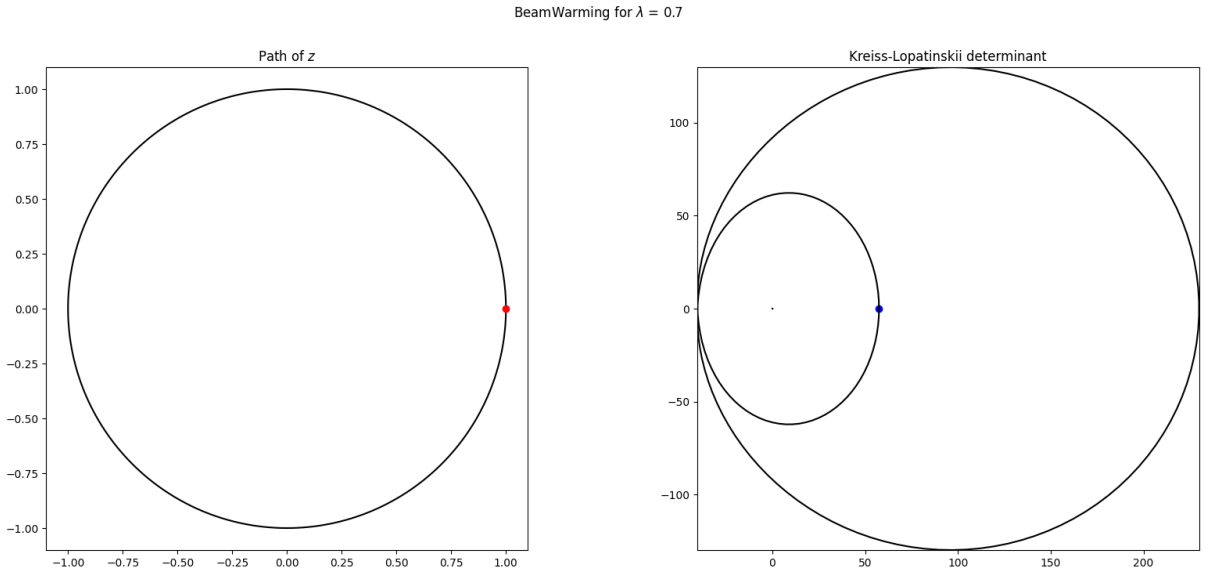
- the scheme with only one boundary is stable
- the Kreiss-Lopatinskii determinant $\Delta(z)$ doesn't vanish on $\{|z| \geqslant 1\}$.

Contents

Main theorem 1

Case $p = 0$ (where p is the number of right points in the scheme)

We draw $\Delta(\mathbb{S})$.



Theorem (B.Boutin, PLB, N.Seguin[BLBS22])

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

$$\begin{cases} U_j^{n+1} = \dfrac{\lambda(\lambda-1)}{2} U_{j-2}^n + \lambda(2-\lambda) U_{j-1}^n + \dfrac{(\lambda-1)(\lambda-2)}{2} U_j^n, \\ U_{-1}^n = \dfrac{1}{2}(U_2^n - 2U_1^n + U_0^n) + g_{-1}^n, \\ U_{-2}^n = 2(U_2^n - 2U_1^n + U_0^n) + g_{-2}^n. \end{cases} \tag{4}$$

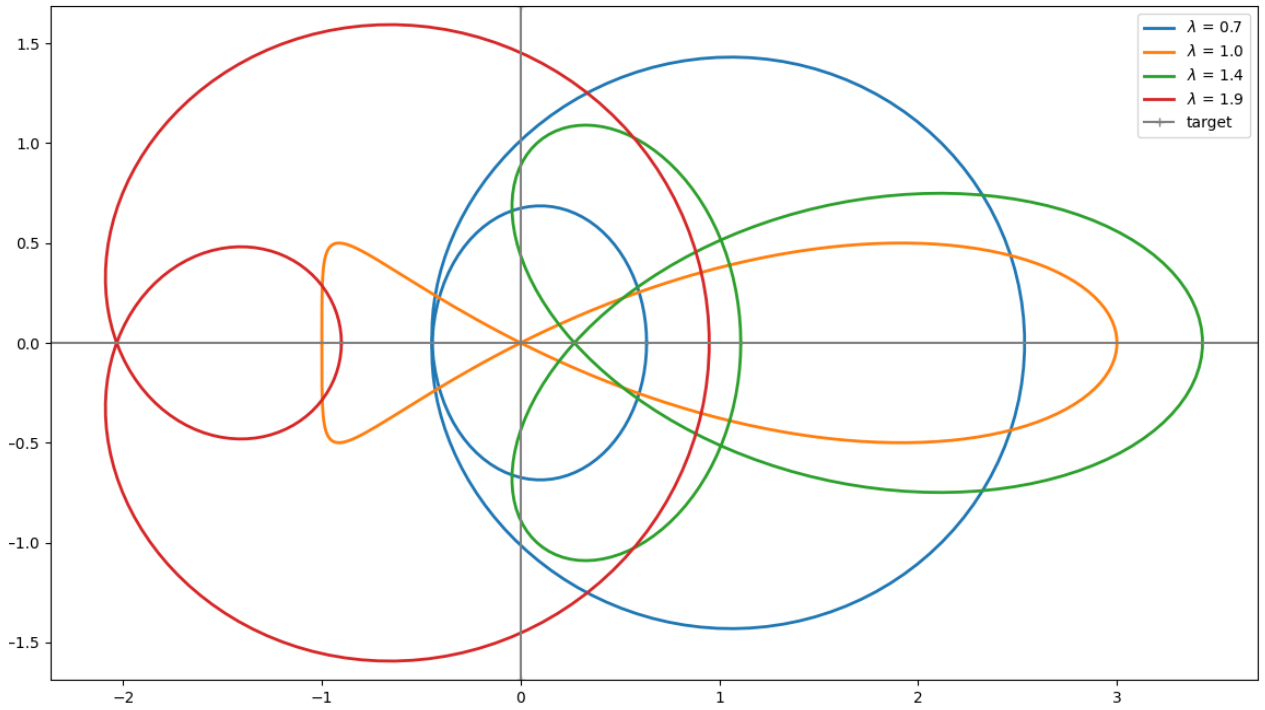
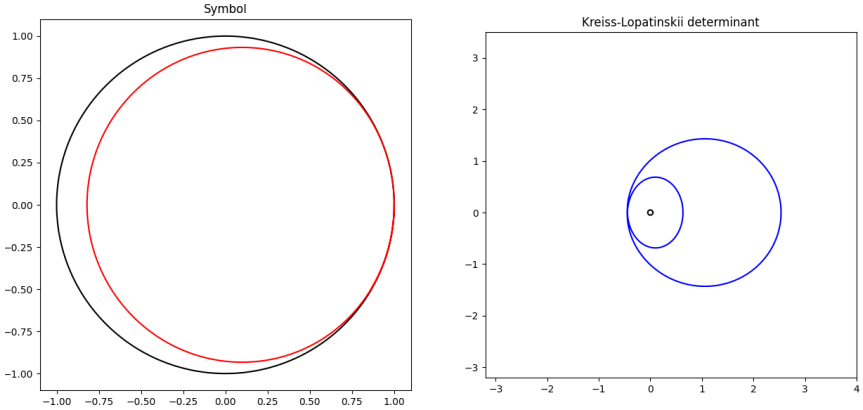
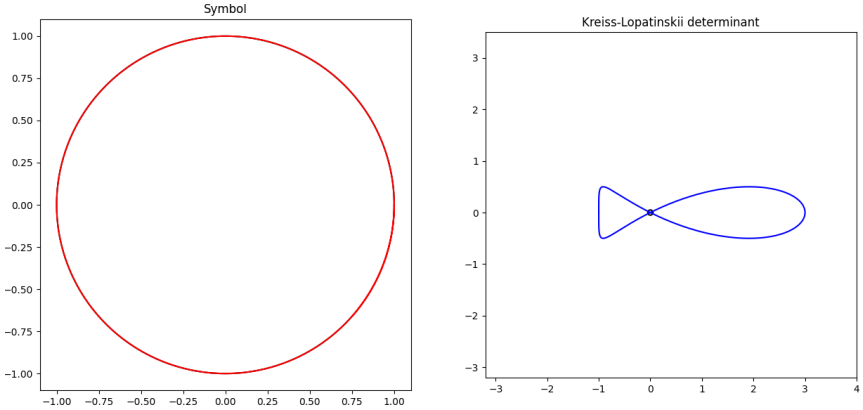


Figure: Kreiss-Lopatinskii determinant for the Beam-Warming scheme (4)

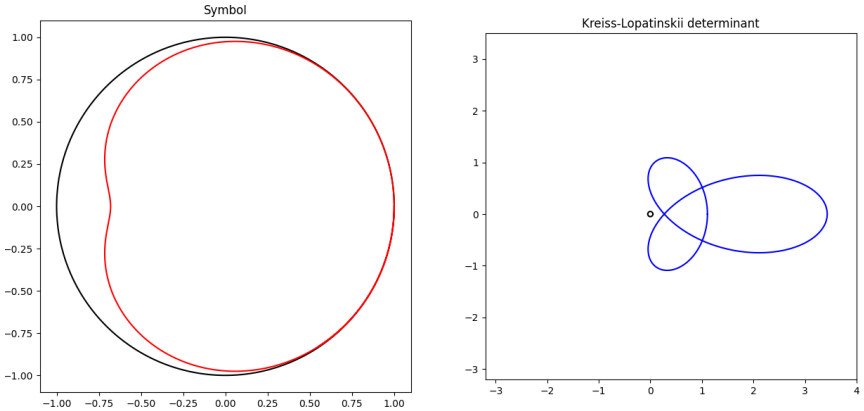
$\lambda = 0.7$



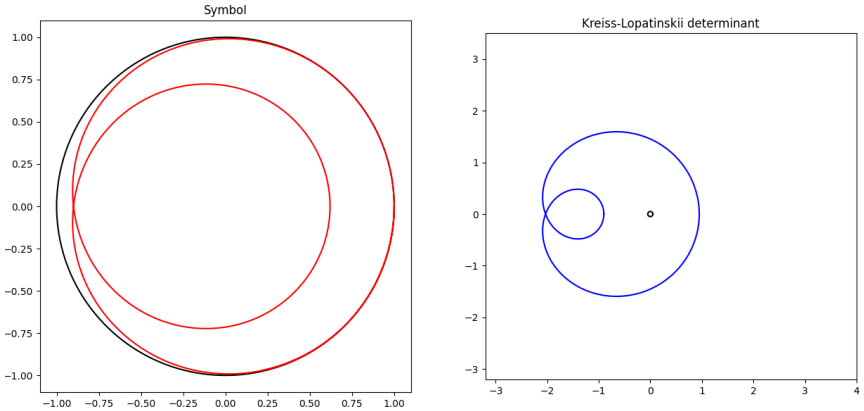
$\lambda = 1$



$\lambda = 1.4$



$\lambda = 1.9$



Main theorem 2

General case (for any p)

Theorem (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

Main theorem 2

General case (for any p)

Theorem (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

Third Order scheme with S1ILW3

$$\begin{cases} U_j^{n+1} = \left(\frac{\lambda^3}{6} - \frac{\lambda}{6}\right) U_{j-2}^n + \left(\lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{2}\right) U_{j-1}^n \\ \qquad + \left(1 - \frac{\lambda}{2} - \lambda^2 + \frac{\lambda^3}{2}\right) U_j^n + \left(\frac{\lambda^2}{2} - \frac{\lambda^3}{6} - \frac{\lambda}{3}\right) U_{j+1}^n, \\ U_{-1}^n = -(U_1 - U_0) + \frac{1}{2}(U_2^n - 2U_1^n + U_0^n) + g_{-1}^n, \\ U_{-2}^n = -2(U_1 - U_0) + 2(U_2^n - 2U_1^n + U_0^n) + g_{-2}^n \end{cases} \tag{5}$$

Main theorem 2

General case (for any p)

Theorem (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

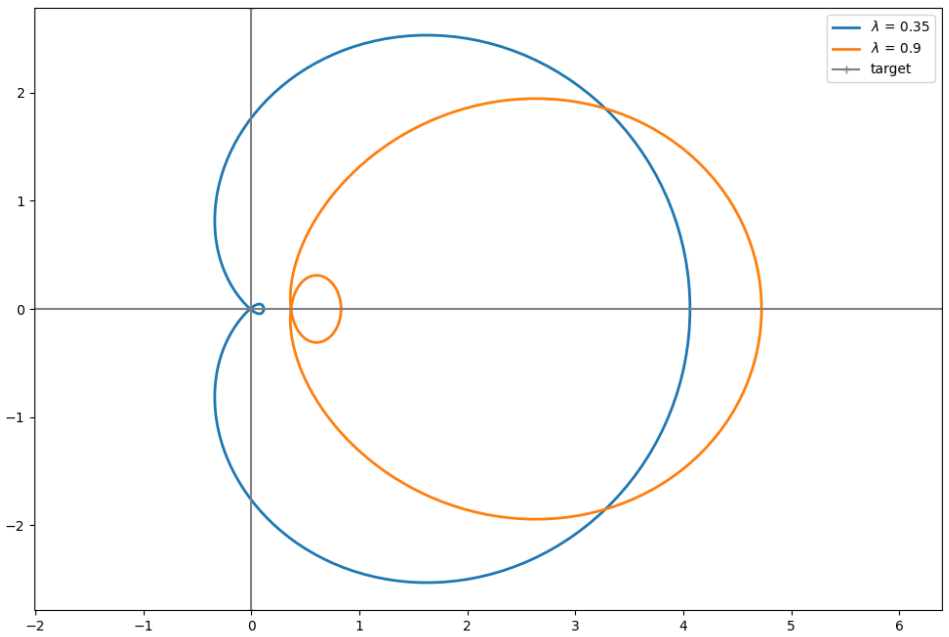


Figure: Kreiss-Lopatinskii determinant for the ThirdOrder scheme with SILW3.

Contents

$\mathcal{Z}$-transform and characteristic equation

We recall the interior equation of the scheme

$$U_j^{n+1} = \sum_{k=-r}^{p} a_k U_{j+k}^n \quad \forall j \in \mathbb{N}, \forall n \in \mathbb{N}.$$

We use the $\mathcal{Z}$-transform and obtain the following recursive sequence

$$z\widetilde{U}_j(z) = \sum_{k=-r}^{p} a_k \widetilde{U_{j+k}}(z) \quad \forall j \in \mathbb{N}, \forall |z| > 1,$$

whose characteristic equation is

$$z\kappa^r = \sum_{j=-r}^{p} a_j \kappa^{r+j}$$

Hersh lemma

Characteristic equation

$$z\kappa^r = \sum_{j=-r}^{p} a_j \kappa^{r+j} \tag{5}$$

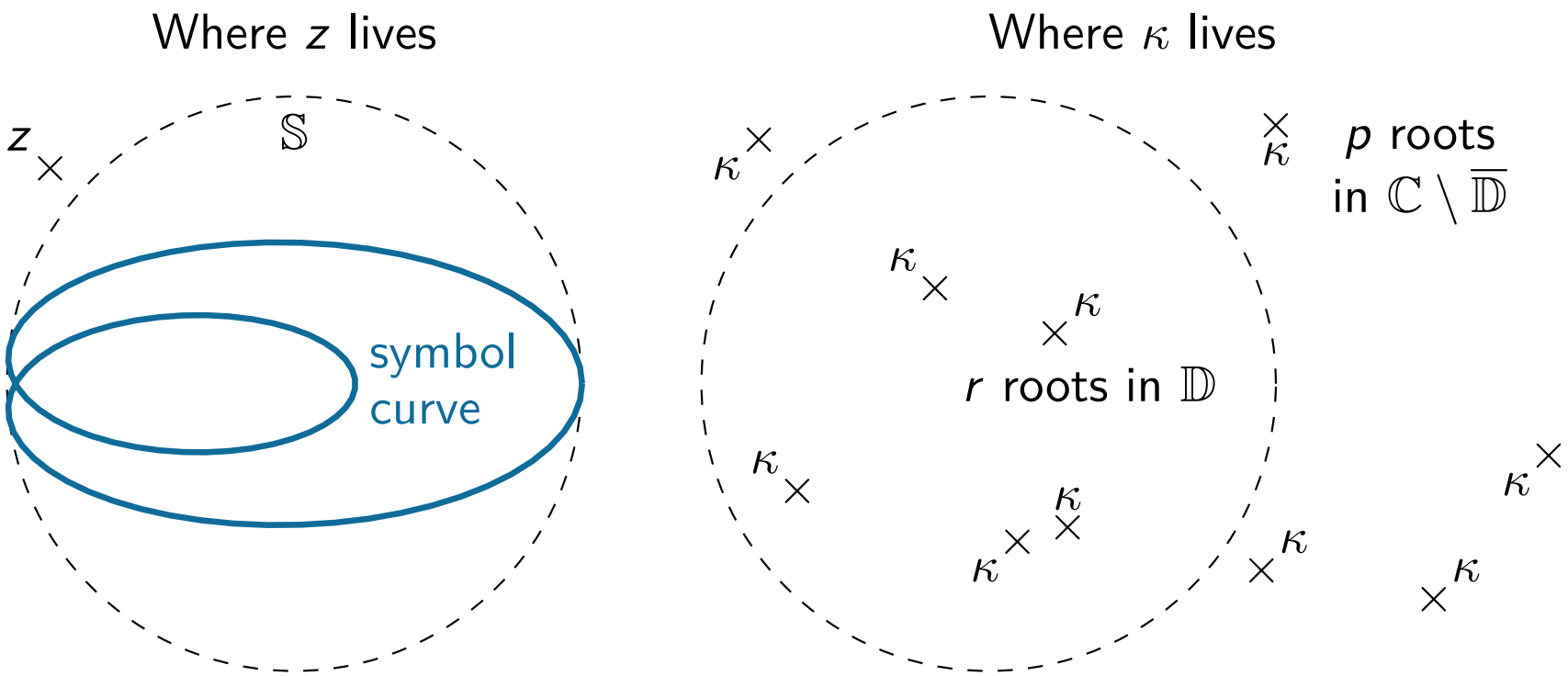
Lemma (Hersh)

If the scheme is Cauchy-stable and if $|z| > 1$, then the characteristic equation (5):

- has no root on the unit circle $\mathbb{S}$,
- has r roots (with multiplicity) in $\mathbb{D}$,
- has p roots (with multiplicity) in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

We select only the r roots (with multiplicity) in the unit disk to have the solution $(\widetilde{U}_j(z))_{j\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$, *i.e.* $\sum_{j=0}^{+\infty} \Delta x |\widetilde{U}_j(z)|^2 < \infty$.

Hersh lemma illustration



Space of solutions in ℓ^2 in space

For the sake of simplicity, we suppose that the roots κ of the characteristic equation from the unit disk are simple.
For $|z| > 1$, we denote $\mathcal{E}_s(z)$ the space of solutions in ℓ^2 in space. By Hersh lemma, its dimension is r because there are r roots κ inside the unit disk.

$$\mathcal{E}_s(z) = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ \kappa_1 \\ \kappa_1^2 \\ \kappa_1^3 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ \kappa_2 \\ \kappa_2^2 \\ \kappa_2^3 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ \kappa_r \\ \kappa_r^2 \\ \kappa_r^3 \\ \vdots \end{pmatrix} \right\}$$

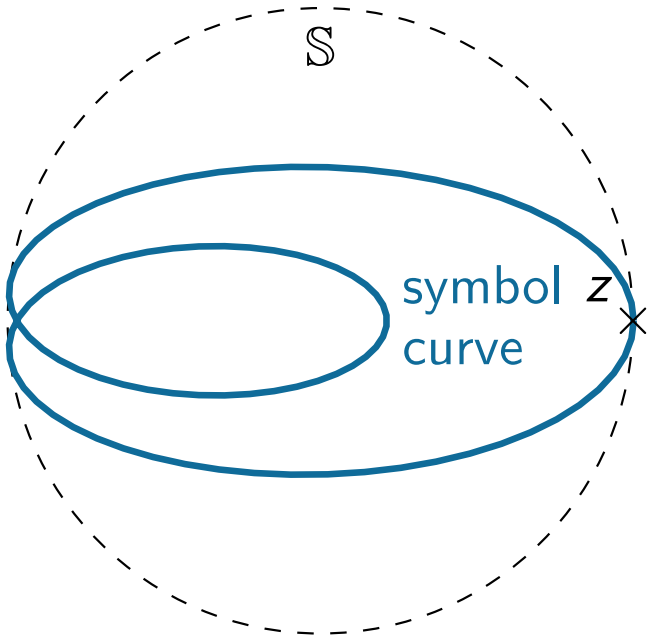
We denote $K_{i,j}(z) \in \mathcal{M}_{j-i+1,r}(\mathbb{C})$ the extraction of the components between row i and j included.

$$K_{i,j}(z) = \begin{pmatrix} \kappa_1^i(z) & \kappa_2^i(z) & \dots & \kappa_r^i(z) \\ \kappa_1^{i+1}(z) & \kappa_2^{i+1}(z) & \dots & \kappa_r^{i+1}(z) \\ \vdots & & & \vdots \\ \kappa_1^j(z) & \kappa_2^j(z) & \dots & \kappa_r^j(z) \end{pmatrix}$$

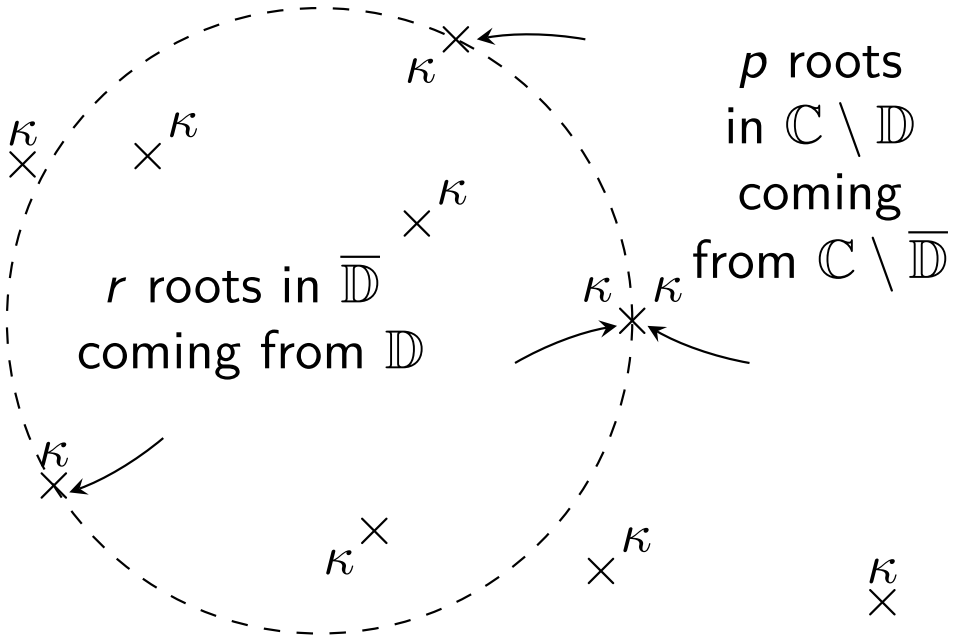
We extend this space to the domain $|z| = 1$ ([Cou13]).

Extension to $\mathbb{C} \setminus \mathbb{D}$

Where z lives



Where κ lives



Boundary consideration

The scheme (with $g_j^n = 0$) can be seen as the following semi-infinite Quasi-Toeplitz matrix:

$$z \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \widetilde{U_2}(z) \\ \widetilde{U_3}(z) \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m} & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & \cdots & 0 \\ \beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,m} & 0 & \cdots & 0 \\ a_{-r} & \cdots & a_0 & \cdots & a_p & & 0 \\ & \ddots & & \ddots & & \ddots & \end{pmatrix} \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \widetilde{U_2}(z) \\ \widetilde{U_3}(z) \\ \vdots \end{pmatrix}$$

The boundary is expressed in the following equality:

$$z \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \vdots \\ \widetilde{U_{r-1}}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m} \\ \vdots & & & \vdots \\ \beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,m} \end{pmatrix}}_{B} \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \vdots \\ \widetilde{U_{m-1}}(z) \end{pmatrix}$$

Kreiss-Lopatinskii determinant

Writing $(\widetilde{U}_j(z))$ in the basis of $\mathcal{E}_s(z)$, to have uniqueness of solutions, the following determinant has to be non zero

$$\Delta_{KL}(z) = \det(zK_{0,r-1}(z) - BK_{0,m-1}(z)).$$

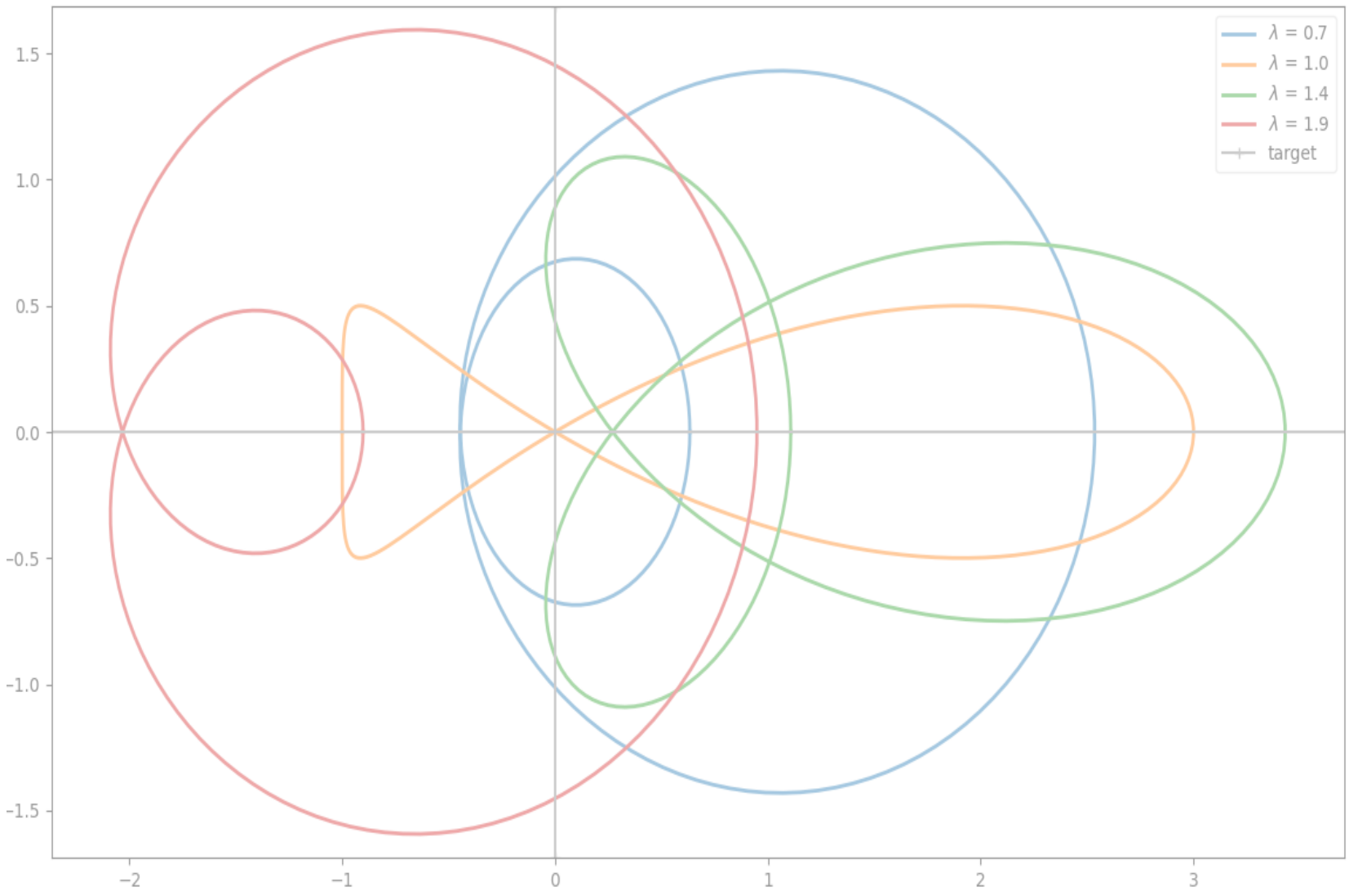
Definition (Intrinsic Kreiss-Lopatinskii determinant)

For all $|z| \geqslant 1$, we define *intrinsic Kreiss-Lopatinskii determinant* by

$$\Delta : z \mapsto \frac{\det(zK_{0,r-1}(z) - BK_{0,m-1}(z))}{\det K_{0,r-1}(z)}.$$

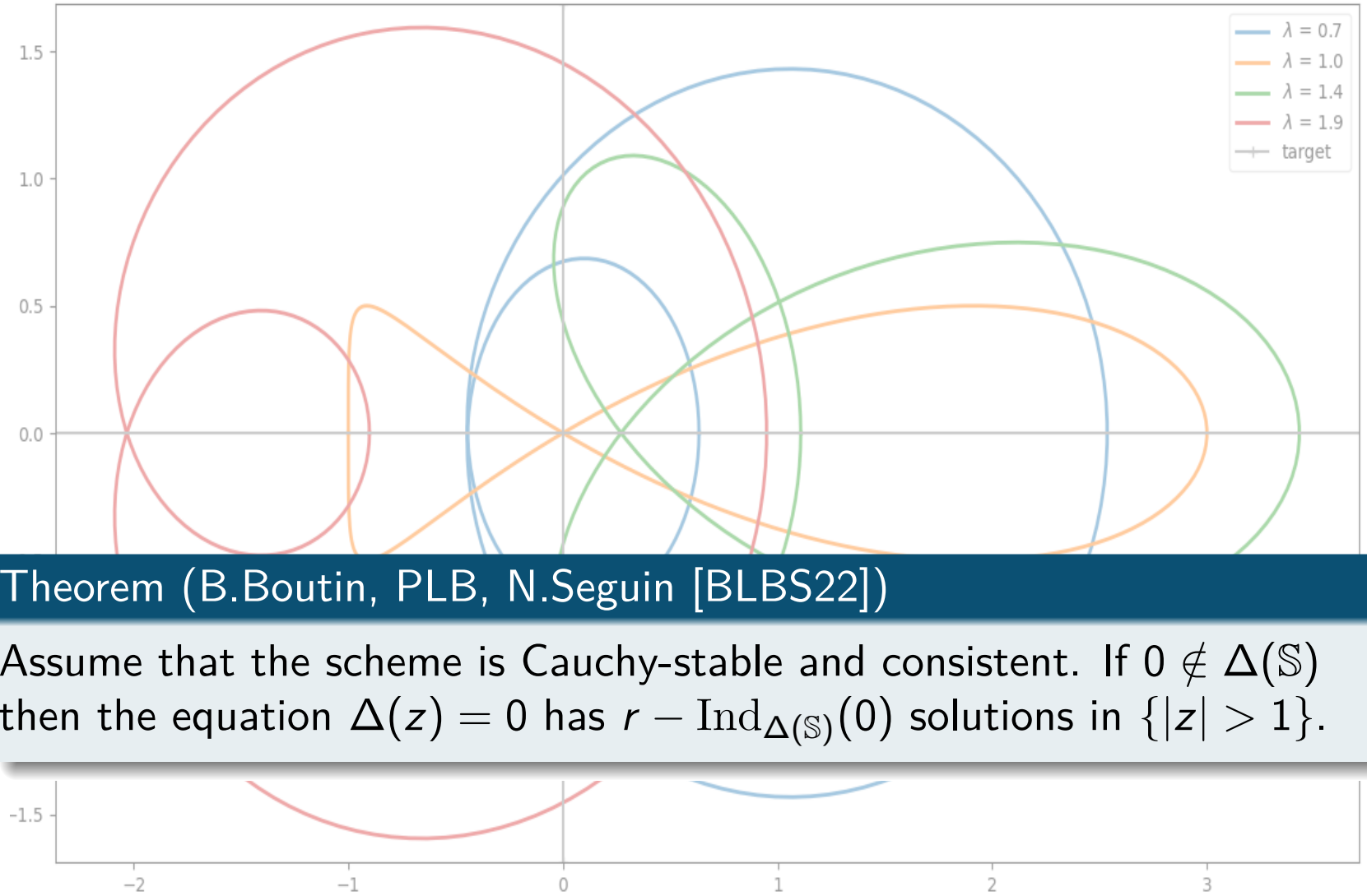
Main result 1

Case $p = 0$



Main result 1

Case $p = 0$

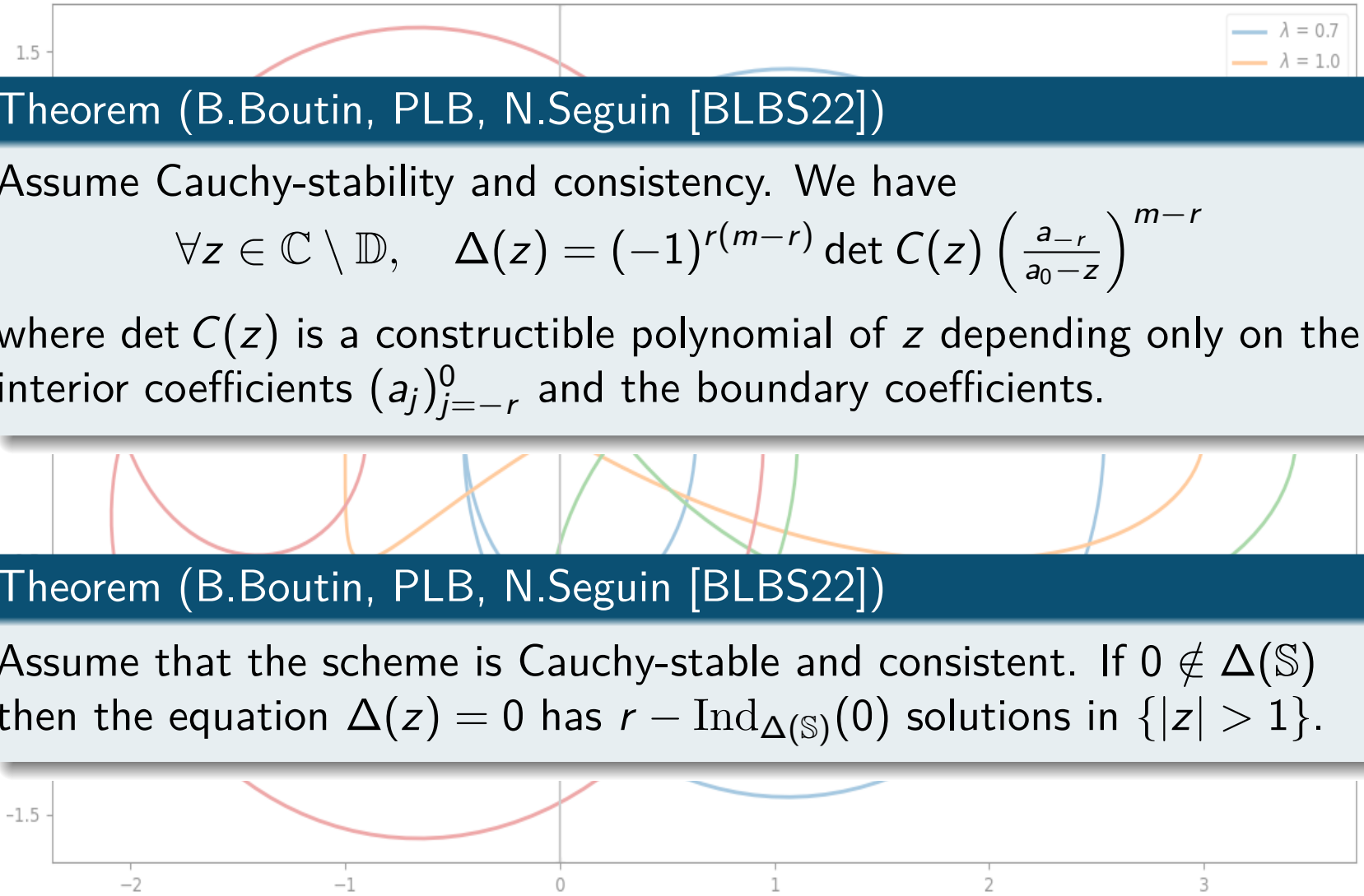


Theorem (B.Boutin, PLB, N.Seguin [BLBS22])

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

Main result 1

Case $p = 0$



Theorem (B.Boutin, PLB, N.Seguin [BLBS22])

Assume Cauchy-stability and consistency. We have

$$\forall z \in \mathbb{C} \setminus \mathbb{D}, \quad \Delta(z) = (-1)^{r(m-r)} \det C(z) \left(\frac{a_{-r}}{a_0 - z} \right)^{m-r}$$

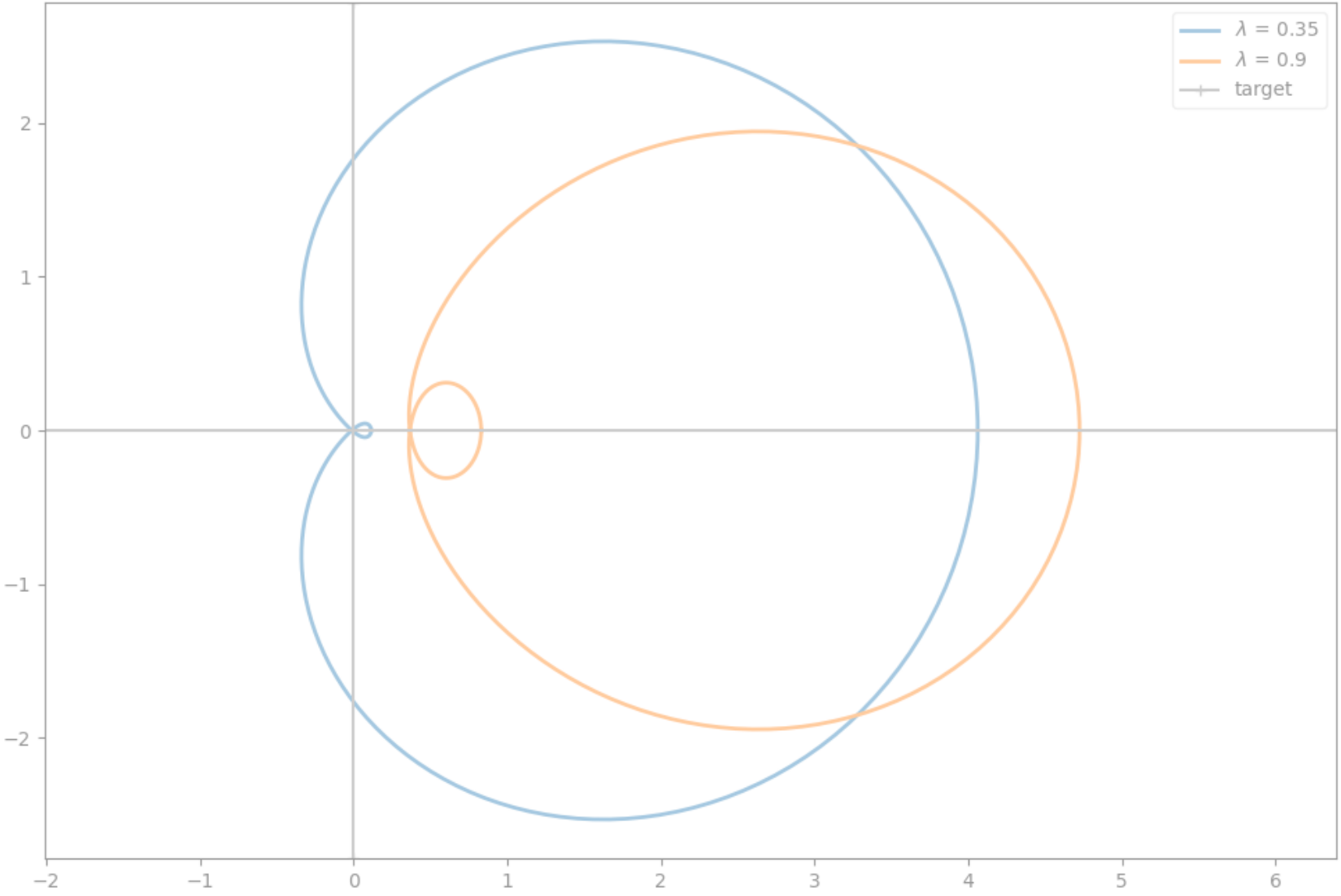
where $\det C(z)$ is a constructible polynomial of z depending only on the interior coefficients $(a_j)_{j=-r}^{0}$ and the boundary coefficients.

Theorem (B.Boutin, PLB, N.Seguin [BLBS22])

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

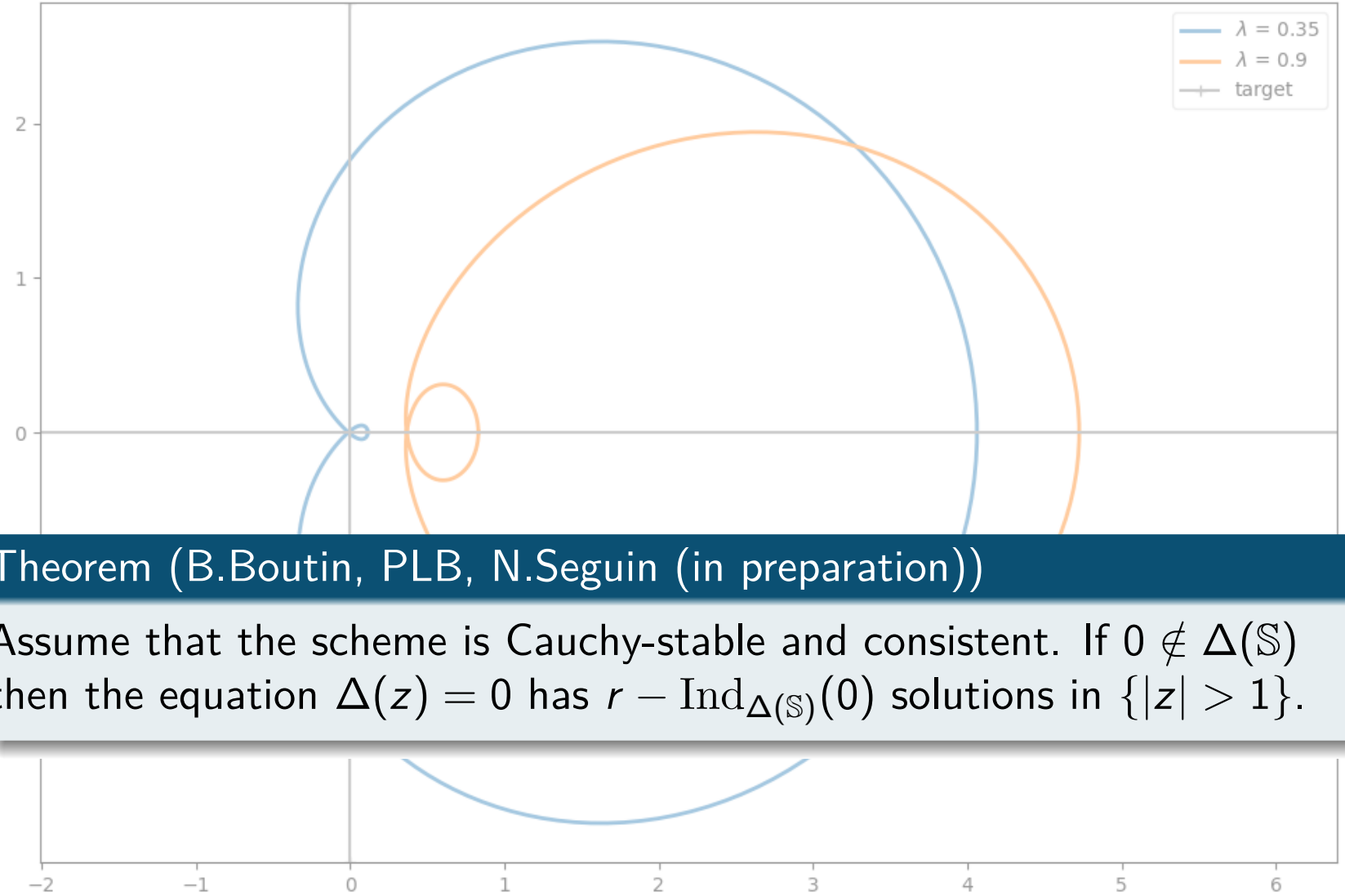
Main result 2

General case



Main result 2

General case

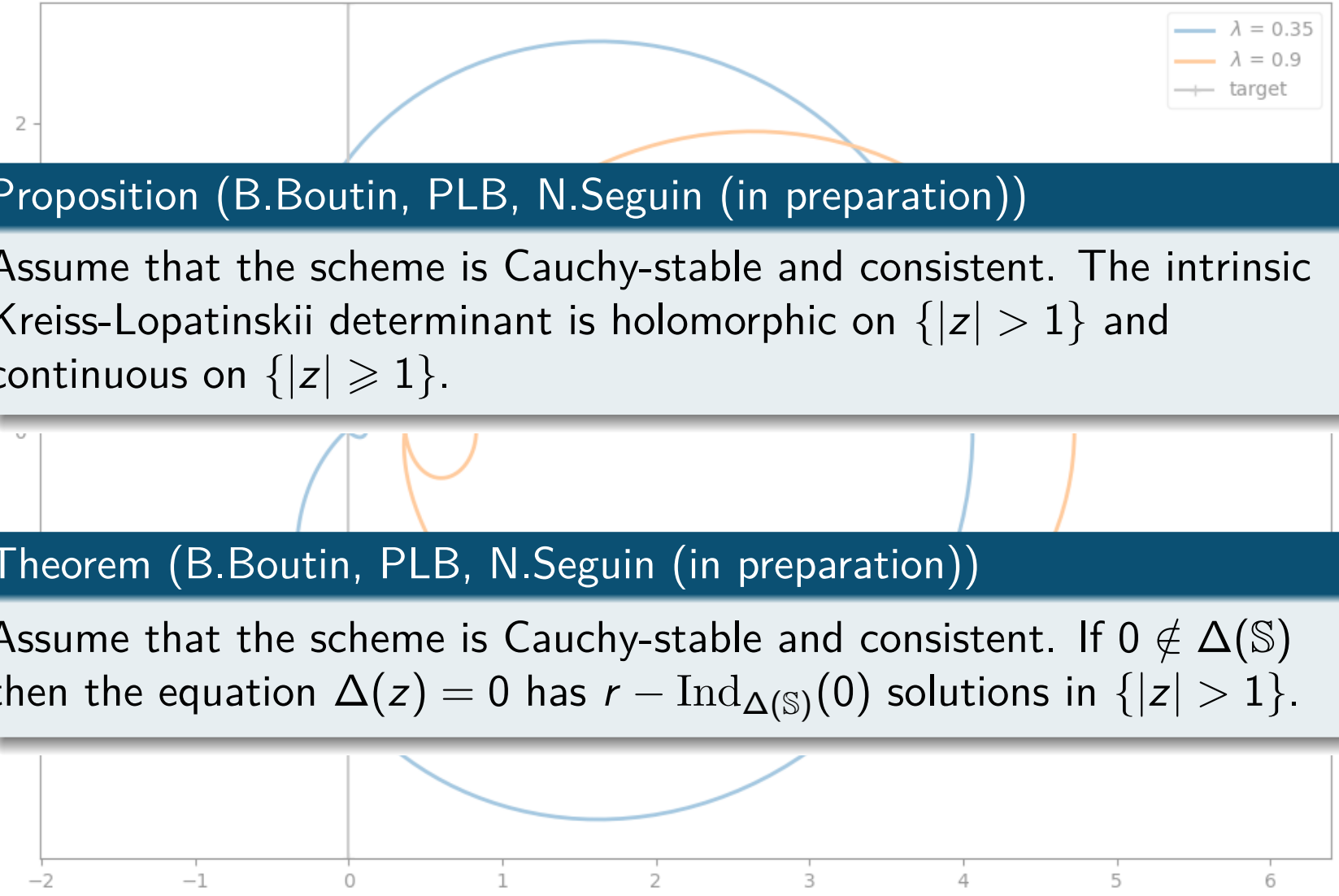


Theorem (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

Main result 2

General case



Proposition (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. The intrinsic Kreiss-Lopatinskii determinant is holomorphic on $\{|z| > 1\}$ and continuous on $\{|z| \geqslant 1\}$.

Theorem (B.Boutin, PLB, N.Seguin (in preparation))

Assume that the scheme is Cauchy-stable and consistent. If $0 \notin \Delta(\mathbb{S})$ then the equation $\Delta(z) = 0$ has $r - \mathrm{Ind}_{\Delta(\mathbb{S})}(0)$ solutions in $\{|z| > 1\}$.

Sketch of the proof

We have

$$\Delta(z) = \frac{\det(zK_{0,r-1}(z) - BK_{0,m-1}(z))}{\det(K_{0,r-1}(z))}$$
$$= z^r \det\left(I_r - \frac{BK_{0,m-1}(z)K_{0,r-1}^{-1}(z)}{z}\right).$$

The function $z \mapsto K_{0,m-1}(z)K_{0,r-1}^{-1}(z)$ is holomorphic on $\{|z| > 1\}$, continuous on $\{|z| \geqslant 1\}$ and bounded on $\{|z| \geqslant 1\}$ (technical proof).

Let us take the continuous function

$$\widetilde{\Delta} : \begin{array}{ccc} \overline{\mathbb{D}} \setminus \{0\} & \to & \mathbb{C} \\ z & \mapsto & \Delta(1/z) \end{array}$$

meromorphic on $\mathbb{D}$ with a pole at 0 of order r.

Sketch of the proof

$$\widetilde{\Delta} : \begin{array}{ccc} \overline{\mathbb{D}} \setminus \{0\} & \to & \mathbb{C} \\ z & \mapsto & \Delta(1/z) \end{array}$$

Use the Residue theorem on $\widetilde{\Delta}$ to get

$$\mathrm{Ind}_{\widetilde{\Delta}(\mathbb{S})}(0) = \#\mathrm{zeros}_{\widetilde{\Delta}}(\mathbb{D}) - \#\mathrm{poles}_{\widetilde{\Delta}}(\mathbb{D})$$

which leads to

$$\#\mathrm{zeros}_{\Delta}(\mathbb{C} \setminus \overline{\mathbb{D}}) = \underbrace{\#\mathrm{poles}_{\widetilde{\Delta}}(\mathbb{D})}_{r} - \mathrm{Ind}_{\Delta(\mathbb{S})}(0).$$

$\square$

Contents

How do we compute the Kreiss-Lopatinskii determinant ?

How do we compute the Kreiss-Lopatinskii determinant ?

$$z \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \vdots \\ \widetilde{U_{r-1}}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m} \\ \vdots & & & \vdots \\ \beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,m} \end{pmatrix}}_{B} \begin{pmatrix} \widetilde{U_0}(z) \\ \widetilde{U_1}(z) \\ \vdots \\ \widetilde{U_{m-1}}(z) \end{pmatrix}$$

How do we compute the Kreiss-Lopatinskii determinant ?

$$z\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{r-1}}(z)\end{pmatrix} = \underbrace{\begin{pmatrix}\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m}\\ \vdots & & & \vdots\\ \beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,m}\end{pmatrix}}_{B}\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{m-1}}(z)\end{pmatrix}$$

But, for all $j \in \mathbb{N}$, we have

$$a_p\widetilde{U_{j+p+r}}(z) + \cdots + a_1\widetilde{U_{j+1+r}}(z) + (a_0 - z)\widetilde{U_{j+r}}(z) + \cdots + a_{-r}\widetilde{U_j}(z) = 0.$$

How do we compute the Kreiss-Lopatinskii determinant ?

$$z\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{r-1}}(z)\end{pmatrix} = \underbrace{\begin{pmatrix}\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m}\\ \vdots & & & \vdots\\ \beta_{r,1} & \beta_{r,2} & \cdots & \beta_{r,m}\end{pmatrix}}_{B}\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{m-1}}(z)\end{pmatrix}$$

But, for all $j \in \mathbb{N}$, we have

$$a_p\widetilde{U_{j+p+r}}(z) + \cdots + a_1\widetilde{U_{j+1+r}}(z) + (a_0 - z)\widetilde{U_{j+r}}(z) + \cdots + a_{-r}\widetilde{U_j}(z) = 0.$$

We can express every $\widetilde{U_0}(z), \widetilde{U_1}(z), \ldots, \widetilde{U_{m-1}}(z)$ in terms of $\widetilde{U_0}(z), \widetilde{U_1}(z), \ldots, \widetilde{U_{r+p-1}}(z)$. Hence,

$$z\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{r-1}}(z)\end{pmatrix} = \mathfrak{B}(z)\begin{pmatrix}\widetilde{U_0}(z)\\ \widetilde{U_1}(z)\\ \vdots\\ \widetilde{U_{r+p-1}}(z)\end{pmatrix} \text{ with } \mathfrak{B}(z) \in \mathcal{M}_{r,r+p}(\mathbb{C})$$

Case $p = 0$

If $p = 0$ then the matrix $\mathfrak{B}(z)$ is a square matrix.
We have

$$\Delta(z) = \frac{\det(zK_{0,r-1}(z) - \mathfrak{B}(z)K_{0,r-1}(z))}{\det K_{0,r-1}(z)}$$
$$= \det(zI_r - \mathfrak{B}(z))$$

with $\mathfrak{B}(z)$ easily computable and depending only on z, the coefficients $(a_j)_{j=-r}^{0}$ and the matrix B.

Moreover, no need to compute the roots κ of the characteristic equation.

General case

If $p \neq 0$ then the matrix $\mathfrak{B}(z)$ is not a square matrix.
Let us take the polynomial of degree r whose roots are the κ from the inside.

$$\prod_{j=1}^{r}(X - \kappa_j(z)) = X^r + \sigma_{r-1}(z)X^{r-1} + \cdots + \sigma_1(z)X + \sigma_0(z)$$

$$\cdots$$

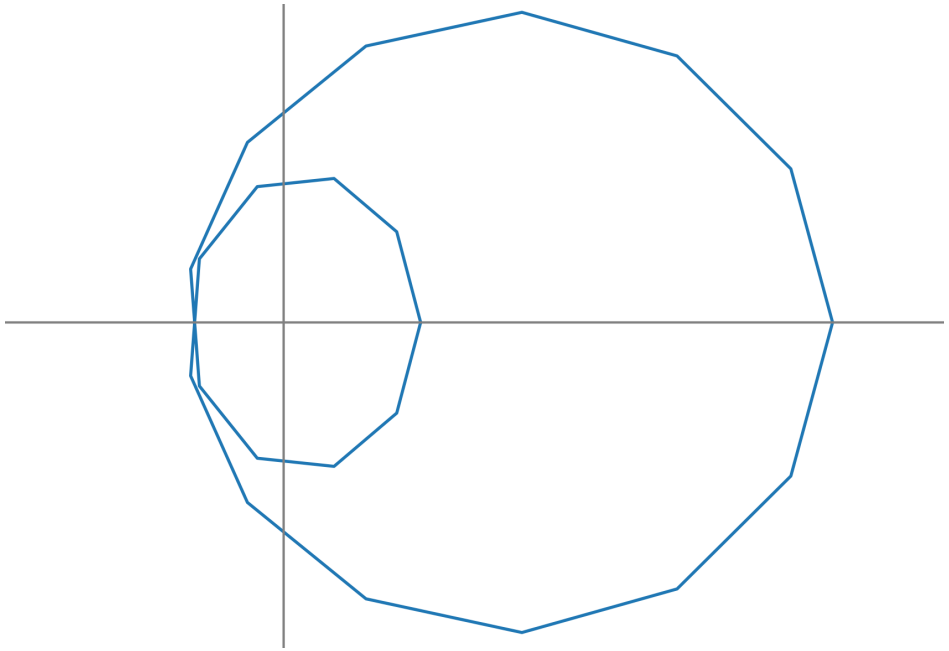
symmetric functions of $(\kappa_j(z))_j$

Then we can do the same transformation with this polynomial and obtain

$$\Delta(z) = \det(zI_r - \widetilde{\mathfrak{B}}(\sigma_{r-1}(z), \ldots, \sigma_0(z)))$$

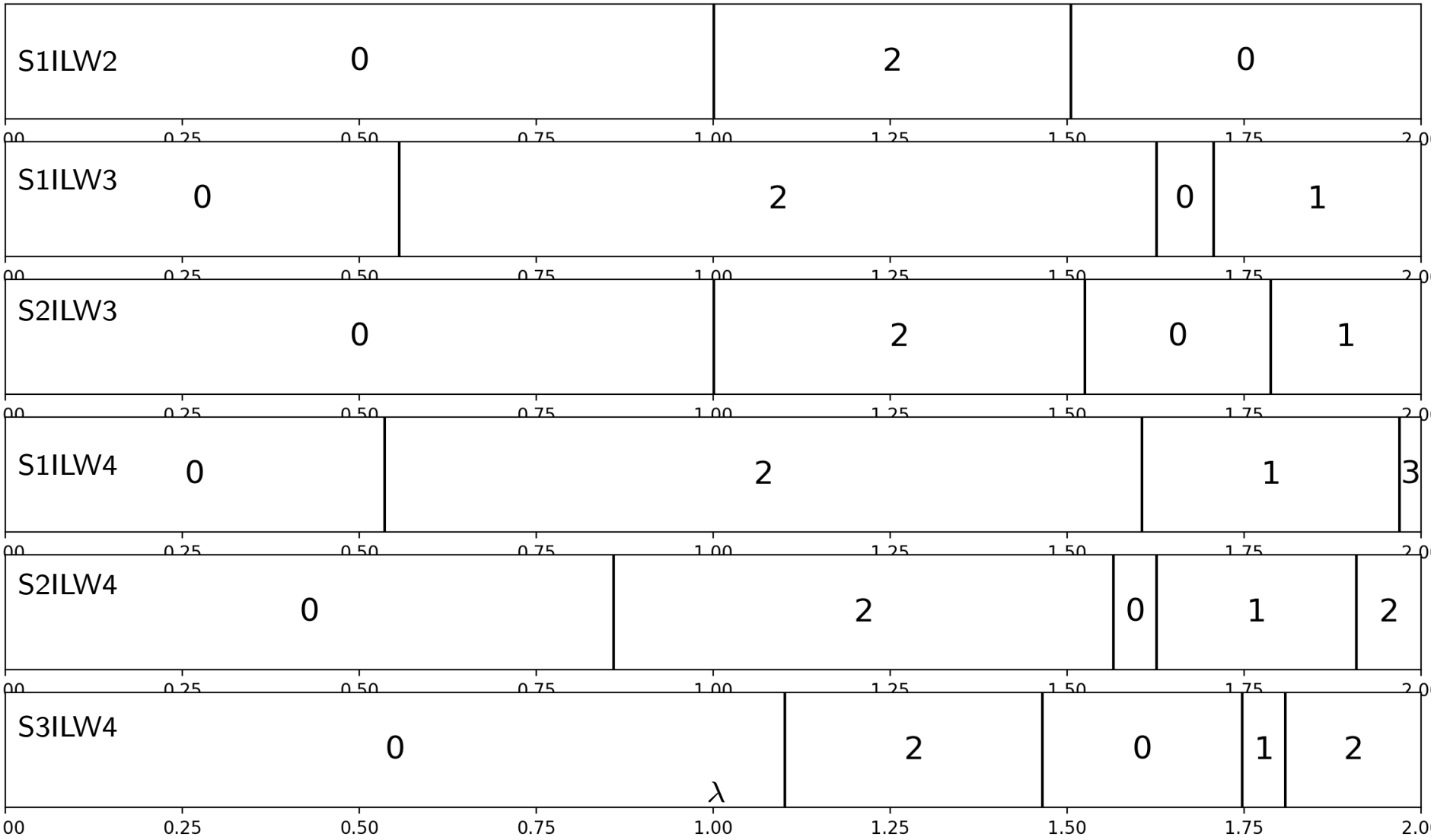
Winding number

The curve we draw is a polygonal line. We count the number of loops around the origin.



See [ZM13] for results of robustness.

Beam-Warming example



Number of zeros of Kreiss-Lopatinskii determinant for Beam-Warming scheme with different SILW boundary with respect to λ.

Conclusion

Conclusion:

- Explicit use of the Kreiss-Lopatinskii determinant ([GKO13]) for one time step explicit scheme.
- Numerical procedure to check the stability of a problem defined on $\mathbb{N}$ with $f = 0$ and $g \neq 0$.

In prospect:

- Link with [CF21] where $f \neq 0$ and $g = 0$
- Find inequality of convergence for Simplified Inverse Lax-Wendroff boundary condition ([BNS+21])
- Explicit the Kreiss-Lopatinskii determinant for multistep scheme (Leapfrog) ([Tre84])
- Study implicit problem (Crank Nicolson)
- Study in higher dimension (dimension 2) ([DDJ18])
- Make rigourous the numerical computation (with interval arithmetics for instance)

Bibliographie I

Benjamin Boutin, Pierre Le Barbenchon, and Nicolas Seguin.
On the stability of totally upwind schemes for the hyperbolic initial boundary value problem.
2022.

B. Boutin, T.H.T. Nguyen, A. Sylla, S. Tran-Tien, and J.-F. Coulombel.
High order numerical schemes for transport equations on bounded domains.
ESAIM: Proceedings and Surveys, 70:84–106, 2021.

Jean-François Coulombel and Grégory Faye.
Sharp stability for finite difference approximations of hyperbolic equations with boundary conditions, 2021.

Bibliographie II

Jean-François Coulombel.
Stability of finite difference schemes for hyperbolic initial boundary value problems.
In *HCDTE lecture notes. Part I. Nonlinear hyperbolic PDEs, dispersive and transport equations*, volume 6 of *AIMS Ser. Appl. Math.*, page 146. Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2013.

Gautier Dakin, Bruno Després, and Stéphane Jaouen.
Inverse Lax–Wendroff Boundary Treatment for Compressible Lagrange-Remap Hydrodynamics on Cartesian Grids.
Journal of Computational Physics, 353:228–257, 2018.

B. Gustafsson, H.O. Kreiss, and J. Oliger.
Time-Dependent Problems and Difference Methods.
Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013.

Bibliographie III

Bertil Gustafsson, Heinz-Otto Kreiss, and Arne Sundström.
Stability theory of difference approximations for mixed initial boundary value problems. II.
Mathematics of Computation, 26(119):649–649, 1972.

Lloyd N. Trefethen.
Instability of difference models for hyperbolic initial boundary value problems.
Communications on Pure and Applied Mathematics, 37(3):329–367, 1984.

Juan Luis García Zapata and Juan Carlos Díaz Martín.
A geometrical root finding method for polynomials, with complexity analysis, 2013.