Perfectly matched layers methods for mixed hyperbolic-dispersive equations

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27 October 2022

Coastal flow models and boundary conditions
IMT, Toulouse

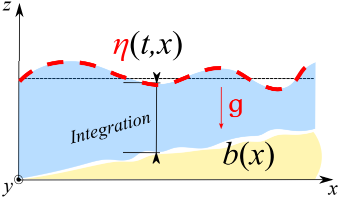
Water waves models

Free-surface incompressible Euler

$t > 0, \vec{x} \in (\mathbb{R}^3, b(\vec{x}) < z < \boldsymbol{\eta}(t, \vec{x}))$

$$\begin{cases} u_t + u \cdot \nabla u = -\dfrac{1}{\rho} \nabla p + \boldsymbol{g} \\ \nabla \cdot u = 0, \quad \boldsymbol{g} = (0, 0, -g) \end{cases}$$

+ kinematic and dynamic boundary conditions

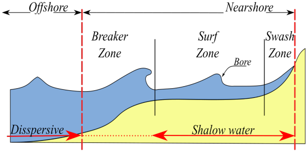


Assumption: constant velocity over vertical $u(t, x, z) = \boldsymbol{v}(t, x)$

Water waves models

$$\mu = H^2/L^2 \text{ (shallowness)},$$

$$\varepsilon = a/H \text{ (nonlinearity)}$$



$$\begin{cases} \dfrac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{v}) = 0, & \text{(Mass Eq)} \\ \dfrac{\partial h\boldsymbol{v}}{\partial t} + \nabla \cdot \left(h\boldsymbol{v} \otimes \boldsymbol{v} + \dfrac{gh^2}{2}\mathcal{I} + p_{NH} \right) = 0, & \text{(Momentum Eq).} \end{cases}$$

model	**NSWE** $\mathcal{O}(\mu)$	$\mathcal{O}(\varepsilon\mu)$	**SGN** $\mathcal{O}(\mu^2)$
Pressure	$p_{NH} = 0$	**Boussinesq**	$p_{NH} = h^2\ddot{h}/3$
ε	no assump		no assump.
Type	hyperbolic		dispersive

Lannes, **2013**

Water waves models

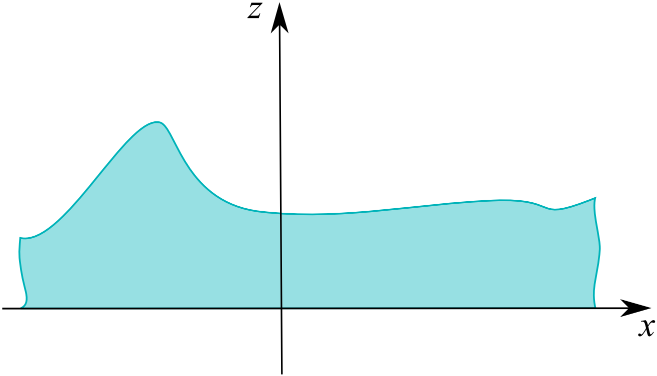
Most costly step for non-hydrostatic models: elliptic problem

First-order hyperbolic equations with dispersive properties for non-hydrostatic free surface flows:

📖 Favrie-Gavrilyuk, **2017** (SGN), Gavrilyuk et al. **2022** (BBM)
! Hyperbolic SGN is rigorously justified in 📖Duchene,**2019**
📖 Escalante et al. (artificial compressibility)**2019**
📖 Richard (compressible and quasi-incompressible)**2021**

Boundary conditions

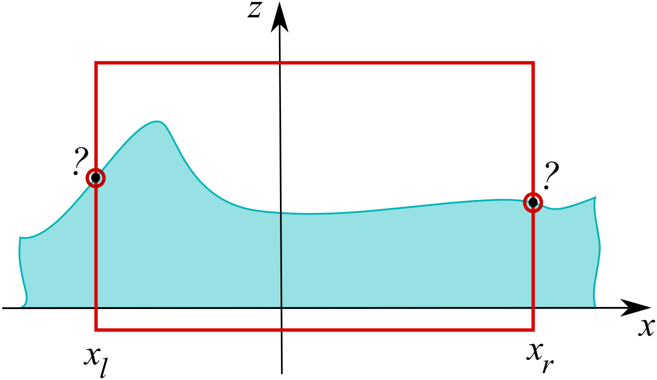
Problems are initially posed on infinite domain $\mathbf{x} \in \mathbb{R}$



Boundary conditions

Problems are initially posed on infinite domain $\boxed{\color{blue}{\mathbf{x}} \in \mathbb{R}} \to \boxed{\mathbf{x} \in \boldsymbol{\Omega}}$

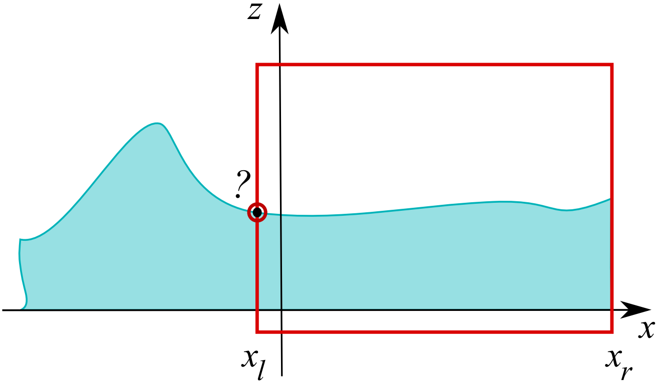
Restriction of the observation area



Boundary conditions

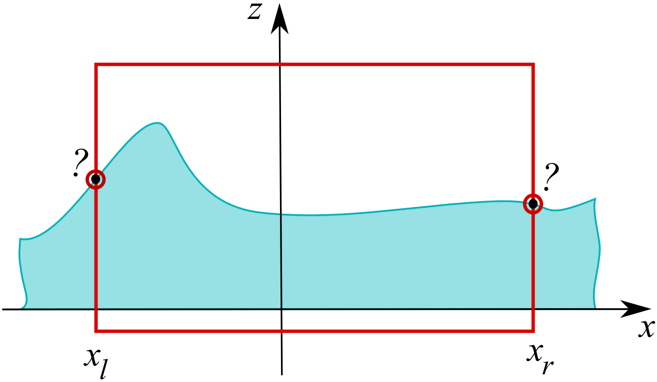
Coupling different models

Generation of the waves for the dispersive water waves problem



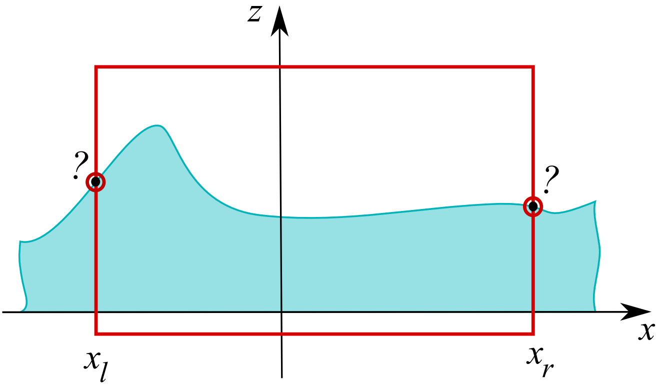
Boundary conditions

Hyperbolic system - *Riemann-invariant form (if exist)*
Dispersive system - ?



Boundary conditions

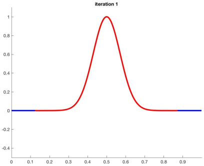
Dispersive system - linear case: non-reflecting TBC, DTBC, PML
Nonlinear case (Coastal engineering, SGN) relaxation zones, sponge layers



Boundary condition

DTBC for dispersive models

Ehrhardt, **2001**, Besse *et al.*, **2016**



Dispersive system, linear case:
KdV, BBM (Besse et al.)
SGN (MK&Noble)

MK, P.Noble (**2020**)

$$w(t,x_{r,l}) = \pm[\mathbb{1} + \partial^2/\partial t]\frac{1}{\sqrt{\mu}}\int_0^t \mathcal{J}_0(s/\sqrt{\mu})\eta(t-s,x_{r,l})ds$$

$$w_{J+1}^{n+1} = F(\mu,\delta x,\delta t,w_{J+1}^n,w_J^n) - 2\delta x\Gamma\sum_{k=2}^{n} s_k(v)w_J^{n-k}.$$

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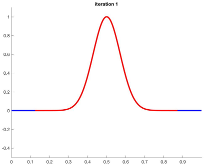

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Motivation: PML methods are simple to implement (1^{st} order)

Goal: Wheither the PML method for dispersive water waves is useful?

Cartesian PML

Cartesian classical *Perfectly Matched Layers* (PML) 📖 Bérenger (**1994**)

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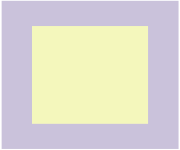

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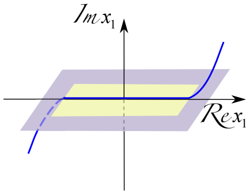
PML change of variables

$$\mathbf{x} \in \mathbb{R}, \quad \tilde{\mathbf{x}} = \mathbf{x}\left(1 + \frac{\sigma(\mathbf{x})}{i\omega}\right)$$



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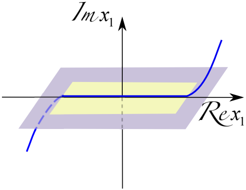

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$$\partial_{\tilde{x}} \to \left(1 + \frac{\sigma(x)}{i\omega}\right)^{-1} \partial_x$$

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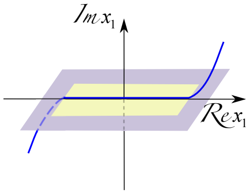

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KdV equation

$$u_t + u\, u_x + \varepsilon u_{xxx} = 0, \quad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

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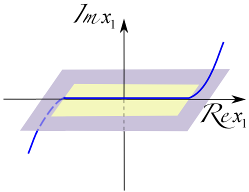

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linear KdV equation

$$u_t + U\, u_x + \varepsilon u_{xxx} = 0 \quad \forall x \in \mathbb{R}, \quad \forall t > 0 \qquad (TD)$$

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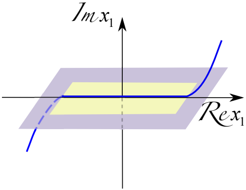

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$$\partial_{\tilde{x}} \to \left(1 + \frac{\sigma(x)}{i\omega}\right)^{-1}\partial_x$$

In Frequency domain (after Fourier transform)

$$-i\omega\hat{u} + U\,\hat{u}_x + \varepsilon\hat{u}_{xxx} = 0 \quad \forall x \in \mathbb{R}$$

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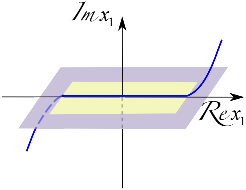

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$$\partial_{\tilde{x}} \to \left(1 + \frac{\sigma(x)}{i\omega}\right)^{-1} \partial_x$$

Artificial truncation by PML: $x \in \Omega, \quad \forall t > 0$

$$-i\omega(1 + \frac{i\sigma}{\omega})u + U\partial_x u + \varepsilon\partial_x\left((1 + \frac{i\sigma}{\omega})^{-1}\partial_x\left((1 + \frac{i\sigma}{\omega})^{-1}\partial_x u\right)\right) = 0$$

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$$-i\omega(1+\frac{i\sigma}{\omega})u + U\partial_x u + \varepsilon\partial_x\left((1+\frac{i\sigma}{\omega})^{-1}\partial_x\left((1+\frac{i\sigma}{\omega})^{-1}\partial_x u)\right)\right) = 0$$

$+$ auxiliary variables u_1 and u_2:

$$\partial_x u = (1+\frac{i\sigma}{\omega})u_1, \quad \partial_x u_1 = (1+\frac{i\sigma}{\omega})u_2,$$

Back to time domain

$$\begin{aligned} &\partial_t u + \sigma u + U\partial_x u + \varepsilon\partial_x u_2 = 0, \\ &\partial_t\left(u_1 - \partial_x u\right) + \sigma u_1 = 0, \quad \partial_t\left(u_2 - \partial_x u_1\right) + \sigma u_2 = 0. \end{aligned} \qquad (TD)_{PML}$$

By applying the initial value theorem, one finds

$$u_1|_{t=0} = \partial_x u|_{t=0}, \quad u_2|_{t=0} = \partial_{xx} u|_{t=0}.$$

Well-posedness and stability

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if and only if ω and $\mathbf{k}$ are related via *dispersion relation*

$\mathfrak{F}(\omega, \mathbf{k}) = 0,$ with solutions $\omega_j(\mathbf{k})$ which are called modes.

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For the PML system $(\mathrm{TD})_{\mathrm{PML}}$ we define perturbed dispersion relation

$$\mathfrak{F}_{pml}(\omega, \mathbf{k}, \sigma) = 0, \text{ with modes } \tilde{\omega}_j(\mathbf{k}, \sigma)$$

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$$\mathfrak{F}_{pml}(\omega, \mathbf{k}, \sigma) = 0, \text{ with modes } \tilde{\omega}_j(\mathbf{k}, \sigma)$$

We search for solutions with an exponential behaviour and the system $(TD)_{PML}$ is stable if and only if $\Im(\tilde{\omega}_j) \leq 0$ for all $\sigma \geq 0$.

Stability condition and inverse waves

We introduce notions of the *phase velocity* $\mathbf{v}_p$ and the *group velocity* $\mathbf{v}_g$ (general case $\mathbf{k} \in \mathbb{R}^3$):

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Necessary stability conditions Bécache(2003)

$$\text{If } \forall \mathbf{k} \in \mathbb{R}^3, \quad (\mathbf{v}_\phi(\mathbf{k}) \cdot \mathbf{e}_j)(\mathbf{v}_g(\mathbf{k}) \cdot \mathbf{e}_j) \geq 0,$$

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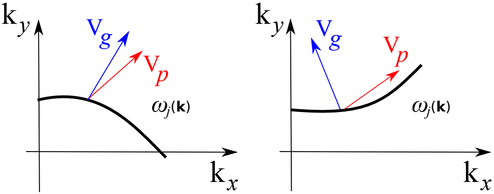

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Backward waves in the direction $\mathbf{e}_j$: $\exists\, \mathbf{k} : (\mathbf{v}_\phi(\mathbf{k}) \cdot \mathbf{e}_j)(\mathbf{v}_g(\mathbf{k}) \cdot \mathbf{e}_j) \leq 0,$
Otherwise: forward propagating waves.

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If there are backward propagating waves in the PML direction
the PML system is unstable.

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Necessary stability conditions in the 1D case $v_g(k)v_\phi(k) \geq 0$

The linear KdV equation

$$u_t + U\,u_x + \varepsilon u_{xxx} = 0, \quad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

Proposition

- If $U = 0$, equations $(TD)_{PML}$ are always unstable.
- If $\varepsilon U < 0$, equations $(TD)_{PML}$ are stable if and only if $k^2 \geq 16\dfrac{|U|}{|\varepsilon|}$.
- If $\varepsilon U > 0$, equations $(TD)_{PML}$ are stable if and only if $k^2 \leq \dfrac{U}{3\varepsilon}$.

Proof. The dispersion relation of $(TD)_{PML}$: Following 📖 Bécache**2003**

dispersion relation for KdV with $k \to k/(1 + \frac{i\sigma}{\omega})$

$$(\omega + i\sigma)^3 = kU(\omega + i\sigma)^2 - \varepsilon k^3 \omega^2.$$

If $k = 0$, $\omega = -i\sigma$ and the condition $\Im(\omega) \leq 0$ is satisfied.
If $k \neq 0$ $\qquad \omega^2(\omega - \omega_0(k)) = 0, \qquad \omega_0(k) = kU - \varepsilon\, k^3.$
Two roots are bifurcating from 0 and one root bifurcates from $\omega = \omega_0(k)$.

The linear KdV equation

From straigforward computations, a necessary condition is

$$(U - \varepsilon k^2)(U - 3\varepsilon k^2) > 0$$

Here $v_g(k) = U - 3\varepsilon k^2$ and $v_\phi(k) = U - \varepsilon k^2$.

$$v_g(k)v_\phi(k) \geq 0.$$

So we recover the classical condition in the PML framework.

We have proved that $\Im(\omega) \leq 0$ for $\sigma > 0$ small enough, under conditions on k claimed in the proposition.

We show then that for any $\sigma > 0$, there are no real solutions, which means that $\Im(\omega) \neq 0$.

We conclude that these conditions are sufficient to guarantee stability, using continuity of the roots of a complex polynomial with respect to its coefficients. This end the proof.

The linear KdV equation

Discretization

We consider a centered space FD with a Crank Nicolson in time scheme:

$$x_j = j\delta x,\ j \in \mathbb{Z}, t_n = n\delta t,\ n \in \mathbb{N}$$

$$\begin{aligned}
&2\frac{v_j^n - u_j^n}{\delta t} + \sigma v_j^n + U\frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} + \varepsilon\frac{v_{2,j+1}^n - v_{2,j-1}^n}{2\delta x} = 0,\\
&\frac{2}{\delta t}\left(\left(v_{1,j}^n - \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x}\right) - \left(u_{1,j}^n - \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)\right) + \sigma v_{1,j}^n = 0,\\
&\frac{2}{\delta t}\left(\left(v_{2,j}^n - \frac{v_{1,j+1}^n - v_{1,j-1}^n}{2\delta x}\right) - \left(u_{2,j}^n - \frac{u_{1,j+1}^n - u_{1,j-1}^n}{2\delta x}\right)\right) + \sigma v_{2,j}^n = 0,
\end{aligned}$$

with $v_{k,j}^n = \frac{u_{k,j}^{n+1} + u_{k,j}^n}{2}$ for $k = 0, 1, 2$ and $u_{0,j}^n = u_j^n$.

The linear KdV equation

Discretization

ℓ^2 stability: $u^n_{k,j} := z^n e^{i\,jK\delta x}\hat{u}_k$.

Proposition

The scheme is stable only under the assumption: $\varepsilon U > 0$ and $\delta x \geq \sqrt{\frac{3\varepsilon}{U}}$.

We recover corresponding dispersion relation if

$$-i\omega := \frac{2}{\delta t}\frac{z-1}{z+1}, \quad k := \frac{\sin(K\delta x)}{\delta x}.$$

$$|z|^2 = \frac{(1+\frac{\Im(\omega)\delta t}{2})^2 + (\frac{\Re(\omega)\delta t}{2})^2}{(1-\frac{\Im(\omega)\delta t}{2})^2 + (\frac{\Re(\omega)\delta t}{2})^2}.$$

$|z| \leq 1$ is equivalent to $\Im(\omega) \leq 0$.

The condition is satisfied only if $k^2 \geq 16\frac{|U|}{|\varepsilon|}$ in the case $\varepsilon U < 0$

$k^2 \leq \frac{|U|}{3|\varepsilon|}$ in the case $\varepsilon U > 0$ (satisfied if $\delta x \geq \sqrt{\frac{3\varepsilon}{U}}$).

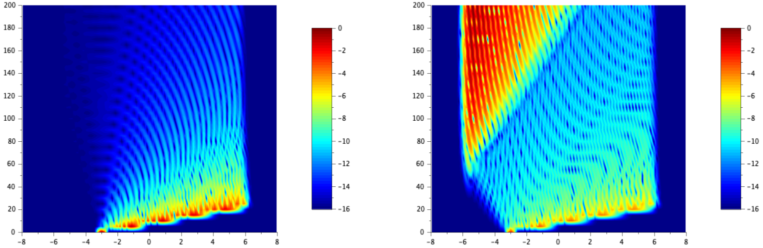
The linear KdV equation

Numerical simulation

Case: $\varepsilon U > 0$
Initial condition $u_0(x) = \exp\left(-40\,(x+3)^2\right)$,$u_1 = u_0'$ and $u_2 = u_0''$.
The domain is $[-8,8] \times [0,200]$, $\delta x = 0.05$, $\delta t = \delta x$.

$$\sigma(x) = 2\left(\max(0, \frac{x-5}{3})^4 + \max(\frac{-x-5}{3}, 0)^4\right)$$

$\varepsilon = U\delta x^2/4$ (stable case) $\varepsilon = U\delta x^2/2$ (unstable case)

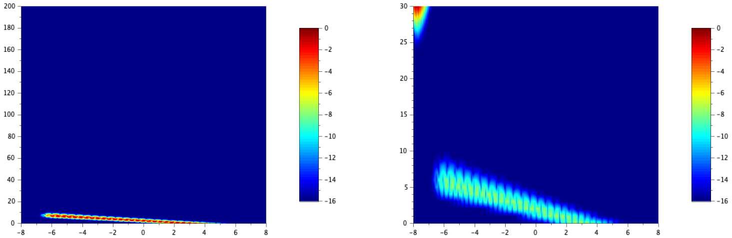


Represention of the function $v(t,x) = \log(1 + 1000|u(t,x)|)$.

The linear KdV equation

Case: $\varepsilon U < 0$
Initial condition $u_0(x) = exp(-(x-3)^2)\sin(2x)$.

$\varepsilon = 16|U|\delta x^2$ $\varepsilon = 32|U|\delta x^2$.



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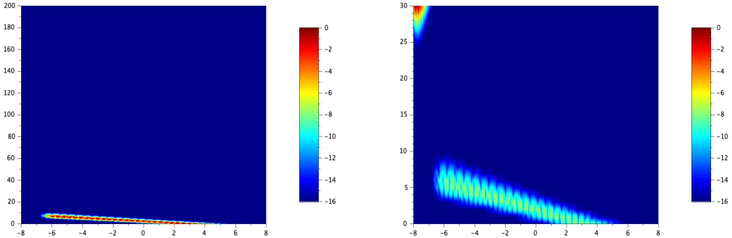

Message I

We recover in this analysis the classical stability condition
Since the phase and group velocities do not always have the same sign the PML is not always stable.

A hyperbolic KdV system

We now consider a relaxation of the original Korteweg-de Vries equation.

$$u_t + u\,u_x + \varepsilon\psi_x = 0, \quad p_t - \frac{p_x - \psi}{\tau} = 0, \quad \psi_t + \frac{u_x - p}{\tau} = 0,$$

ε – the dispersion parameter, $\tau > 0$ – the relaxation parameter.

Also: Euler-Lagrange equations for a given Lagrangian 📖<hal>

Formally, $\tau \to 0$, the function u turns out to be an approximate solution of the KdV equation. Indeed, p, ψ expand as

$$p = u_x + \tau u_{txx} + O(\tau^2), \quad \psi = u_{xx} + \tau\,(u_{txxx} - u_{tx}) + O(\tau^2).$$

By inserting this expansion we have

$$(u - \tau u_{xx} + \tau u_{xxxx})_t + u\,u_x + \varepsilon u_{xxx} = O(\tau^2).$$

which is the Benjamin-Bona-Mahoney (BBM) regularization of the KdV.

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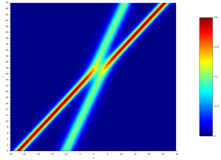
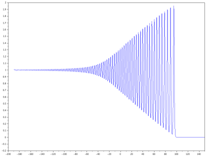
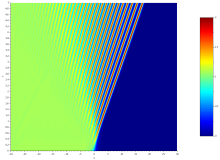






A hyperbolic KdV system

A classical PML are easily derived for first order systems:

PML I

$$u_t + \sigma u + U\,u_x + \varepsilon\psi_x = 0, \quad p_t + \sigma p - \frac{p_x - \psi}{\tau} + \frac{\sigma}{\tau}\phi = 0,$$
$$\psi_t + \sigma\psi + \frac{u_x - p}{\tau} - \frac{\sigma}{\tau}q = 0, \quad q_t = p, \qquad \phi_t = \psi.$$

An "alternative" approach is to neglect (forget!) the source terms:

PML II

$$u_t + \sigma u + U\,u_x + \varepsilon\psi_x = 0, \quad p_t + \sigma p - \frac{p_x - \psi}{\tau} = 0, \quad \psi_t + \sigma\psi + \frac{u_x - p}{\tau} = 0.$$

PML II satisfies the energy estimate:

$$\left(\frac{u^2}{2\tau} + \varepsilon\frac{p^2}{2} + \varepsilon\frac{\psi^2}{2}\right)_t + \sigma\left(\frac{u^2}{\tau} + \varepsilon\psi^2 + \varepsilon p^2\right) + \left(U\frac{u^2}{2\tau} + \frac{\varepsilon}{\tau}\psi u - \frac{\varepsilon p^2}{2\tau}\right)_x = 0.$$

The system is strongly stable, however recall that it is not an exact PML system!

A hyperbolic KdV system

We carry out a Fourier transform in space on PML I:

$$\widehat{V}_t + \mathbb{A}(\xi)\widehat{V} = 0,$$

We study the eigenvalues of $\mathbb{A}$, the characteristic equation associated to $\mathbb{A}$ is given by

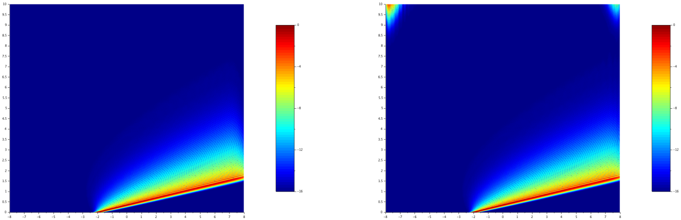
$$(\sigma - X + i\xi U)\left(\tau X^2(\sigma - X)(\tau(\sigma - X) - i\xi) + (\sigma - X)^2\right) + \varepsilon\xi^2 X^2\left(\tau(\sigma - X) - i\xi\right) = 0.$$

$\xi \to \infty$ (high frequency limit):
the roots are nothing but the characteristic speeds of the original system $+$ an additional root $x = 0$ (double).

Conclusion: if $\varepsilon U > 0$, the system is not stable.

A hyperbolic KdV system

Initial wave: $u_0(x) = exp(-40(x+2)^2)$



PML II PML I

$v(t,x) = \log(1 + 1000|u(t,x)|)$ in the (x,t)
in the case $\varepsilon U > 0$, $U = 1$, $\varepsilon = 5\delta x^2$
(unstable for the original KdV and for PML I)

Message II

Although not an exact PML II absorb outgoing waves without numerical instabilities.

Application to abcd-model

We consider the hyperbolic-dispersive systems which models water wave propagation BBM-Boussinesq type model (also known as abcd-model):

abcd

$$\begin{aligned} &(1 - b\partial_x^2)\partial_t \eta + \partial_x u + a\partial_x^3 u = 0, \\ &(1 - d\partial_x^2)\partial_t u + \partial_x \eta + c\partial_x^3 \eta = 0, \end{aligned} \qquad \forall (t,x) \in [0,T] \times [x_\ell, x_r].$$

📖Bona, Chen and Saut **(2002)**

By-product: KdV dynamic is included in this model (properly chosed initial data creates approximate one-way propagating waves)

Application to abcd-model

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$$\begin{aligned}&(1-b\partial_x^2)\partial_t\eta+\partial_x u+a\partial_x^3 u=0,\\&(1-d\partial_x^2)\partial_t u+\partial_x\eta+c\partial_x^3\eta=0,\end{aligned}\qquad \forall(t,x)\in[0,T]\times[x_\ell,x_r].$$

📖Bona, Chen and Saut **(2002)**

$$\begin{aligned}&\partial_t(\eta-b\eta_2)+\sigma(\eta-b\eta_2)+\partial_x(u+au_2)=0,\\&\partial_t(u-du_2)+\sigma(u-du_2)+\partial_x(\eta+c\eta_2)=0,\\&\partial_t(\eta_1-\partial_x\eta)+\sigma\eta_1=0,\quad \partial_t(\eta_2-\partial_x\eta_1)+\sigma\eta_2=0,\\&\partial_t(u_1-\partial_x u)+\sigma u_1=0,\quad \partial_t(u_2-\partial_x u_1)+\sigma u_2=0.\end{aligned}$$

The initial conditions are given by

$$\eta_i|_{t=0}=\partial_x\eta_{i-1}|_{t=0},\quad u_i|_{t=0}=\partial_x u_{i-1}|_{t=0},\quad i=1,2.$$

Application to abcd-model

Necessary condition

Denote v_g and v_ϕ respectively the group velocity and phase velocity. A necessary condition of stability is written again $v_g(k)v_\phi(k) \geq 0$ for all $k \in \mathbb{R}$.

Proposition

The PML equations associated to the classical Boussinesq equation $(a = b = c = 0, d > 0)$ and the shallow water equations with surface tension $(a = b = d = 0, c < 0)$ are stable.

Proposition

The PML system is stable under the assumption $a = d = 0$ and $b > 0, c < 0$. The PML system is also stable in the case $b = c = 0$ and $d > 0, a < 0$.

Application to abcd-model

The classical linearized Boussinesq approximation:

$a = b = c = 0$ and $d > 0$ (we have fixed $d = 1/3$)

Discretization: centered FD in space with a Crank Nicolson in time

$$
\begin{aligned}
&2\frac{h_j^n - \eta_j^n}{\delta t} + \sigma h_j^n + \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x} = 0,\\
&\frac{2}{\delta t}\big((v_j^n - dv_{2,j}^n) - (u_j^n - du_{2,j}^n)\big) + \sigma(v_j^n + v_{2,j}^n) + \frac{h_{j+1}^n - h_{j-1}^n}{2\delta x} = 0,\\
&\frac{2}{\delta t}\left(\left(v_{1,j}^n - \frac{v_{j+1}^n - v_{j-1}^n}{2\delta x}\right) - \left(u_{1,j}^n - \frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right)\right) + \sigma v_{1,j}^n = 0,\\
&\frac{2}{\delta t}\left(\left(v_{2,j}^n - \frac{v_{1,j+1}^n - v_{1,j-1}^n}{2\delta x}\right) - \left(u_{2,j}^n - \frac{u_{1,j+1}^n - u_{1,j-1}^n}{2\delta x}\right)\right) + \sigma v_{2,j}^n = 0,
\end{aligned}
$$

Application to abcd-model

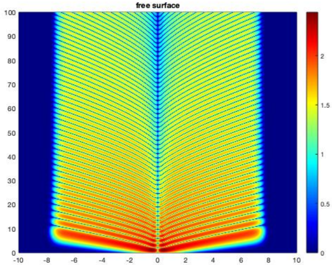
Bidirectionnel wave propagation

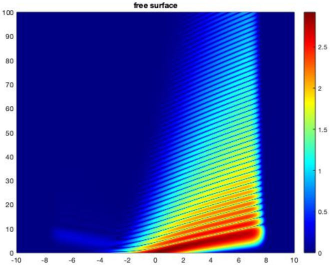
$$\eta(t=0,x) = exp(-x^2), \quad u(t=0,x) = 0.$$

In order to chose a right propagating wave we need to set:

$$u(t=0,x) = (1 - d\partial_x^2)^{-1/2}\eta(t=0,x).$$

The FFT and inverse FFT allow to calculate the fractional derivative.





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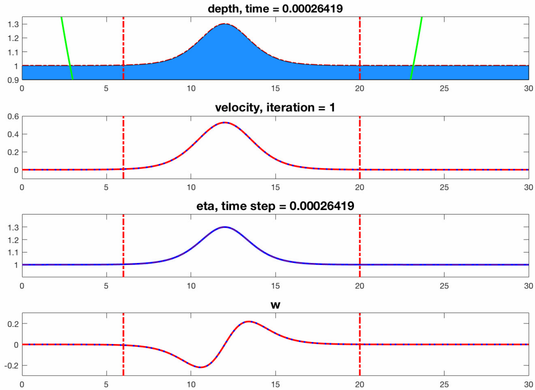
Message III

The PML is always stable when dispersive properties of the model are better suited for this technique, i.e. the condition $v_g(k)v_\phi(k) \geq 0$ is always satisfied.

Perspectives

Nonlinear test cases. Generation condition.



Conclusions

Results on PML stability for linearised water wave problem:
– Linear KdV (not always stable)
– Hyperbolic (with sources, withour sources)
– abcd-model (always stable for Boussinesq)

I. Dispersive properties of the model are important for stability of PML

II. If the dispersive properties of the model do not fits to the necessary stability condition

Change the model
Construct a non-classical PML