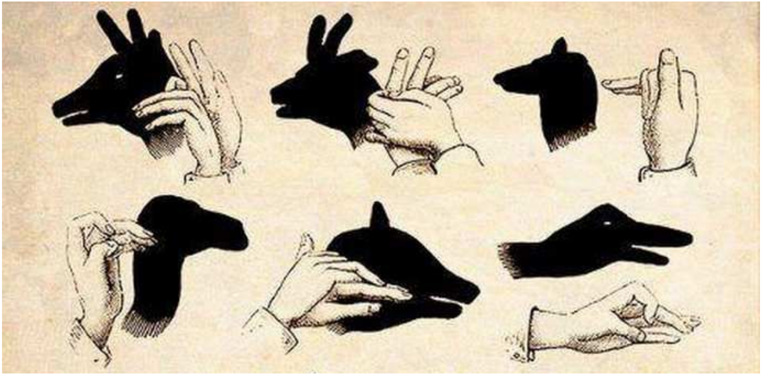
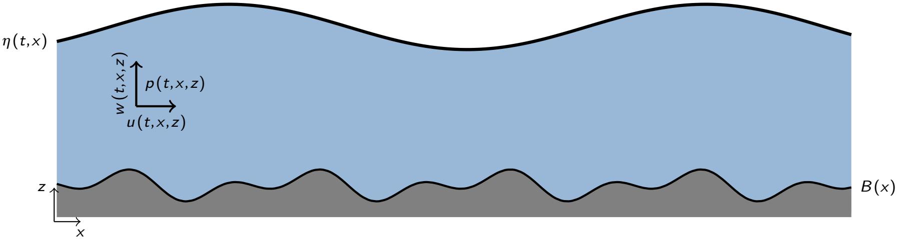
# On the projection structure of the time-discrete Green-Naghdi model

Martin Parisot - Cardamom Inria Bordeaux



**Coastal flow models and boundary conditions**
**October 24-27$^{th}$ 2022**




Incompressible free surface Euler model ($E$): for $(x,z) \in \left\{ \begin{array}{c} (x,z) \in \mathbb{R}^d \times \mathbb{R} \\ B \leq z \leq \eta \end{array} \right\}$

$$\partial_t \eta + u_\eta \cdot \nabla \eta - w_\eta = 0 \qquad \nabla \cdot u + \partial_z w = 0$$

$$\partial_t u + u \cdot \nabla u + w \partial_z u = -\nabla p \qquad u_B \cdot \nabla B - w_B = 0$$

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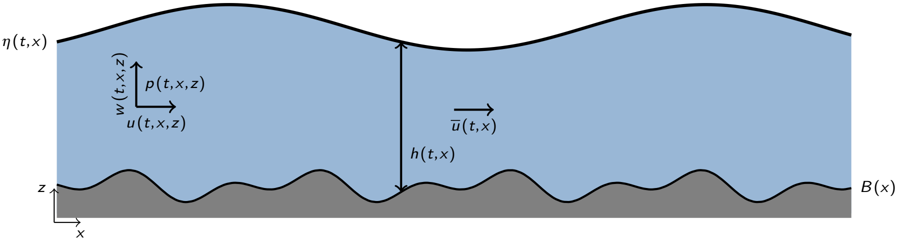


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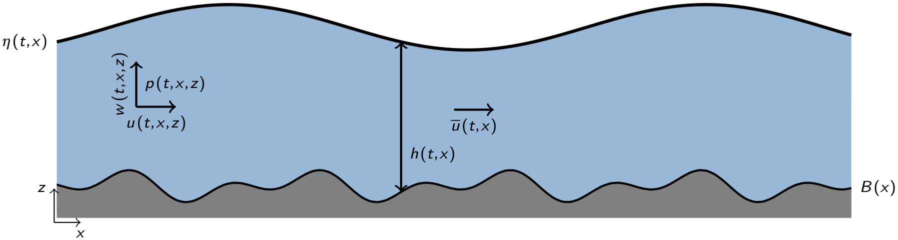
## Vertically integrated model: $\mathscr{H}_{yp}$: $\partial_z u \ll 1$

[Fernández-Nieto, Parisot, Penel, Sainte-Marie'18]

$$\partial_t h + \nabla \cdot (h \overline{u}) = 0$$

$$(\text{with } h = \eta - B)$$

$$\left(\text{and } \overline{u} = \frac{1}{h} \int_B^\eta u \, \mathrm{d}z\right)$$

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$$\partial_t w + u \cdot \nabla w + w \partial_z w = -\partial_z p - g \boxed{= -\partial_z q} \qquad \text{with } q(t,x,z) := p - (P_a + g(\eta - z))$$

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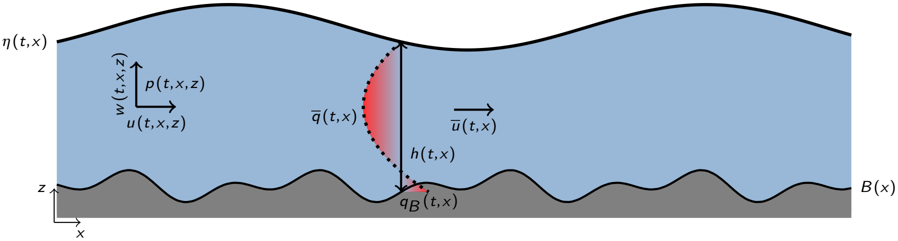


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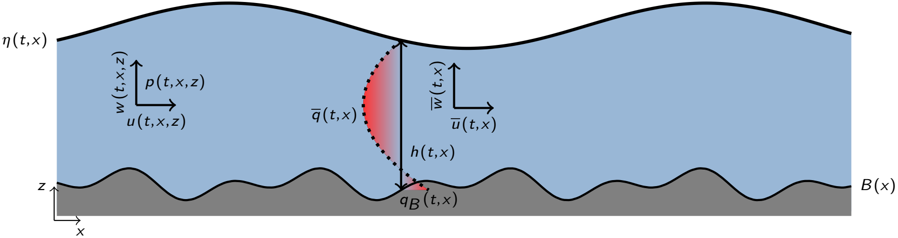
$\left(\text{with } \overline{q} = \frac{1}{h} \int_B^\eta q \, \mathrm{dz}\right)$

$\left(\text{and } q_B = q_{|_{z=B}}\right)$

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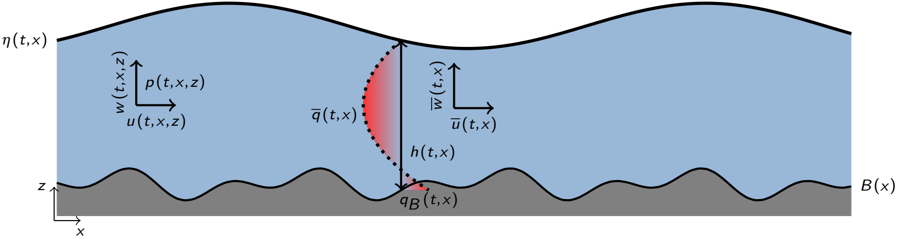


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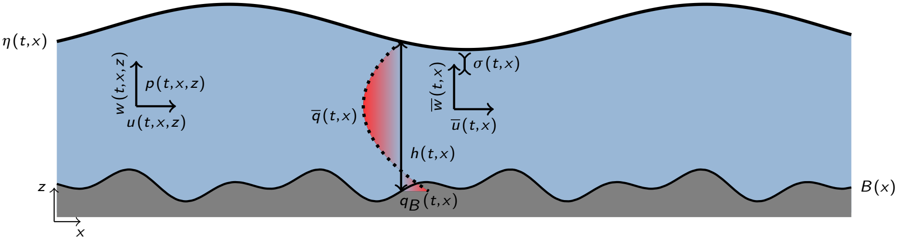


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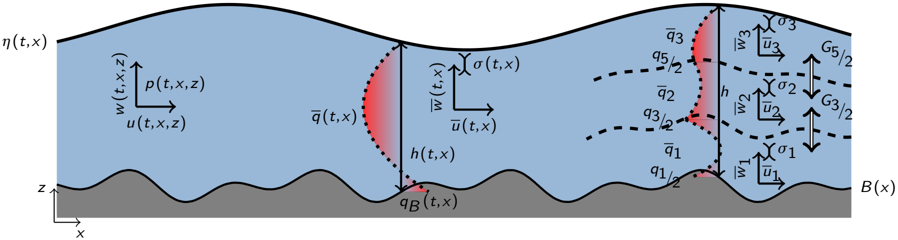
$$\partial_t (h\sigma) + \nabla \cdot (h\sigma\ \overline{u}) = \sqrt{3}(2\overline{q} - q_B) \qquad \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \left( = \pm \left( \frac{1}{h} \int_B^\eta (w - \overline{w})^2 \, \mathrm{d}z \right)^{\frac{1}{2}} \right)$$

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$$\begin{aligned} \partial_t \eta + u_\eta \cdot \nabla \eta - w_\eta &= 0 \qquad & \nabla \cdot u + \partial_z w = 0 \\ \partial_t u + u \cdot \nabla u + w \partial_z u &= -\nabla p \qquad & u_B \cdot \nabla B - w_B = 0 \\ \partial_t w + u \cdot \nabla w + w \partial_z w + g &= -\partial_z p & \end{aligned}$$

## Green-Naghdi model $(GN)$: $\mathscr{H}_{yp}$: $\partial_z u \ll 1$

[Fernández-Nieto, Parisot, Penel, Sainte-Marie'18]

$$\begin{aligned} \partial_t h + \nabla \cdot (h \overline{u}) &= 0 \\ \partial_t (h \overline{u}) + \nabla \cdot (h \overline{u} \otimes \overline{u}) + g h \nabla (h + B) &= -\nabla (h \overline{q}) - q_B \nabla B \\ \partial_t (h \overline{w}) + \nabla \cdot (h \overline{w}\ \overline{u}) &= q_B \qquad & \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \\ \partial_t (h \sigma) + \nabla \cdot (h \sigma\ \overline{u}) &= \sqrt{3} (2 \overline{q} - q_B) \qquad & \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \end{aligned}$$

Correction step $(CS)$: in a splitting $(SW)+(CS)$

Find $U\in\mathbb{A}_h$, $\overline{q}$ and $q_B$ such that for $\delta_t>0$ we have

$$U=U^*-\delta_t\Psi_h(\overline{q},q_B) \qquad \text{with} \qquad \Psi_h(Q_1,Q_2)=\frac{1}{h}\begin{pmatrix}\nabla(hQ_1)+Q_2\nabla B\\ -Q_2\\ -\sqrt{3}(2Q_1-Q_2)\end{pmatrix}$$

with the **set of admissible functions**: $\mathbb{A}_h=\left\{V=\begin{pmatrix}\overline{u}\\ \overline{w}\\ \sigma\end{pmatrix} \middle| \begin{array}{c}\overline{w}=\overline{u}\cdot\nabla B-\frac{h}{2}\nabla\cdot\overline{u}\\ \sigma=-\frac{h}{2\sqrt{3}}\nabla\cdot\overline{u}\end{array}\right\}$

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- We define the **scalar product** $\langle a,b\rangle_h:=\int_\Omega ab\ h\,\mathrm{d}x$ $\quad(\Omega=\mathbb{R}^d$ for unbounded domains$)$

  and the associated **norm** $\|a\|_h=\sqrt{\langle a,a\rangle_h}$.
- We define the spaces $L_h^2=\{a\mid\|a\|_h<\infty\}$ and $H_h^1=\left\{\frac{f}{h}\in L_h^2\mid\frac{\nabla(hf)}{h}\in L_h^2\right\}$.

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$\mathscr{H}_{yp}$: Assume that $B$ is **Lipschitz** and for any $t \geq 0$, $h(t, \bullet)$ is a **measure** (see as a **parameter**).

**Proprosition:** *The "projection" structure* *[Noelle, Parisot, Tscherpel'22]*

For any $U^*(t, \bullet) \in L^2_h$, there exist a unique $U \in \mathbb{A}_h$, $\overline{q} \in H^1_h$ and $q_B \in L^2$ solution of $(CS)$ defined by

$$U = \Pi_h[\mathbb{A}_h](U^*) \qquad \text{and} \qquad \begin{pmatrix} \overline{q} \\ q_B \end{pmatrix} = \Psi_h^{-1}\left( \frac{U^* - U}{\delta_t} \right)$$

with $\Pi_h[\mathscr{E}] : \left(L^2_h\right)^{d+2} \mapsto \mathscr{E}$ the $\langle \bullet, \bullet \rangle_h$-**orthogonal projection** on the linear subspace $\mathscr{E}$.

**Proof:**

- The mapping $\Psi_h : H^1_h \times L^2_h \mapsto \mathbb{Q}_h$ is invertible.
- For any $V \in \mathbb{A}_h$ and $\Phi \in \mathbb{Q}_h$ we have $\langle V, \Phi \rangle_h = 0$ ........ $\mathbb{Q}_h = \mathbb{A}_h^\perp$

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**Lemma**: *Regularity*

For any $V = (\overline{u}, \overline{w}, \sigma)^\perp \in \mathbb{A}_h$, we have $\overline{u} \in H(div)$ $\quad$ actually $h \nabla \cdot \overline{u} \in L^2_h$

**Green-Naghdi model ($GN$):** "Variational" formulation

Find $(h,U)\in L^2\times\mathbb{A}_h$ such that for any $(\zeta,V)\in L^2\times\mathbb{A}_h$ we have

$$\int_{\mathbb{R}} \left(\zeta\partial_t h+\nabla\cdot(h\zeta\ \overline{u})\right)\mathrm{d}x = \int_{\mathbb{R}} h\overline{u}\cdot\nabla\zeta\,\mathrm{d}x$$

$$\int_{\mathbb{R}} V\left(\partial_t(hU)+\nabla\cdot(hU\ \overline{u})\right)\mathrm{d}x = -\int_{\mathbb{R}} h\overline{u}\cdot\nabla\phi(h)\,\mathrm{d}x$$

with the potential of the conservative forces $\phi(h)=g(h+B)$.

**Proprosition:** *Energy conservation*

For smooth enough solution, the mechanical energy is preserved, i.e.

$$\partial_t\left(\int_{\mathbb{R}}\mathscr{E}(h)\,\mathrm{d}x+\frac{\|U\|_h^2}{2}\right)=0 \qquad \text{with} \qquad \partial_h\mathscr{E}=\phi(h).$$

Proof: Corollary of the energy conservation of $(SW)$ + "projection" property.

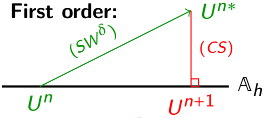
What is the use?

1. The use for a robust and efficient approximation
   - High order schemes
   - Entropy-satisfying scheme
2. The use for the modeling of boundary conditions
   - Boundary condition of the time-discrete model
   - Well-balanced scheme
   - Adaptive scheme
3. The hyperbolic "projection" model for free surface flows
   - General framework
   - Coupling of reduced models
   - Improvement of the dispersion relation
   - Perspectives

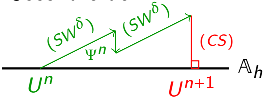
## High order schemes for $(GN)$:

[Guermond, Minev, Shen'06]

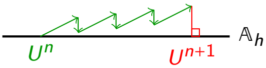
**First order:**

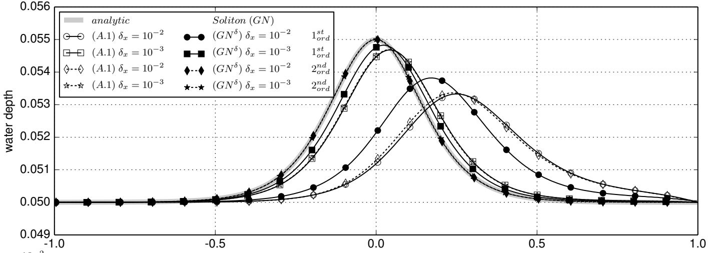


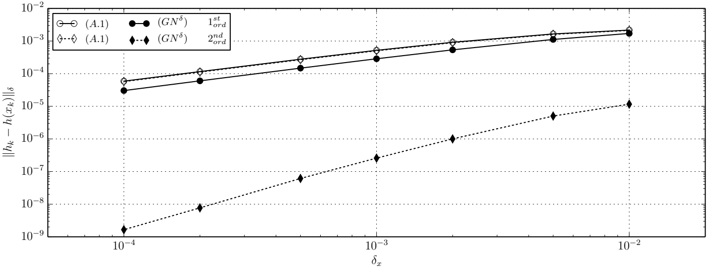

**Second order:**




**Fourth order:**



**Proprosition:** *Entropy-satisfying scheme* *[Parisot'19]*

Assume we have an **entropy-satisfying** scheme $(SW^\delta)$ of $(SW)$,

*ex: HLLC[Bouchut'04] + Improved hydrostatic reconstruction[Berthon&al'19]*

$\langle a_\star, b_\star \rangle^\delta_{h_\star} = \sum_k a_k b_k h_k \mathfrak{m}_k$

$\mathscr{E}(h_\star) = g \sum_k h_k \left(\frac{h_k}{2} + B_k\right)$

$$\mathscr{E}(h_\star) + \frac{\left|\| U^*_\star \|^\delta_{h_\star}\right|^2}{2} \leq \mathscr{E}(h^n_\star) + \frac{\left|\| U^n_\star \|^\delta_{h_\star}\right|^2}{2}.$$

Then any **projection scheme** with the same scalar product satisfies the following **entropy dissipation** law

$$\frac{\left(\| U_\star \|^\delta_{h_\star}\right)^2 - \left(\| U^*_\star \|^\delta_{h_\star}\right)^2}{\delta_t} \leq -\delta_t \left(\| \Psi_\star \|^\delta_{h_\star}\right)^2 \qquad \text{with} \qquad \Psi_\star = \frac{U^*_\star - U_\star}{\delta_t}.$$

Proof: Pythagorean theorem.

How to build a projection scheme?

**PROPROSITION:** *Entropy-satisfying scheme* [Parisot'19]

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$$\mathscr{E}(h_\star) + \frac{\left|\|U^*_\star\|^\delta_{h_\star}\right|^2}{2} \leq \mathscr{E}(h^n_\star) + \frac{\left|\|U^n_\star\|^\delta_{h_\star}\right|^2}{2}.$$

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PROOF: Pythagorean theorem.

How to build a projection scheme?

- Choose a discretization of the $L^2_h$-scalar product $\langle a_\star, b_\star\rangle^\delta_{h_\star}$,
- Choose a discretization of the set of admissible functions $\mathbb{A}^\delta_{h_\star}$,

*ex:* $\mathbb{A}^\delta_{h_\star} := \left\{\overline{w}_k = \overline{u}_k \cdot \nabla^\delta_k B_\star - \frac{h_k}{2}\nabla^\delta_k \cdot \overline{u}_\star \quad , \quad \sigma_k = -\frac{h_k}{2\sqrt{3}}\nabla^\delta_k \cdot \overline{u}_\star\right\}$

*with (in 1D)* $\nabla^\delta_k \phi_\star := \frac{\phi_{k+1}-\phi_{k-1}}{2\delta_k}$

- Determine $\mathbb{Q}^\delta_{h_\star} := \left(\mathbb{A}^\delta_{h_\star}\right)^\perp$,

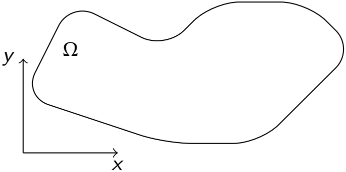
$$\mathbb{Q}^\delta_{h_\star} = \left\{\psi_{1k} = -\frac{1}{h_k}\nabla^\delta_k\left(\frac{h^2_\star}{2}\left(\psi_{2\star} + \frac{\psi_{3\star}}{\sqrt{3}}\right)\right) - \psi_{2k}\nabla^\delta_k B_\star\right\}$$

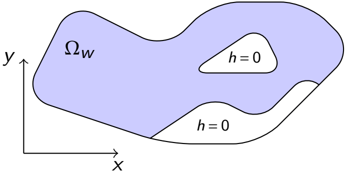
- From $\mathbb{A}^\delta_{h_\star}$ and $\mathbb{Q}^\delta_{h_\star}$ we deduce the projection scheme.

$$\alpha_k \overline{u}_k + \nabla^\delta_k(\mu_\star \overline{u}_\star) - \mu_k \nabla^\delta_k \cdot \overline{u}_\star - \nabla^\delta_k\left(\kappa_\star \nabla^\delta_\star \cdot \overline{u}_\star\right) = \beta_k$$

*with* $\alpha_\star$, $\mu_\star$, $\kappa_\star$ *and* $\beta_k$ *functions of* $h_\star$ *and* $B_\star$

What about bounded domains?



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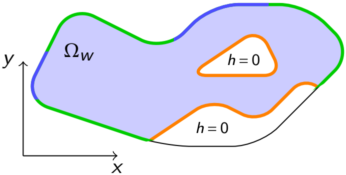


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**Proposition:** *The "projection" structure on a bounded domain* [Noelle, Parisot, Tscherpel'22]

For any $\Gamma_{\overline{q}} \subset \partial\Omega - \Gamma_h$ with finitely many connected components $\quad (\Gamma_{\overline{u}} := \partial\Omega - \Gamma_h - \Gamma_{\overline{q}})$,
any $U^*(t,\bullet) \in L^2_h$, any $\widetilde{u} \in H^{-1/2}(\partial\Omega)$ and any $\widetilde{hq} \in H^{1/2}\left(\Gamma_{\overline{q}}\right)$,
there exist a unique $U \in \mathbb{A}_{h,\Gamma_{\overline{u}}}(\widetilde{u})$, $\overline{q} \in H^1_{h,\Gamma_{\overline{q}}}\left(\widetilde{hq}\right)$ and $q_B \in L^2_h$ sol. of $(CS)$ defined by

$$U = U^r + \Pi_h\left[\mathbb{A}_{h,\Gamma_{\overline{u}}}(0)\right]\left(U^* - U^r - \delta_t \Psi_h(\overline{q}^r, 0)\right) \qquad \text{and} \qquad \begin{pmatrix}\overline{q}\\ q_B\end{pmatrix} = \Psi_h^{-1}\left(\frac{U^* - U}{\delta_t}\right)$$

for any reference functions $U^r \in \mathbb{A}_{h,\Gamma_{\overline{u}}}(\widetilde{u})$ and $\overline{q}^r \in H^1_{h,\Gamma_{\overline{q}}}\left(\widetilde{hq}\right)$.

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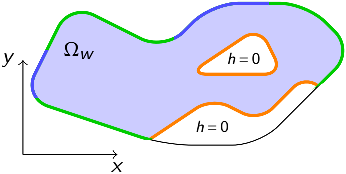


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- Boundary condition to ensure the projection: $\boxed{\left(h_i \overline{q}_i^0 \overline{u}_g^0 + h_g \overline{q}_g^0 \overline{u}_i^0\right)\cdot\mathfrak{n} = 0}$
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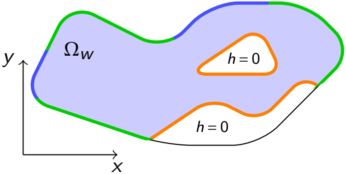


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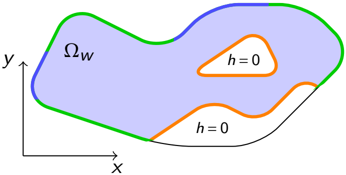


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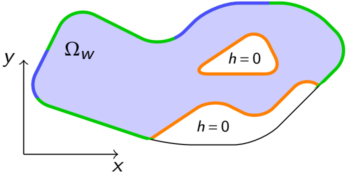


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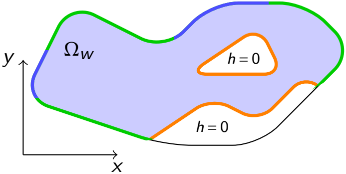


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[Kazakova, Noble'20]

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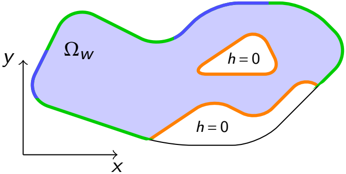


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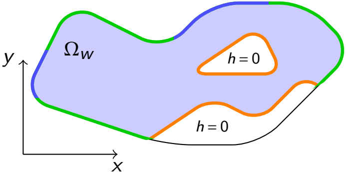


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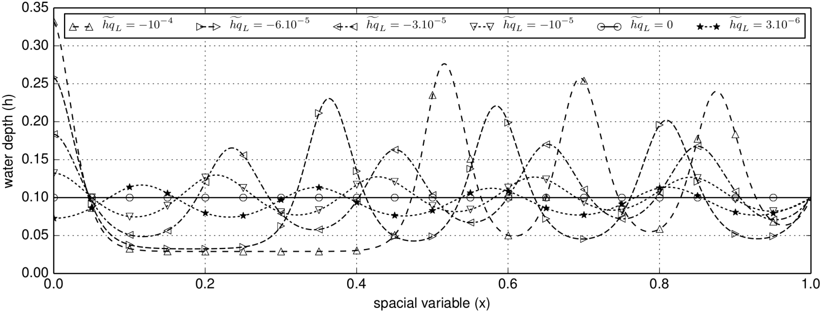


**Proposition:** *Steady strong solutions on flat bottom $B=0$* [Audusse, Parisot, Tscherpel]

For any $M$, $K \geq \frac{3}{2}h_c$, $H \in \left[\underline{h}, \frac{h_c^3}{\underline{h}^2}\right]$ and $|H'| = \sqrt{3}\left(\frac{h_c^3-\underline{h}^3}{h_c \underline{h}^2}\right)^{\frac{3}{2}} \operatorname{sech}^2(\chi)\tanh(\chi)$

with $h_c = \sqrt[3]{\frac{M^2}{g}}$, $\underline{h} = \frac{h_c^3}{4h_r^2}\left(1+\sqrt{1+8\left(\frac{h_r}{h_c}\right)^3}\right)$, $\chi = \log\left(\frac{\sqrt{h_c^3-\underline{h}^2 H}+\sqrt{h_c^3-\underline{h}^3}}{\underline{h}\sqrt{H-\underline{h}}}\right)$

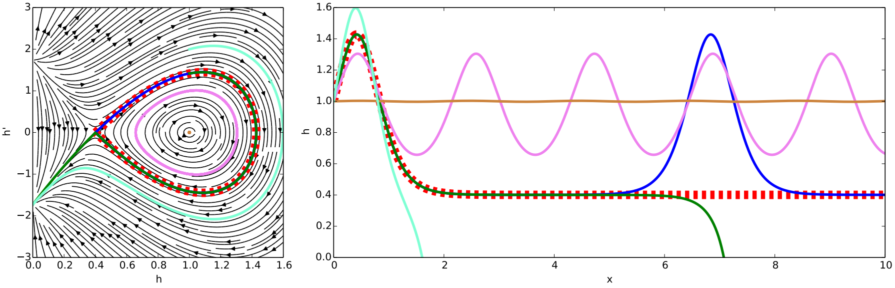
and $h_r = \frac{K}{3}\left(1+2\max_{k\in\{0,1,2\}}\left(\cos\left(\frac{1}{3}\arccos\left(1-2\left(\frac{3h_c}{2K}\right)^3\right)+\frac{2k\pi}{3}\right)\right)\right)$

there exists a **unique steady solution** of the Green-Naghdi model such that

$$h\overline{u} = M, \qquad h + \frac{\overline{u}^2+\overline{w}^2+\sigma^2+2\overline{q}}{2g} = K, \qquad h(0) = H \qquad \text{and} \qquad h'(0) = H'.$$

**Proof:**

Thanks to the Bernoulli's principle, we get the **ODE**: $h'' = \frac{|h'|^2-1}{2h} + \frac{3gh}{M^2}(K-h)$



Steady undular wave
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## WELL-BALANCED SCHEME: in 1D

[Audusse, Parisot, Tscherpel]

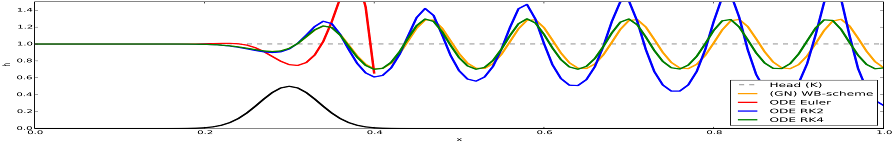
- Consider a **scheme for the ODE** to compute a **face value** in each cell.
- Use a **well-balanced scheme** for the hyperbolic part based on **face reconstruction**

$$W_k^{n*} = W_k^n - \frac{\delta_t}{\delta_x}\left(\mathscr{F}\left(\underline{W_k^n}_R, \underline{W_{k+1}^n}_L\right) - \mathscr{F}\left(\underline{W_{k-1}^n}_R, \underline{W_k^n}_L\right)\right) + \frac{\delta_t}{\delta_x}\left(F\left(\underline{W_k^n}_R\right) - F\left(\underline{W_k^n}_L\right)\right)$$

- Construct the **admissible functions** so the solutions of the ODE scheme are **inside**.

$$\text{if } \underline{W_k^n}_R = \underline{W_{k+1}^n}_L, \text{ then } \overline{w}_{k+1/2} = \underline{\overline{w}_k^n}_R \text{ and } \sigma_{k+1/2} = \underline{\sigma_k^n}_R$$

- Adapte the **scalar product** to the face constraints.



## Well-balanced scheme: in 1D

[Audusse, Parisot, Tscherpel]

- Consider a **scheme for the ODE** to compute a **face value** in each cell.
- Use a **well-balanced scheme** for the hyperbolic part based on **face reconstruction**

$$W_k^{n*} = W_k^n - \frac{\delta t}{\delta x}\left(\mathscr{F}\left(\underline{W_k^n}_R, \underline{W_{k+1}^n}_L\right) - \mathscr{F}\left(\underline{W_{k-1}^n}_R, \underline{W_k^n}_L\right)\right) + \frac{\delta t}{\delta x}\left(F\left(\underline{W_k^n}_R\right) - F\left(\underline{W_k^n}_L\right)\right)$$

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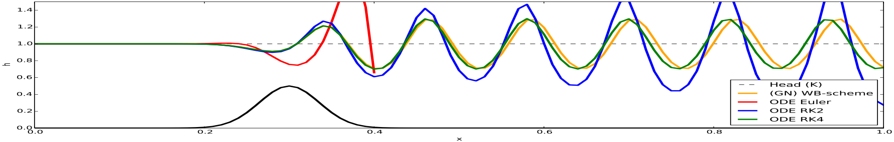


How to impose the boundary conditions?

- At the left we set $h\overline{u} = M > 0$, $h\overline{q} = \widetilde{hq}$ and $\overline{w} = \widetilde{w}$

*Thanks to the condition for the projection* $\boxed{\frac{h_i \overline{q}_g + h_g \overline{q}_i}{2} \frac{\overline{u}_g + \overline{u}_i}{2} = 0}$, *we have* $\overline{u}_g = \overline{u}_i$ *hence* $h_g = \frac{M}{\overline{u}_i}$.

- At the right we impose $K$

*We consider the last cell in the domain and we replace* $\overline{q}$ *to impose* $K$, *then we compute the ODE scheme to estimate the state in the ghost cell, finally, we use* $h_g$ *and* $\overline{u}_g$ *as boundary condition.*

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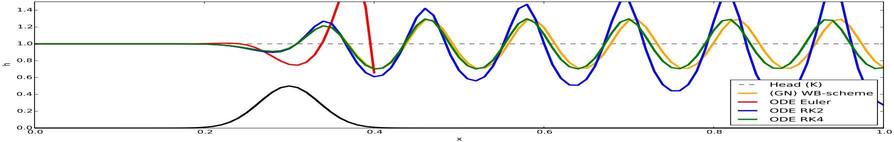


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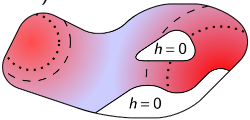
- We recover the parameters of the ODE.

$$H \text{ sol. of } h^3 + \left(\frac{2}{3g}\tilde{w}^2 + B(0) - K\right)h^2 + \frac{\widetilde{hq}}{g}h + \frac{M^2}{2g} = 0$$
$$H' = 2\left(\frac{H\tilde{w}}{M} - B'(0)\right)$$

ADAPTIVE $(GN)/(SW)$: Apply the projection only where (and when) it is **needed**




- Estimate **locally the error** on the constrains after $(SW)$.

$$\varepsilon := \sqrt{h\left(\left|\overline{w}^* + \frac{h}{2}\nabla\cdot\overline{u}^* - \overline{u}^*\cdot\nabla B\right|^2 + \left|\sigma^* + \frac{h}{2\sqrt{3}}\nabla\cdot\overline{u}^*\right|^2\right)}$$

- Find the **projection subdomain** $\Omega_p$ as a function of $\varepsilon$, i.e.

$$\Omega_p := \left\{x\in\Omega_w \mid \varepsilon > \overline{\varepsilon}\right\} \cup \Omega_\varepsilon\left(\varepsilon_0, x_\varepsilon, \sigma_\varepsilon, \ldots\right)$$ defined thanks to **empirical laws**

with $\varepsilon_0 := \int_\Omega \varepsilon\,\mathrm{d}x$, $x_\varepsilon := \int_\Omega x\frac{\varepsilon}{\varepsilon_0}\,\mathrm{d}x$, $\sigma_\varepsilon := \left(\int_\Omega (x-x_\varepsilon)^2\frac{\varepsilon}{\varepsilon_0}\,\mathrm{d}x\right)^{1/2}\ldots$

▸ Adaptive

$ex\ :\ \Omega_\varepsilon := \left\{x\in\mathbb{R}^d \textit{ such that } |x-x_\varepsilon| < C\sigma_\varepsilon\right\}$

**M2 internship:** with M. Kazolea and M. Chavent.

- deeper analysis of the empirical laws.
- domain decomposition for non-connexe projection with clustering methods.
- application to 2D/realistic cases.

Consider an hyperbolic model $$\partial_t \begin{pmatrix} H \\ U \end{pmatrix} + A\begin{pmatrix} H \\ U \end{pmatrix}\nabla\begin{pmatrix} H \\ U \end{pmatrix} = 0$$

with $H(t,x) \in \mathbb{R}_+^m$ and $U(t,x) \in \mathbb{R}_+^n$
with an underlying **energy dissipation** law: $$\partial_t \left(\mathscr{E}(H) + \langle U, U\rangle_H\right) \le 0.$$

### The unified "projection" model:

[Kazolea, Parisot]

For a given **set of admissible functions** $\mathbb{A}_H \subset L^2_H$,
find $H$, $U \in \mathbb{A}_H$ and $Q \in \mathbb{Q}_H$

$$\partial_t \begin{pmatrix} H \\ U \end{pmatrix} + A\begin{pmatrix} H \\ U \end{pmatrix}\nabla\begin{pmatrix} H \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_H(Q) \end{pmatrix},$$

with and the **duality property**: $\langle V, \Psi_H(Q)\rangle_H = 0$ for any $V \in \mathbb{A}_H$ and $Q \in \mathbb{Q}_H$.

Consider an hyperbolic model

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The unified "projection" model: [Kazolea, Parisot]

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with and the **duality property**: $\langle V, \Psi_H(Q) \rangle_H = 0$ for any $V \in \mathbb{A}_H$ and $Q \in \mathbb{Q}_H$.

Is the model well-posed for any subset $\mathbb{A}_H \subset L^2_H$ ?

- From a **numerical** point of view, there is **no impact** on the method.
- From a **analytical** point of view, it preserves the **energy dissipation law**
- From a **modeling** point of view, this opens the way to interesting possibilities.

$H$: variable density, multilayer...

$U$: layerwise, enstrophy, vorticity...

| | $H, U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| $(GN)$ | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \\ \sigma \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 & 0 \\ g & \overline{u} & 0 & 0 \\ 0 & 0 & \overline{u} & 0 \\ 0 & 0 & 0 & \overline{u} \end{pmatrix}$ | $\mathbb{A}_h^{\mathrm{GN}} = \left\{ \begin{array}{c} \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \\ \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \end{array} \right\}$ |

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| $(NH)$ [BMSmS’15] | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 \\ g & \overline{u} & 0 \\ 0 & 0 & \overline{u} \end{pmatrix}$ | $\mathbb{A}_h^{\mathrm{NH}} = \left\{ \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \right\}$ |
| $(SW)$ | $h, \overline{u}$ | $\begin{pmatrix} \overline{u} & h \\ g & \overline{u} \end{pmatrix}$ | $L_h^2$ |

| | $H, U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| $(GN)$ | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \\ \sigma \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 & 0 \\ g & \overline{u} & 0 & 0 \\ 0 & 0 & \overline{u} & 0 \\ 0 & 0 & 0 & \overline{u} \end{pmatrix}$ | $\mathbb{A}_h^{\mathrm{GN}} = \left\{ \begin{array}{c} \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \\ \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \end{array} \right\}$ |
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| “coupled” model Sinus | $h, \begin{pmatrix}\overline{u}\\ \overline{w}\\ \sigma\end{pmatrix}$ | $\begin{pmatrix}\overline{u} & h & 0 & 0\\ g & \overline{u} & 0 & 0\\ 0 & 0 & \overline{u} & 0\\ 0 & 0 & 0 & \overline{u}\end{pmatrix}$ | $\left\{ \begin{array}{ll} \mathbb{A}_h^{\mathrm{SW}} & \text{if } x \in \Omega^{\mathrm{SW}}(x,h) \\ \mathbb{A}_h^{\mathrm{GN}} & \text{if } x \in \Omega^{\mathrm{GN}}(x,h) \\ \dots & \end{array} \right.$ |

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| "coupled" model ▸ Sinus | $h,\begin{pmatrix}\overline{u}\\ \overline{w}\\ \sigma\end{pmatrix}$ | $\begin{pmatrix}\overline{u} & h & 0 & 0\\ g & \overline{u} & 0 & 0\\ 0 & 0 & \overline{u} & 0\\ 0 & 0 & 0 & \overline{u}\end{pmatrix}$ | $\left\{ \begin{array}{ll} \mathbb{A}_h^{\mathrm{SW}} & \text{if } x\in\Omega^{\mathrm{SW}}(x,h) \\ \mathbb{A}_h^{\mathrm{GN}} & \text{if } x\in\Omega^{\mathrm{GN}}(x,h) \\ \dots & \end{array} \right.$ |
| weakly non-linear Boussinesq, Abbott [Peregrine'67] | $h,\begin{pmatrix}\overline{u}\\ \overline{w}\\ \sigma\end{pmatrix}$ | $\begin{pmatrix}\overline{u} & h & 0 & 0\\ g & \overline{u} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ | $\mathbb{A}_H^{\mathrm{GN}}$ |

| | $H, U$ | $A$ | $\mathbb{A}_h$ |
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| $(SW)$ | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \\ \sigma \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 & 0 \\ g & \overline{u} & 0 & 0 \\ 0 & 0 & \overline{u} & 0 \\ 0 & 0 & 0 & \overline{u} \end{pmatrix}$ | $\mathbb{A}_h^{\mathrm{SW}} = \left\{ \begin{array}{c} \overline{w} = 0 \\ \sigma = 0 \end{array} \right\}$ |
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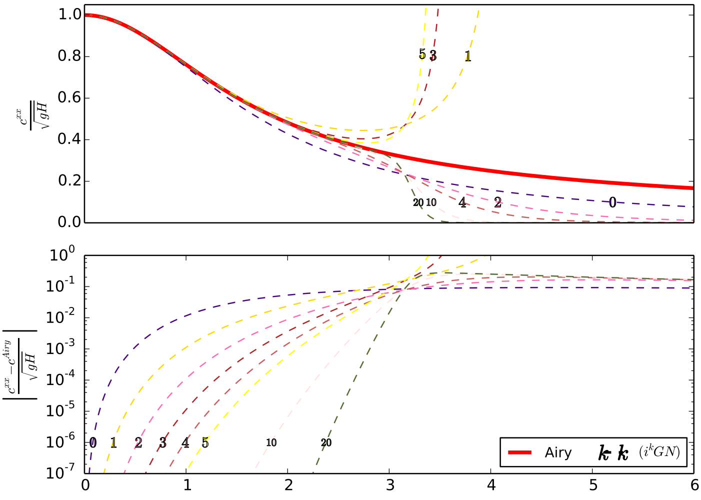
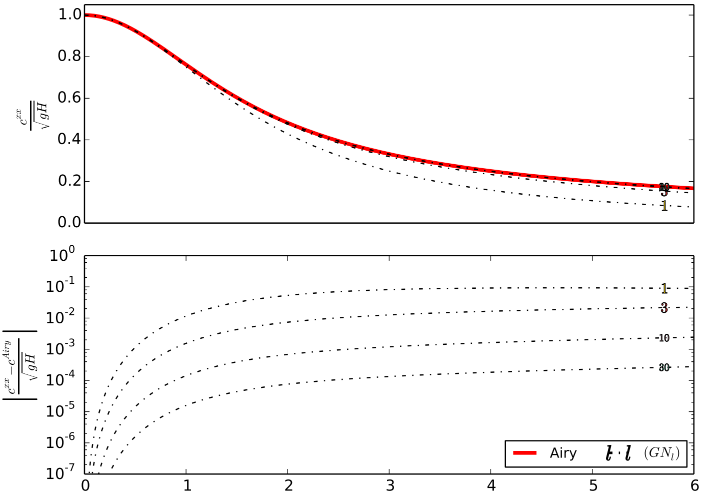
| unidirection KdV, BBM | $, \begin{pmatrix} \overline{u} \\ \overline{w} \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & 0 \\ 0 & \overline{u} \end{pmatrix}$ | $\left\{ \overline{w} = -\partial_x \overline{u} \right\}$ |

| | $H, U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| high order dispersion<br>Taylor expansion<br>[Madsen, Sørensen'92] | $h, \begin{pmatrix} \overline{u} \\ \overline{w}_0 \\ \vdots \\ \overline{w}_N \end{pmatrix}$ | $\left(\begin{array}{cc\|c} \overline{u} & h & 0 \\ g & \overline{u} & \\ \hline 0 & & \overline{u}\ \mathrm{I_d} \end{array}\right)$ | $\left\{ \overline{w}_i = (-1)^i \alpha_i h^{2i+1} \Delta^i \nabla \cdot \overline{u} \right\}$ |



| | $H,U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| high order dispersion rational fraction | $h, \begin{pmatrix} \overline{u} \\ \overline{w}_{-M} \\ \vdots \\ \overline{w}_N \end{pmatrix}$ | $\left(\begin{array}{cc\|c} \overline{u} & h & 0 \\ g & \overline{u} & \\ \hline 0 & & \overline{u}\ \mathrm{I_d} \end{array}\right)$ | $\left\{ \begin{array}{c} (-1)^i \beta_i h^{2i} \Delta^i \overline{w}_{-i} = -h\nabla\cdot\overline{u} \\ \overline{w}_i = (-1)^i \alpha_i h^{2i+1} \Delta^i \nabla\cdot\overline{u} \end{array} \right\}$ |



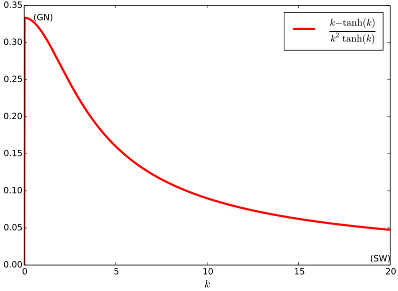
| | $H, U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| fully dispersive [Duchêne, Israwi, Talhouk'16] | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \\ \sigma \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 & 0 \\ g & \overline{u} & 0 & 0 \\ 0 & 0 & \overline{u} & 0 \\ 0 & 0 & 0 & \overline{u} \end{pmatrix}$ | $\left\{ \begin{array}{c} \overline{w} = \beta(h)\overline{u} \cdot \nabla B - \alpha(h) h \nabla \cdot \overline{u} \\ \sigma = -\gamma(h) h \nabla \cdot \overline{u} \end{array} \right\}$ |

**Proposition:** *Fully dispersive model*

Let $\alpha(h) = \widetilde{\alpha}(|\widetilde{k}\widetilde{H}|)$, $\beta(h) = \widetilde{\beta}(|\widetilde{k}\widetilde{H}|)$ and $\gamma(h) = \widetilde{\gamma}(|\widetilde{k}\widetilde{H}|)$ with $\widetilde{k}(h)$ and $\widetilde{H}(h)$.

Setting $\widetilde{\alpha}^2(k) + \widetilde{\gamma}^2(k) = \dfrac{k - \tanh(k)}{k^2 \tanh(k)}$ and $\widetilde{\beta}(k) = \dfrac{\widetilde{\alpha}(0)}{\widetilde{\alpha}(k)}$

and **if $\widetilde{k}\widetilde{H}$ correspond to the wave number**,
the model **exactly** recover the Airy's dispersion relation.



| | $H, U$ | $A$ | $\mathbb{A}_h$ |
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| fully dispersive [Duchêne, Israwi, Talhouk'16] | $h, \begin{pmatrix} \overline{u} \\ \overline{w} \\ \sigma \end{pmatrix}$ | $\begin{pmatrix} \overline{u} & h & 0 & 0 \\ g & \overline{u} & 0 & 0 \\ 0 & 0 & \overline{u} & 0 \\ 0 & 0 & 0 & \overline{u} \end{pmatrix}$ | $\left\{ \begin{array}{c} \overline{w} = \beta(h)\overline{u}\cdot\nabla B - \alpha(h) h \nabla\cdot\overline{u} \\ \sigma = -\gamma(h) h\nabla\cdot\overline{u} \end{array} \right\}$ |

**Proprosition:** *Fully dispersive model*

Let $\alpha(h) = \widetilde{\alpha}(|\widetilde{k}\widetilde{H}|)$, $\beta(h) = \widetilde{\beta}(|\widetilde{k}\widetilde{H}|)$ and $\gamma(h) = \widetilde{\gamma}(|\widetilde{k}\widetilde{H}|)$ with $\widetilde{k}(h)$ and $\widetilde{H}(h)$.

Setting $\widetilde{\alpha}^2(k) + \widetilde{\gamma}^2(k) = \frac{k - \tanh(k)}{k^2 \tanh(k)}$ and $\widetilde{\beta}(k) = \frac{\widetilde{\alpha}(0)}{\widetilde{\alpha}(k)}$

and **if $\widetilde{k}\widetilde{H}$ correspond to the wave number**,
the model **exactly** recover the Airy's dispersion relation.

$\alpha$, $\beta$ and $\gamma$ are space functions, not of the wave number.

| | $H, U$ | $A$ | $\mathbb{A}_h$ |
|---|---|---|---|
| fully dispersive [Duchêne, Israwi, Talhouk'16] | $h, \begin{pmatrix}\overline{u}\\ \overline{w}\\ \sigma\end{pmatrix}$ | $\begin{pmatrix}\overline{u} & h & 0 & 0\\ g & \overline{u} & 0 & 0\\ 0 & 0 & \overline{u} & 0\\ 0 & 0 & 0 & \overline{u}\end{pmatrix}$ | $\left\{\begin{array}{c}\overline{w} = \beta(h)\overline{u}\cdot\nabla B - \alpha(h) h\nabla\cdot\overline{u}\\ \sigma = -\gamma(h) h\nabla\cdot\overline{u}\end{array}\right\}$ |

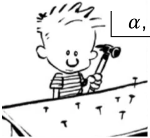
**Proprosition:** *Fully dispersive model*

Let $\alpha(h) = \widetilde{\alpha}(|\widetilde{k}\widetilde{H}|)$, $\beta(h) = \widetilde{\beta}(|\widetilde{k}\widetilde{H}|)$ and $\gamma(h) = \widetilde{\gamma}(|\widetilde{k}\widetilde{H}|)$ with $\widetilde{k}(h)$ and $\widetilde{H}(h)$.

Setting $\widetilde{\alpha}^2(k) + \widetilde{\gamma}^2(k) = \frac{k - \tanh(k)}{k^2 \tanh(k)}$ and $\widetilde{\beta}(k) = \frac{\widetilde{\alpha}(0)}{\widetilde{\alpha}(k)}$

and **if $\widetilde{k}\widetilde{H}$ correspond to the wave number**,

the model **exactly** recover the Airy's dispersion relation.



$\alpha$, $\beta$ and $\gamma$ are space functions, not of the wave number.

- Use a **local mean-square approximation**, i.e.

$$\widetilde{h}(t,\widetilde{x},x) := \widetilde{H}(t,\widetilde{x}) + \widetilde{\eta}(t,\widetilde{x})\cos\left(\widetilde{\psi}(t,\widetilde{x}) - \widetilde{k}(t,\widetilde{x})\cdot x\right)$$

with $\widetilde{H}(t,\widetilde{x})$, $\widetilde{\eta}(t,\widetilde{x})$, $\widetilde{\psi}(t,\widetilde{x})$ and $\widetilde{k}(t,\widetilde{x})$ minimizing the error

$$\int_\Omega \left|\widetilde{h}(t,\widetilde{x},x) - h(t,x)\right|^2 \mu(|x-\widetilde{x}|)\,\mathrm{d}x$$

where $\mu(d)$ is a weight (typically Gaussian or Gate).

## The advantage of using the "projection" formulation

- From a **numerical** point of view, it produces schemes in **bounded domains**:
  **robust**: entropy-satisfying or well-balanced
  **efficient**: cheaps high order and adaptive.
- From a **modeling** point of view, it opens the way to **improved models**
  **coupling**: waves breaking and boundary condition
  **dispersion**: fully dispersive model usable in the context of applications.

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## The need of the "projection" formulation

- Establish a **fully continuous** justification.

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# Thank You