Smooth branche

nonlinear Schrödinger equation

WORKSHOP Coastal flow models and boundary conditions

Toulouse, October 2022

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Partly joint works with E. Pacherie

The Nonlinear Schrödinger/Gross-Pitaevskii equation

$$i\partial_t \Psi + \Delta\Psi = \Psi(|\Psi|^2 - 1) \qquad \text{in } \mathbb{R}^2$$

- with $|\Psi| \to 1$ at spatial infinity.
- Bose-Einstein condensate, superfluidity, Nonlinear Optics, etc.
- Energy

$$E(\Psi) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\Psi|^2 + \frac{1}{2}(1 - |\Psi|^2)^2 \, dx$$

- Momentum

$$\vec{P}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^2} \langle \nabla\Psi | i\Psi \rangle \, dx$$

Hydrodynamical formulation

- Madelung's transform $\Psi = A\mathrm{e}^{i\varphi} = \sqrt{\rho}\mathrm{e}^{i\varphi}$

$$\begin{cases} \partial_t \rho + 2\nabla \cdot (\rho \nabla \varphi) = 0 \\ \partial_t \varphi + |\nabla \varphi|^2 + (\rho - 1) = \dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \end{cases}$$

$\rightsquigarrow$ free wave equation of speed $\mathfrak{c}_s = \sqrt{2}$

$$\begin{cases} \partial_t \tilde{\rho} + 2\nabla \cdot (\nabla \tilde{\varphi}) = 0 \\ \partial_t \tilde{\varphi} + \tilde{\rho} = 0 \end{cases}$$

- the momentum

$$\vec{P}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla \Psi | i \Psi \rangle \, dx = \frac{1}{2} \int_{\mathbb{R}^2} \rho \nabla \varphi \, dx$$

Travelling waves

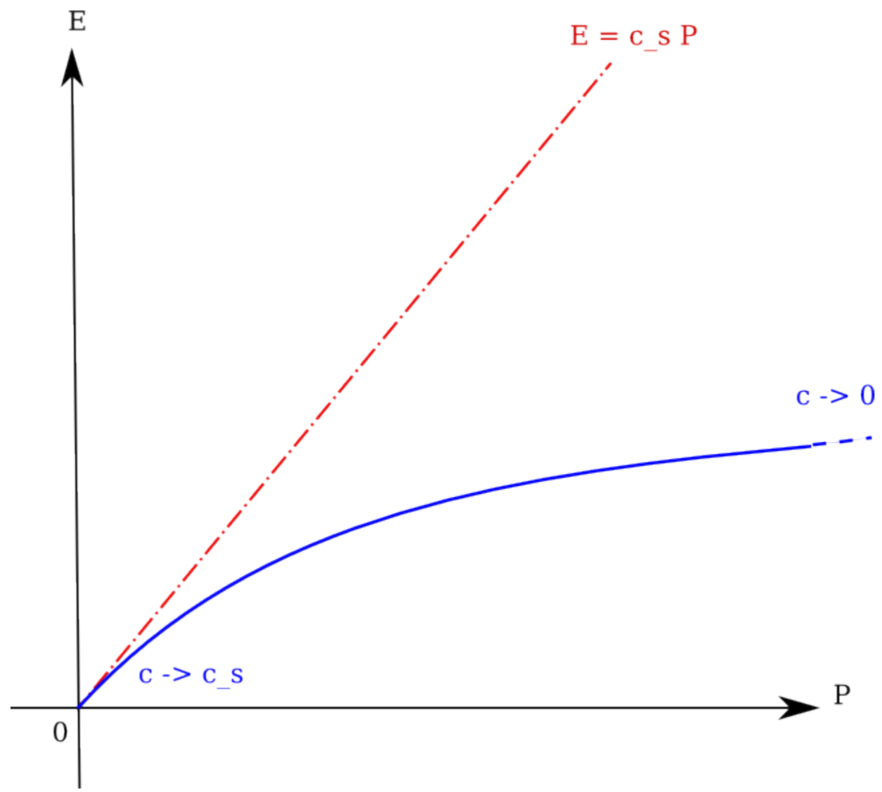
$$\Psi\Big(t,(x_1,x_2)\Big) = u(x_1 - ct, x_2), \qquad \text{speed } c$$

$$\updownarrow$$

$$ic\partial_{x_1}u - \Delta u + u(|u|^2 - 1) = 0 \qquad\qquad (\mathrm{TW}_c).$$

Numerical study of *Jones-Roberts* in 2D and 3D axi. (1982).

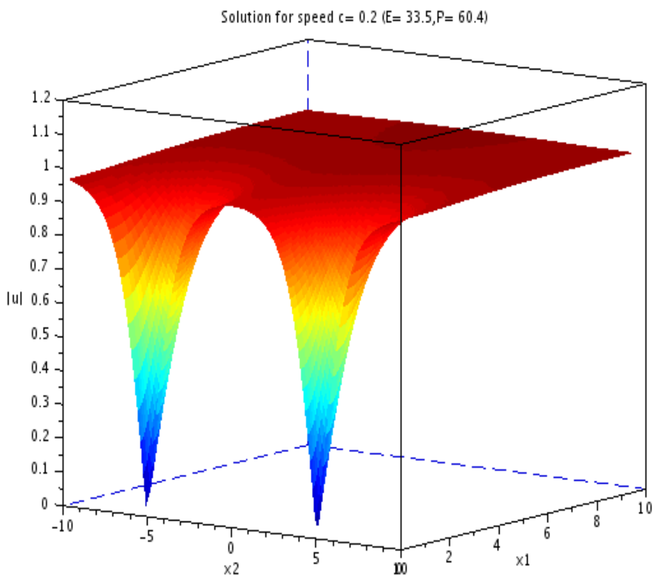
Travelling waves: the Jones-Roberts branch

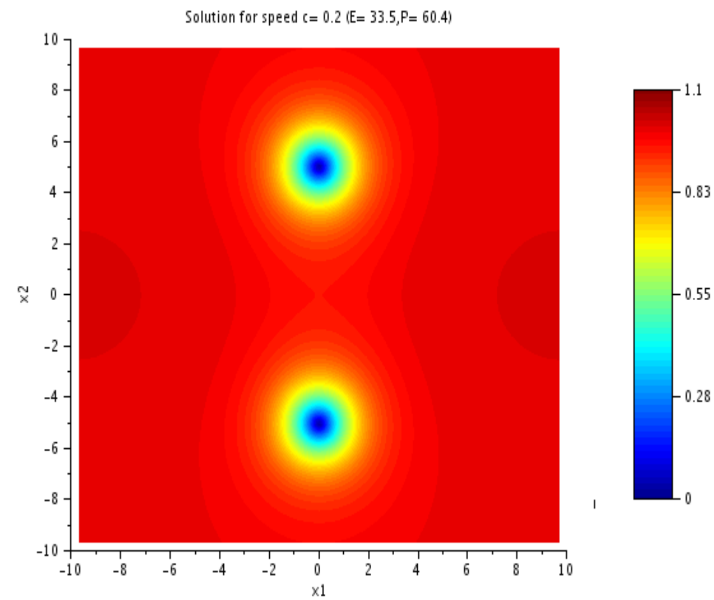


Energy E vs momentum P diagrams

Hamilton group relation $c = \frac{dE}{dP}$

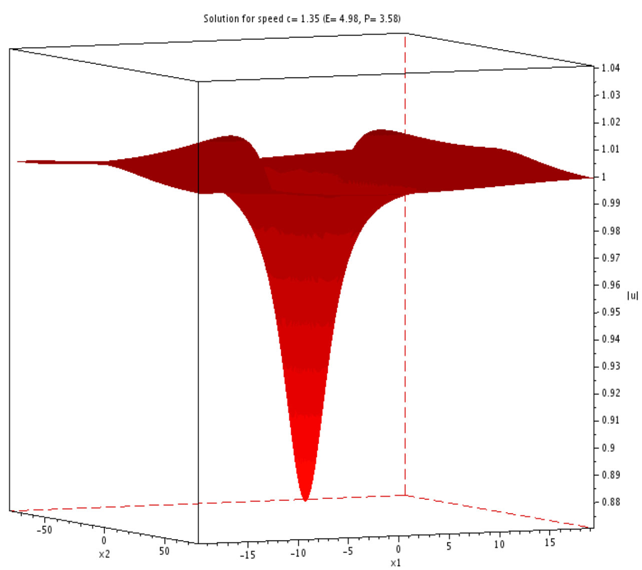
Travelling waves: the Jones-Roberts branch

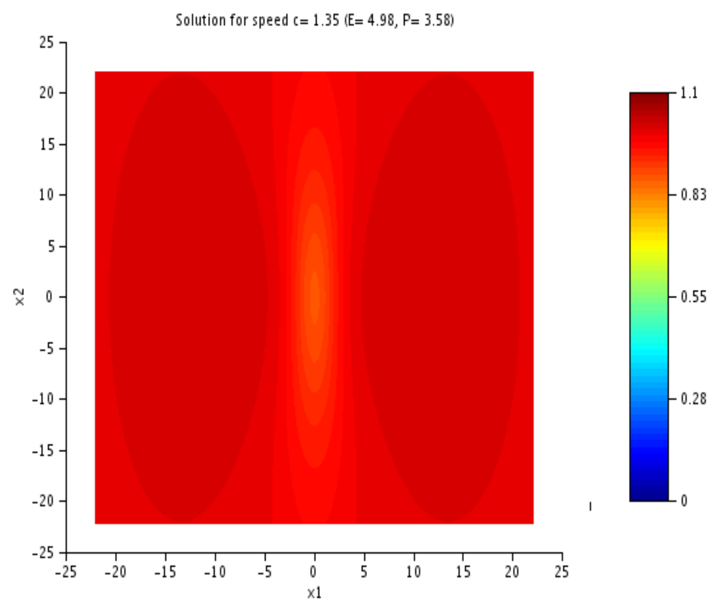




$$c = 0.2 \ll 1$$

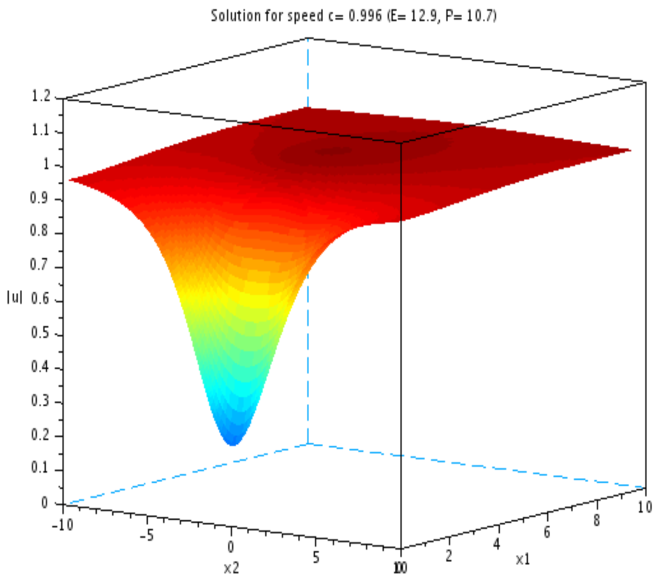
Travelling waves: the Jones-Roberts branch

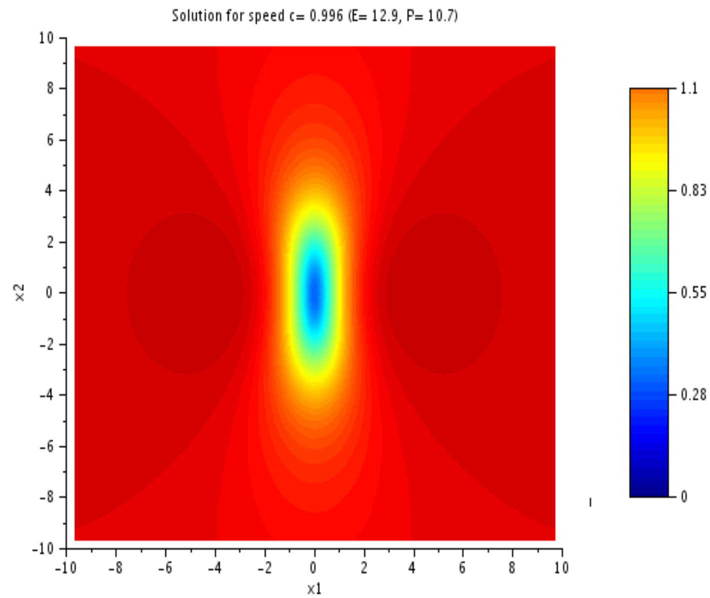




$$c = 1.35 \approx \sqrt{2}$$

Travelling waves: the Jones-Roberts branch





$$c = 1 \approx \sqrt{2}$$

Existence results for the travelling waves

• *Béthuel-Saut (1999)*: existence (2D) for small c (mountain pass lemma).

• *Béthuel-Gravejat-Saut (2009)*: existence for GP in 2D and 3D ($\min E$ at fixed P).

• *C.-Mariș (2017)*: compactness of minimizing sequences for NLS in 2D and 3D ($\min E$ at fixed P and ...) $\rightsquigarrow$ orbital stability.

• *Bellazini-Ruiz (2019)*: existence for almost all speed $c \in]0, \sqrt{2}[$.

Open questions

• Uniqueness of minimizers of E at fixed P (up to the natural invariances)?

• Are the correspondances $c \leftrightarrow P \leftrightarrow E$ bijective? Smooth? Can we parametrize the branches by the speed c?

• Do the different variational methods yield the same solutions?

$\Rightarrow$ two existence results of smooth branches for $c \ll 1$ and $c \approx \sqrt{2}$

Vortex branch: statement of the main results

Theorem 1 *C.-Pacherie (2021)* There exists $c_0 > 0$ such that

(i) for $0 < c < c_0$, $\exists$ travelling wave U_c such that

$$U_c(x) = V_1(x_1, x_2 - d_c)V_{-1}(x_1, x_2 + d_c) + o(1)$$

where $\|o(1)\|_{L^{2+0} \cap L^\infty} \to 0$, $\|\nabla o(1)\|_{L^{1+0} \cap L^\infty} \to 0$ and

$$d_c \sim 1/c;$$

(ii) for $2 < p \leqslant \infty$, the mapping

$$]0, c_0(p)[\ni c \mapsto U_c - 1 \in \{h \in L^p,\ \nabla h \in L^{p-1}\}$$

is of class $\mathcal{C}^1$.

Vortex branch: statement of the main results

(iii) for $c \in]0, c_0'[$, we have the Hamilton group relation

$$\frac{d}{dc}E(U_c) = c\frac{d}{dc}P(U_c);$$

(iv) when $c \to 0$, there holds

$$E(U_c) \sim -2\pi \ln c \qquad \text{and} \qquad P(U_c) \sim \frac{2\pi}{c}.$$

$\rightsquigarrow$ Existence result in *Liu-Wei (2020)* without smoothness in c with several vortices and in *Wei et al.* for many other models.

Vortex branch: statement of the main results

Theorem 2 *C.-Pacherie (2020)* There exists $0 < c_0' \leqslant c_0$ s. t.

- the mapping $]0, c_0'] \ni c \mapsto P(U_c) \in [P(U_{c_0'}), +\infty[$ is a smooth diffeomorphism;

- for $0 < c < c_0'$, the set of minimizer for

$$E_{\min}(P(U_c)) = \inf\{E(u) | P(u) = P(U_c)\}$$

is exactly the orbit

$$\{e^{i\alpha} U_c(\cdot - X) | X \in \mathbb{R}^2,\ \alpha \in \mathbb{R}\}.$$

$\rightsquigarrow$ Orbital stability of U_c.

$\rightsquigarrow$ the correspondance $c \leftrightarrow P \leftrightarrow E$ for minimizers is smooth and bijective for c small (P, E large).

Rarefaction pulse branch: formal derivation of KP-I

- $c = \sqrt{2-\varepsilon^2} \to \sqrt{2}$

- small amplitude - long wavelength ansatz

$$u(x) = \left(1 + \varepsilon^2 A_\varepsilon(z)\right) \exp\left(i\varepsilon\phi_\varepsilon(z)\right), \qquad z_1 \stackrel{\text{def}}{=} \varepsilon x_1, \quad z_2 \stackrel{\text{def}}{=} \varepsilon^2 x_2$$

$$A_\varepsilon \to A, \qquad \phi_\varepsilon \to \phi \qquad \text{as} \quad \varepsilon \to 0$$

$$(TW_{\sqrt{2-\varepsilon^2}}) \quad \rightsquigarrow \quad \begin{cases} \sqrt{2}A = \partial_{z_1}\phi \\ \frac{1}{2}\partial_{z_1}A - \frac{1}{2}\partial_{z_1}^3 A + 6A\partial_{z_1}A + \partial_{z_2}^2\partial_{z_1}^{-1}A = 0. \end{cases}$$

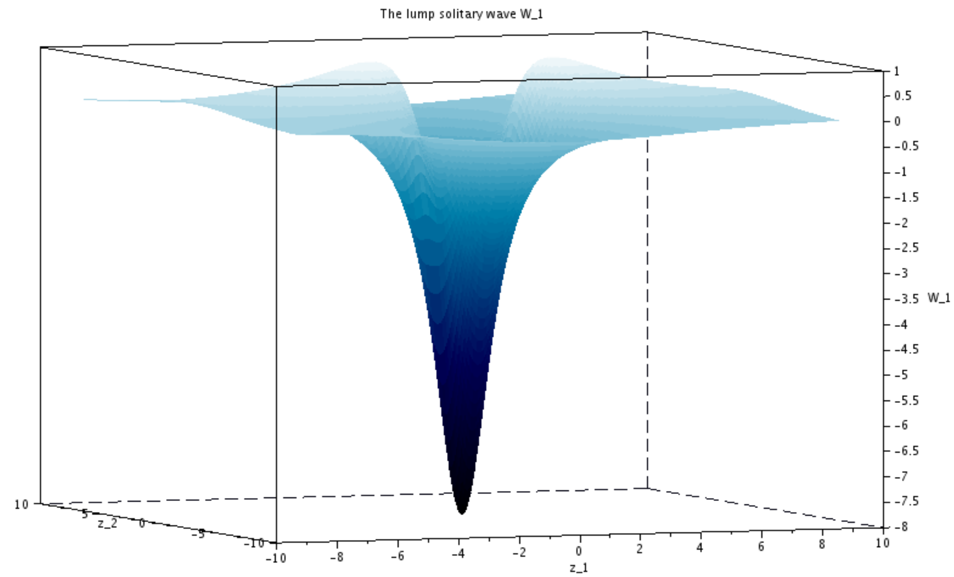
Rarefaction pulse branch: KP-I solitary waves

• Existence of ground state solitary waves *De Bouard-Saut (1996, 1997)* and orbital stability.

• The KP-I equation is integrable (Lax Pair) $\rightsquigarrow$ explicit solitary waves: rational functions called *lumps*. The first lump *Manakov-Zakharov-Bordag-Its-Matveev (1977)*

$$\mathcal{W}_1(z) = -2\frac{3 - z_1^2 + z_2^2/2}{(3 + z_1^2 + z_2^2/2)^2}.$$

Rarefaction pulse branch: KP-I solitary waves



The lump solitary wave $\mathcal{W}_1$.

Is it "the" ground state?

Rarefaction pulse branch: statement of the main results

Theorem 3 *C.* For $1 < p \leqslant \infty$ given, there exist $\varepsilon_*(p) > 0$ small and a $\mathcal{C}^1$ mapping

$$]0, \varepsilon_*(p)] \ni \varepsilon \mapsto (A_\varepsilon, \phi_\varepsilon) \in W^{1,p} \times W^{1,p+1}$$

such that:

(i) $\forall 0 < \varepsilon \leqslant \varepsilon_*(p)$,

$$U_{c(\varepsilon)}(x) \stackrel{\text{def}}{=} \Big(1+\varepsilon^2 A_\varepsilon(z)\Big) \exp\Big(i\varepsilon\phi_\varepsilon(z)\Big), \qquad z_1 = \varepsilon x_1, \quad z_2 = \varepsilon^2 x_2,$$

is a travelling wave for NLS of speed $c(\varepsilon) = \sqrt{\mathfrak{c}^2 - \varepsilon^2}$;

(ii) when $\varepsilon \to 0$,

$$\|A_\varepsilon - \mathcal{W}_1\|_{W^{1,p}} + \|\phi_\varepsilon - \mathfrak{c}\partial_{z_1}^{-1}\mathcal{W}_1\|_{W^{1,p+1}} \leqslant C(p)\varepsilon^2|\ln\varepsilon|^2 \to 0.$$

Rarefaction pulse branch: statement of the main results

(iii) for $c \in]c(\varepsilon_*(p)), \mathfrak{c}[$, we have the Hamilton group relation

$$\frac{d}{dc}E(U_c) = c\frac{d}{dc}P(U_c)$$

(iv) when $\varepsilon \to 0$, there holds

$$E(U_{c(\varepsilon)}) \sim \mathfrak{c}P(U_{c(\varepsilon)}) \approx \varepsilon,$$

$$E(U_{c(\varepsilon)}) - c(\varepsilon)P(U_{c(\varepsilon)}) \approx \varepsilon^3.$$

Rarefaction pulse branch: statement of the main results

• Variational methods: convergence to *some* ground state *Béthuel-Gravejat-Saut (2008)* and *C.-Mariș (2014)*

• Same kind of result obtained by *Liu-Wang-Wei-Yang*, with much worse error estimates

• (immediate) extension to the Euler-Korteweg model

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla(g(\rho)) = \nabla\Big(\kappa(\rho)\Delta\rho + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\Big), \end{cases} \tag{EK}$$

(NLS = EK with $\kappa(\rho) = 1/\rho$)

$\to$ Existence (variational) of travelling waves *Audiard (2017)* of small energy in 2d.

About the proofs

• Not variational, but from an implicit function type argument (or Liapounov-Schmidt reduction)

• Relies on decay estimates on the kernel

$$\mathcal{K} = \mathscr{F}^{-1}\Big(\frac{1}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}\Big)$$

$\rightsquigarrow$ *Gravejat (2004-2008)*

OK for $c \leqslant 1$:

$$|\nabla^2\mathcal{K}(x)| \leqslant \frac{C}{|x|^{1/2}(1+|x|)^{3/2}}, \qquad |\nabla^3\mathcal{K}(x)| \leqslant \frac{C}{|x|^{3/2}(1+|x|)^{3/2}}$$

Rarefaction pulse branch: estimates on the kernel

... but anisotropy when $c \approx \sqrt{2}$

Kernel in the z-variable

$$\mathcal{K}^\varepsilon(z) = \mathscr{F}^{-1}\Big(\frac{1}{\xi_1^2 + 2\xi_2^2 + (\xi_1^2 + \varepsilon^2\xi_2^2)^2}\Big).$$

Proposition *C.*

- For $|z| \geqslant 1$, $|\nabla\mathcal{K}^\varepsilon(z)| \leqslant \frac{C}{|z|}$, $|\nabla^2\mathcal{K}^\varepsilon(z)| \leqslant \frac{C}{|z|^2}$, $|\nabla^3\mathcal{K}^\varepsilon(z)| \leqslant \frac{C}{|z|^3}$.

- $\int_{D(0,1)}|\nabla\mathcal{K}^\varepsilon(z)| + |\nabla\partial_{z_1}\mathcal{K}^\varepsilon(z)| + |\partial^3_{z_1}\mathcal{K}^\varepsilon(z)|\,dz \leqslant C$ and $\int_{D(0,1)}|\partial_{z_1}\partial^2_{z_2}\mathcal{K}^\varepsilon(z)|\,dz \leqslant C\frac{|\ln\varepsilon|}{\varepsilon}$, $\int_{D(0,1)}|\partial^3_{z_2}\mathcal{K}^\varepsilon(z)|\,dz \leqslant C\frac{|\ln\varepsilon|}{\varepsilon^2}$.

About the proofs

• Not variational, but from an implicit function type argument (or Liapounov-Schmidt reduction)

• Relies on decay estimates on the kernel

$$\mathcal{K} = \mathscr{F}^{-1}\Big(\frac{1}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}\Big)$$

• Non-degeneracy results:

Rarefaction pulse branch: non-degeneracy of the $\mathcal{W}_1$ lump

Theorem 4 *Liu-Wei (2019).* If w is smooth, $w \to 0$ at infinity and

$$\partial_{z_1}^2 w - \partial_{z_1}^4 w + \partial_{z_1}^2 (w\mathcal{W}_1) + \partial_{z_2}^2 w = 0,$$

then

$$w \in \mathrm{Span}(\partial_{z_1}\mathcal{W}_1, \partial_{z_2}\mathcal{W}_1).$$

Furthermore, the linearized operator

$$\mathfrak{L} : w \mapsto w - \partial_{z_1}^2 w + w\mathcal{W}_1 + \partial_{z_2}^2 \partial_{z_1}^{-2} w$$

has exactly one negative eigenvalue.

The proof uses Bäcklund transform $\Rightarrow$ direct proof using Darboux transform ?

Rarefaction pulse branch: last difficulties

• Work with symmetric functions $\rightsquigarrow$ removes the kernel of $\mathfrak{L}$.

• Formal derivation of KP-I from NLS by expansion in ε... but *no* expansion in ε for A_ε and ϕ_ε due to the $\partial_{z_1}^{-1}$ operator (*C. (2014)*)

$\rightsquigarrow$ *no* implicit function theorem from $\varepsilon = 0$

$\rightsquigarrow$ fix point problem with ε-dependent norms taking into account the preparedness assumption $\sqrt{2}A \approx \partial_{z_1}\phi$

$\rightsquigarrow$ *necessary* (small) loss in the inversion of the linearized operator $\Rightarrow$ ansatz = approximate solution + error.

The vortex branch: vortices

Vortex of degree n: particular *stationary* solution of (GP) of the form

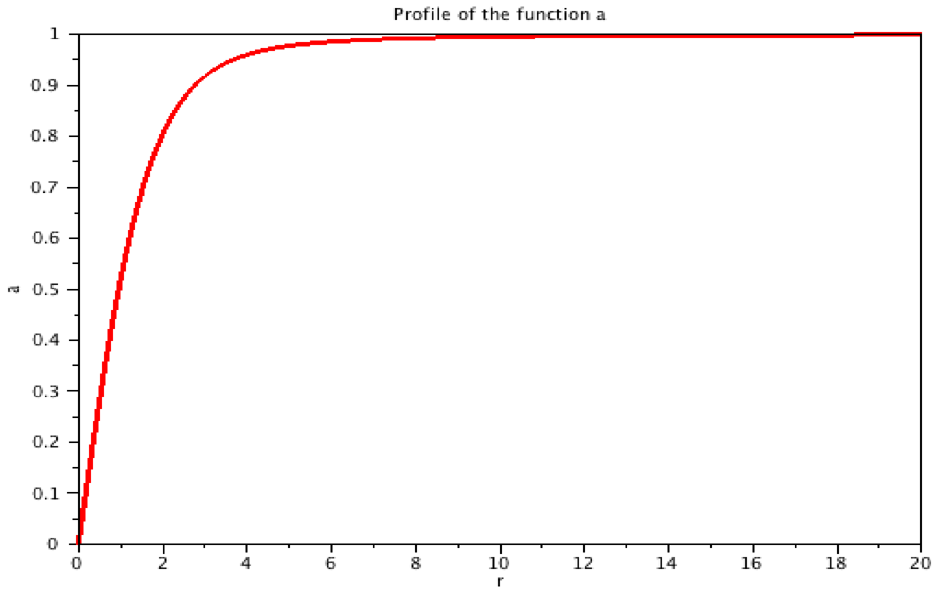
$$V_n(x) = \mathfrak{a}_n(r)\mathrm{e}^{in\theta}$$

n = degree = winding number $\in \mathbb{Z}$

$$\mathfrak{a}_n'' + \frac{\mathfrak{a}_n'}{r} - \frac{n^2}{r^2}\mathfrak{a}_n + \mathfrak{a}_\mathfrak{n}(1 - \mathfrak{a}_n^2) = 0$$

with $\mathfrak{a}_n(0) = 0$ and $\mathfrak{a}_n(+\infty) = 1$.

Vortices



Profile $\mathfrak{a}_1$ of the vortex of degree one

Vortex branch: about the proof

- Linearization near the vortex of degree one

Theorem *Del Pino-Felmer-Kowalczyk (2004)* Assume that

$$\|w\|_{V_1}^2 = \int_{\mathbb{R}^2} |\nabla w|^2 + 2\langle V_1 | w\rangle^2 + (1 - |V_1|^2)|w|^2 \, dx < \infty.$$

Then,

$$0 \leqslant \mathcal{Q}_{V_1}(w) = \int_{\mathbb{R}^2} |\nabla w|^2 + 2\langle V_1 | w\rangle^2 - (1 - |V_1|^2)|w|^2 \, dx$$

and

$$\mathcal{Q}_{V_1}(w) = 0 \iff w \in \mathrm{Span}(\partial_1 V_1, \partial_2 V_1)$$

Remark: $\mathcal{L}_{V_1}(iV_1) = 0$ but $\|iV_1\|_{V_1}^2 = +\infty$.

Vortex branch: about the proof

• Liapounov-Schmidt reduction Del Pino-Kowalczyk-Musso (2006): Ginzburg-Landau model (Dirichlet/Neumann boundary condition), and extensively used by Wei *et al.*

• Ansatz $U_c = \chi V_1 V_{-1}(1+\zeta) + (1-\chi)V_1 V_{-1} \mathrm{e}^{\zeta}$

• Norms: for some $\sigma \in]0,1[$,

$$\begin{aligned}\|\zeta\|_* = \|V_1 V_{-1}\zeta\|_{\mathcal{C}^1(\{\tilde{r}\leqslant 3\})} + \|\tilde{r}^{1+\sigma}(\zeta_1, \tilde{r}\nabla\zeta_1)\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ + \|\tilde{r}^{\sigma}(\zeta_2, \tilde{r}\nabla\zeta_2)\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\end{aligned}$$

where $\tilde{r} = \min(|x-(0,d)|, |x-(0,-d)|)$.

$\rightsquigarrow$ existence and smoothness in c.

Vortex branch: further results

Theorem *C.-Pacherie (2022)* For c small, the travelling wave U_c enjoys:

- if u " $\perp$ " to $\partial_1 U_c$, $\partial_2 U_c$ and $\partial_c U_c$, then

$$\mathcal{Q}_{U_c}(u) = \langle \mathcal{L}_{U_c}(u) | u \rangle \geqslant \alpha c^{2+0} \|u\|^2_{\mathrm{coer}},$$

with

$$\|u\|^2_{\mathrm{coer}} = \int_{\mathbb{R}^2} |U_c|^4 |\nabla(u/Q_c)|^2 + |U_c|^4 |\mathrm{Re}(u/Q_c)|^2$$

- in the linear energy space,

$$\ker(\mathcal{L}_{U_c}) = \mathrm{Span}(\partial_1 U_c, \partial_2 U_c)$$

- the mapping $c \mapsto U_c$ is of class $\mathcal{C}^2$

- $\mathcal{L}_{U_c}$ has exactly one negative eigenvalue

Vortex branch: constrained minimization

• Improvement of the coercivity for symmetric functions (constrained minimizers are symmetric)

• Proof that any minimizer satisfies the required hypotheses $\rightsquigarrow$ tools from Ginzburg-Landau theory *Béthuel-Brézis-Hélein (1994)*, *Sandier (1998)*, *Béthuel-Orlandi-Smets (2004)*: concentration of the Jacobian

$$JU = \langle i\partial_1 U | \partial_2 U \rangle = \frac{1}{2}\mathrm{curl}\langle iU | \nabla U \rangle$$

• Proof of the orbital stability indirect: use of *C.-Mariș (2017)*; coercivity norm not sufficient

Conclusions

Main results

• Existence of smooth branches of travelling waves for $c \ll 1$ with vortices and of rarefaction pulses (KP-I equation) for $c \approx \mathfrak{c}$.

• Uniqueness of constrained minimizers for large momentum

Future work

• Rigorous justification of the $(-2, +2)$ and $(-3, +3)$ branches obtained in *C-Scheid (2018)*? Need coercivity (at least non degeneracy) around V_2 and V_3.

• Construction of a branch for speeds $c \approx \sqrt{2}$ related to the multilumps solitary waves of the KP-I equation *C-Scheid (2018)* $\rightsquigarrow$ extension of the analysis of Liu-Wei.

Thank you for your attention!