Majda and ZND models for detonation: Nonlinear stability vs. formation of singularities

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Introduction I

The motivating question is: Given a shock solution $\underline{u}$ to a hyperbolic balance law of the form

$$u_t + f(u)_x = g(u),$$

with $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ and f, g suitable functions and given a small smooth perturbation v with compact support disjoint from the initial discontinuity of $\underline{u}$, does the solution to the balance law with initial data $\underline{u} + v$ form any new singularities?

Introduction II

This is a rather new topic, with many recent results such as Duchêne-Rodrigues '20, Yang-Zumbrun '20, Faye-Rodrigues '22, Blochas-Rodrigues '22.

Important physical examples of balance laws include relaxation models, such as the Saint-Venant equations [Liu '87, Johnson-Noble-Rodrigues-Yang-Zumbrun '19] in fluids.

Introduction III

Damping estimates are a powerful tool in the stability of shocks, as they allow one to control higher order derivatives in weighted Sobolev norms by lower order derivatives. These estimates were introduced in Zumbrun '04 in the context of compressible Navier-Stokes, with later works such as Z '10, JNZ '11, JNRZ '14, and RZ '16 applying them to other problems in fluids, gas dynamics, and certain abstract systems.

A damping estimate is an estimate of the form

$$\frac{d}{dt}\mathcal{E}(v) \leq -\theta\mathcal{E}(v) + C||v||^2_{L^2_\alpha},$$

where $\mathcal{E}(v)$ is an energy equivalent to $||v||^2_{H^2_\alpha}$ and L^2_α and H^2_α refer to the weighted Sobolev spaces with weight $x \to e^{\alpha|x|}$.

Introduction IV

By Sobolev embedding, a damping estimate prevents the formation of secondary shocks so long as the weighted L^2 norm of the perturbation remains under control. Compared with classical results on the formation of singularities in conservation laws such as Lax '73, John '74, Liu '79 among others, one sees that damping estimates depend importantly on properties of g.

Introduction V

We will focus on two models of combustion, an inviscid variation of the original Majda model (which we will call the Majda model for simplicity) and Zeldovich-von Neumann-Doering (ZND) models. These two models take the abstract form

$$\begin{aligned} U_t + f(U)_x &= \vec{q} k \phi(T) z, \\ z_t &= -k\phi(T) z, \end{aligned}$$

where $U, \vec{q} \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

Introduction VI

The shocks we are interested in here are called right going detonation waves, which are traveling shocks with speed $\sigma > 0$ satisfying

$$\lim_{x \to \pm\infty} (U, z)(x, t) = (U_\pm, z_\pm),$$

with $z_- = 0$ and $z_+ = 1$. We additionally want U and z to converge exponentially fast to their endstates.

Introduction VII

For the Majda model, one takes U and q to be scalars with U a lumped variable representing features of velocity/temperature/density and q the heat released by the reaction.

- Levy '92: weak entropy solutions converge to the shock wave
- Lyng-Raoofi-Texier-Zumbrun '07: nonlinear stability for viscous Majda model assuming spectral stability
- Jung-Yang-Zumbrun '21, Jung-Yao '12, Liu-Yu '99: spectral stability for viscous Majda

Introduction VIII

We write the ZND model in Lagrangian coordinates as

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ E_t + (pu)_x &= qk\phi(T)z, \\ z_t &= -k\phi(T)z, \end{aligned}$$

where v represents specific volume, u is velocity, p is pressure, and E is specific gas-dynamical energy. The form of the pressure and temperature is not particularly important.

Introduction IX

For a very brief summary of stability results for the ZND model

- Erpenbeck '60's: formal stability results for multidimensional shocks
- Zumbrun '12: ZND is linearly stable if $q \ll 1$
- Zumbrun '11: linear stability of reactive Navier-Stokes (rNS) for all small viscosities implies linear stability for ZND
- Texier-Zumbrun '12: linear stability of rNS implies nonlinear stability, however, this is not known for ZND

Damping estimate for the Majda model

The positive result for the Majda model is:

Theorem

If q is sufficiently small, then there exists a $\delta_0 > 0$, $\vartheta > 0$, $C > 0$ and $\epsilon > 0$ such that for every $(v_0, \zeta_0) \in H^{\frac{5}{2}}(\mathbb{R}) \cap H^2_\epsilon(\mathbb{R})$ supported away from 0 with $||v_0||_{H^2_\epsilon} + ||\zeta_0||_{H^2_\epsilon} < \delta_0$, the solution (u, z) to the Majda model with initial data $(\underline{u} + v_0, \underline{z} + \zeta_0)$ satisfies

$$||u(t, \cdot + \psi(t)) - \underline{u}||_{H^2_\epsilon(\mathbb{R}^*)} + ||z(t, \cdot + \psi(t)) - \underline{z}||_{H^2_\epsilon(\mathbb{R}^*)} + \\ + |\psi'(t) - \sigma| \leq C(||v_0||_{H^2_\epsilon} + ||\zeta_0||_{H^2_\epsilon})e^{-\vartheta t},$$

where $\psi(t)$ is a C^1 function denoting the position of the shock.

Sketch I

Let

$$v(t,x) = u(t,\psi(t)+x) - \underline{u}(x),$$
$$\zeta(t,x) = z(t,\psi(t)+x) - \underline{z}(x).$$

As long as the unweighted Lipschitz norms of v, ζ remain small, then one can write down evolution equations for v and ζ on $\mathbb{R}^+$ and $\mathbb{R}^-$. Formally assuming smoothness of the restrictions of v and ζ to $\mathbb{R}^\pm$, one can differentiate the evolution equations with respect to x up to two times.

Sketch II

Specifically, one has the systems

$$
\begin{aligned}
v_t &= (\psi' - f'(\underline{u} + v))v_x + (\psi' - \sigma + f'(\underline{u}) - f'(\underline{u} + v))\underline{u}' + kq\zeta, \\
\zeta_t &= (\psi' - \sigma)\underline{z}' - k\zeta + \psi'\zeta,
\end{aligned}
$$

on $\mathbb{R}^-$ and on $\mathbb{R}^+$ one has

$$
\begin{aligned}
v_t &= (\psi' - f'(\underline{u} + v))v_x, \\
\zeta_t &= \psi'\zeta_x.
\end{aligned}
$$

Sketch III

Introducing weights

$$\varrho_-(x) = \exp(-\epsilon x - \int_0^x Ce^{\theta y} dy),$$
$$\varrho_+(x) = \exp(\epsilon x + \int_0^x Ce^{-\theta y} dy),$$

for $\epsilon > 0$, $\theta > 0$ and $C > 0$ to be determined; we see that these weights induce Sobolev spaces equivalent to the weight $x \to e^{\epsilon |x|}$. The additional exponential is used to absorb terms coming from the underlying shock as well as control boundary terms. Let $E_\pm(w)$ denote the quantity

$$E_\pm(w) = \int_{\mathbb{R}^\pm} w^2(x) \varrho_\pm(x) dx.$$

Boundary terms

There are good and bad boundary terms depending on whether or not the signal is incoming or outgoing to the shock. Consider a transport equation

$$u_t - u_x = 0,$$

and consider a weighted L^2_α estimate for a solution u. By integration by parts, one finds that

$$\frac{d}{dt}\frac{1}{2}||u(t)||^2_{L^2_\alpha} = +\frac{u(0^+)^2}{2} - \frac{1}{2}\alpha\int_0^\infty u(x)^2 e^{\alpha x} dx - \\ - \frac{u(0^-)^2}{2} + \frac{1}{2}\alpha\int_{-\infty}^0 u(x)^2 e^{-\alpha x} dx.$$

Sketch IV

From here, one performs the energy estimates as usual while keeping track of the boundary terms. To finish the argument, one checks that one can choose constants $C_1, \ldots, C_{12}$ so that

$$\frac{1}{2}\frac{d}{dt}(C_1 E_-(v) + C_2 E_-(\zeta) + \ldots) \leq \frac{-\vartheta}{2}(C_1 E_-(v) + C_2 E_-(\zeta) + \ldots),$$

which leads to the final desired energy estimate after applying Sobolev embedding to ensure that the unweighted Lipschitz norm doesn't blow up.

Singularity formation for ZND

We first prove a generalization of a classical result of F. John.

Theorem

Let $A(U)$ be strictly hyperbolic with at least one genuinely nonlinear field. Consider a stationary shock solution $\underline{U}$ of

$$U_t + A(U)U_x = 0.$$

Assume $\underline{U}$ is smooth for $x \neq 0$ and that $\underline{U}$ and its derivatives converges to its endstates exponentially fast. Then for all $\theta > 0$ sufficiently small, there is a nonzero $\hat{U}(x,0)$ with C^2 norm at most θ and supported a distance $\sim \theta^{-1}$ from 0 so that if U is the solution with initial data $\underline{U} + \hat{U}$, then $\hat{U}(x,t) = U(x,t) - \underline{U}(x)$ satisfies

$$\sup_{0 \leq t < T_*} ||\hat{U}(t)||_\infty < \infty \qquad \limsup_{t \to T_*} ||\partial_x \hat{U}(t)||_\infty = \infty,$$

for some $T_* \sim \theta^{-1}$.

Remark

The assumption that the shock is stationary is harmless in the proof, as the assumptions on A are invariant under Galilean changes of coordinates. The exponential convergence of $\underline{U}$ to its endstates can be relaxed somewhat; the price paid is that one can no longer control how far the support of $\hat{U}$ needs to be from the origin.

Motivation: blowup for Burgers equation

For Burgers equation, one can show blowup for all nonzero compactly supported data by using the method of characteristics. Along characteristics, Burgers equation takes the form

$$\frac{du}{dt} = 0,$$

and the derivative $w := u_x$ is subject to

$$\frac{dw}{dt} = -w^2,$$

which blows up in finite time.

John '74 adapts this argument to systems.

Sketch I

We will modify John's argument to suitable variable coefficient systems.

For the shock $\underline{U}$, there is a radius $R > 0$ such that for $|x| \geq R$, the shock is smooth and small enough. So we will choose the support of our initial perturbation $\hat{U}$ so far to the left so that on the desired finite time of existence T, the support of the perturbation remains entirely in the region where $|x| \geq R$ (which can be done by finite propagation speed).

Sketch II

Letting U be the solution with initial data $\underline{U} + \hat{U}$ and subtracting the equation for $\underline{U}$ leads to a system of the form

$$\hat{U}_t + (A(\underline{U} + \hat{U}) - A(\underline{U}))\underline{U}_x + A(\underline{U} + \hat{U})\hat{U}_x = 0.$$

Applying the fundamental theorem of calculus twice and some rearranging leads to

$$\hat{U}_t + (A(\hat{U}) + B(x, \hat{U}))\hat{U}_x = G(x, \hat{U})\hat{U},$$

where for $|x| \geq R$ $B(x, \hat{U})$ and $G(x, \hat{U})$ decay exponentially in x as do all of their derivatives.

Sketch III

John's argument relies on the method of characteristics, and for the evolution equation of the perturbation; the evolution along the i-th characteristic takes the form

$$\frac{d\hat{U}}{dt} = \sum (\lambda_i - \lambda_k) w_k \xi^k - G(x, \hat{U})\hat{U},$$

where the λ_i are the eigenvalues of $\mathcal{A}(x, \hat{U}) := A(\hat{U}) + B(x, \hat{U})$, the ξ^k are the corresponding right eigenvectors, and the w_k are defined by

$$w_k = \eta_k \cdot \hat{U}_x,$$

where η_k is the left eigenvector.

Sketch IV

The evolution of w_i along the i-th characteristic is given by

$$\frac{dw_i}{dt} = \sum_{k,m} \gamma_{ikm} w_k w_m + \sum_k \zeta_{ik} w_k + \kappa_i \hat{U},$$

where each coefficient is known.

Sketch V

The i-th characteristic field is genuinely nonlinear precisely if $\gamma_{iii} \neq 0$, so the core insight of John's argument is that provided the other w_k remain small along some i-th characteristic where w_i satisfies

$$w_i(y,0) = \max_k \sup_x |w_k(x,0)|,$$

for some y, then the perturbation blows up in C^1 norm in finite time by comparison with a suitable Riccati equation.

Sketch VI

The final step is to do a careful analysis of the characteristic equations along each characteristic to show that $||\hat{U}||_\infty$ remains finite until the blowup and that each w_k for $k \neq i$ remains small along the i-th characteristics.

Application to ZND

To apply the blowup theorem to ZND, note that if the perturbation of $\underline{z}$, $\hat{z}$, starts identically zero; then because the ignition function is piecewise constant the perturbation $\hat{z}$ remains zero forever. Effectively, this allows us to eliminate z from the ZND model which reduces the system to gas dynamics.

The same strategy to apply the blowup theorem for ZND will also work to show blowup for the Majda model as well, which shows that the weight is crucial in the stability result; since the perturbation we construct is *very* large in the weighted norm, the weight is what allows the stability result to hold.

Lack of damping for ZND

For the ZND model, there are no damping estimates with a uniform weight on each coordinate. This is because there is an acoustic mode which is traveling to the left.

Proposition

Consider a system of the form

$$u_t + (A(u) + B(x,u))u_x = G(x,u)u,$$

as in the proof of the blowup theorem. Suppose further that the smallest eigenvalue, Λ_n of $A(0)$ is negative. Then for all $\alpha > 0$, then there exists $\varepsilon_0 > 0$ so that for all $0 < \varepsilon < \varepsilon_0$ and all $\delta > 0$ there is a smooth compactly supported initial data u_0 with $||u_0||_{H^2_\alpha} \leq \delta$ and $||u(t)||_{H^2_\alpha} > \epsilon$ for some time in the interval of existence.

Sketch I

Let ξ_n denote the right eigenvector associated to Λ_n and choose ϕ smooth and compactly supported on $[-1, 0]$. Then let

$$v_p := c_p e^{-\alpha p} \phi(\cdot + p)\xi_n,$$

where $c_p > 0$ is a sequence of numbers chosen to go to 0 as $p \to \infty$. The sequence v_p then goes to zero in H^2_α as $p \to \infty$. Let u_p be the solution to the system with initial data v_p.

Sketch II

Let η_n denote the corresponding left eigenvector to Λ_n and consider the evolution equation for the scalar quantity $w_p := \eta_n \cdot u_p$ given by

$$(w_p)_t + \Lambda_n (w_p)_x = \eta_n (A(0) - A(u_p) - B(x, u_p))(u_p)_x - \eta_n G(x, u_p) u_p.$$

Call the right hand side $N(u_p)$.

Sketch III

By contradiction, for each $\varepsilon > 0$ u_p stays in the ball of radius ε centered at 0 in H^2_α. In particular, this is a global solution to the equation. Applying the Duhamel formulation, we can write w_p as

$$w_p(x, T) = w_p(x - \Lambda_n T, T) + \int_0^T N(u_p)(x - \Lambda_n(T - s), s)ds,$$

for $T \geq 0$.

Sketch IV

From here, the plan is to show that the Duhamel term remains small enough up to time T for T large enough (depending on p), as one can check that the linear term is bounded from below (in L^2_α) by

$$||\phi||_{L^2_\alpha} e^{-\Lambda_n T}.$$

Key difference

On the right of the shock, all modes in both models are incoming which are damped by the exponential weight.

On the left side of the shock, both models have an outgoing damped mode, namely $\hat{z}$. So long as the weight is not too strong, this allows for decay.

The key difference between the Majda and ZND models is that the ZND model has an outgoing *undamped* mode on the left of the shock; causing the damping estimate to fail.

Aside: $W^{1,p}$ damping

In the case of conservation laws, one does not have access to unweighted $W^{1,p}$-damping estimates for $p > 1$. What underlies this observation are results of Bressan and coauthors about well-posedness in BV in conjunction with results of Liu regarding convergence to N-waves. The core of the argument is that for any interval I and solution u in $W^{1,p}$ one can bound the variation of u on I by using Hölder's inequality

$$V(u, I) \leq |I|^{1-\frac{1}{p}} ||u_x||_{L^p}.$$

The convergence to N-waves gives access to a point (x_*, t_*) for which the variation of u remains bounded away from 0 on any interval centered at x_*, which means that the $W^{1,p}$ cannot be damped.

Open Problems

- What does nonlinear stability look like for systems, like ZND, which do not admit damping estimates?
- Does the solution blow up in the weighted norm, and if so, what types of blowup can occur?
- For compactly supported initial data, how much longer does the $W^{1,p}$-norm for $p > 1$ take to blow up than the $W^{1,\infty}$-norm?
- How does one detect formation of singularities in $W^{1,1}$?