The boundary value problem for the Schrödinger equation

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Plan

Consider the Schrödinger equation on the half space $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^+$:

$$\begin{cases} i\partial_t u + \Delta u = f, \ (x, y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}_t^+, \\ u|_{t=0} = u_0, \\ B(u|_{y=0}, \partial_y u|_{y=0}) = g. \end{cases}$$

The main result is

Theoreme

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Moreover global Strichartz estimates hold

$$\|u\|_{L^p B^s_{q,2}} \lesssim \|u_0\|_{H^s} + \|g\|_{\mathcal{H}^s} + \|f\|_{B^s}.$$

with $2/p + d/q = d/2$, $p > 2$

A model case : Dirichlet boundary conditions : by Fourier(-Laplace) transform the solution of the pure BVP satisfies

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"Hence"

$$\widehat{u} = e^{-y\sqrt{|\eta|^2+\tau}}\widehat{g},$$

letting $\gamma \to 0$,

$$\boxed{\widehat{u}(\eta, y, \delta) = e^{-y\sqrt{|\eta|^2+\delta}}\widehat{g}(\eta, \delta).}$$

(where $\sqrt{}$ is the appropriate square root).

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For general boundary conditions, the solution is still (should be) of the form

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$$\widehat{\partial_y u}(\eta,0,\delta)=-\widehat{u}(y,0,\delta)\sqrt{|\eta|^2+\delta}.$$

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Consider boundary conditions of the form

$$\widehat{Bu} = a(\eta,\delta)\widehat{u}|_{y=0} + b(\eta,\delta)\widehat{\partial_y u}|_{y=0},$$

then

$$\boxed{\widehat{u}(\eta,0,\delta) = \frac{\widehat{Bu}}{a+b\sqrt{|\eta|^2+\delta}}}.$$

Définition (admissible boundary conditions)

The boundary conditions are said to satisfy the uniform Kreiss-Lopatinskii condition when a, b are holomorphic, homogeneous of respective order $0, -1$:
$a(\lambda\eta, \lambda^2\delta) = a(\eta, \delta)$, and

$$\exists\, \alpha, \beta > 0 : \ \alpha \leq D(B) := |a - b\sqrt{|\eta|^2 + \tau}| \leq \beta.$$

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Relevant examples :

1. Dirichlet, $a = 1, b = 0$,
2. Neuman (forced) $a = 0,\ b = 1/\sqrt{|\eta|^2 + \tau}$,
3. Transparent $a = 1,\ b = -1/\sqrt{|\eta|^2 + \tau}$.

The uniform Kreiss-Lopatinskii condition ensures that it is essentially enough to treat the Dirichlet case.

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By analogy with the Cauchy problem, we want the solution to be C_tL^2, it is explicitly given

$$u(x,y,t)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} e^{i(x\cdot\eta+\delta t)-y\sqrt{|\eta|^2+\delta}}\widehat{g}(\eta,\delta)d\delta d\eta.$$

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Consider only the part of the integral where $|\eta|^2+\delta<0$, and set $\xi=\sqrt{-|\eta|^2-\delta}$,

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}+} e^{i(x\cdot\eta+y\xi)-i(|\eta|^2+\xi^2)t} \widehat{g}(\eta,-|\eta|^2-\xi^2)\, 2\xi \, d\xi d\eta$$

$$\boxed{= e^{it\Delta}\mathcal{F}^{-1}\left(2\xi\,\widehat{g}(\eta,-|\eta|^2-\xi^2)1_{\xi\geq 0}\right).}$$

This part is exactly the Schrödinger evolution operator for a Cauchy problem, hence is C_tL^2 iff the corresponding initial condition is L^2.

Définition

The space $\mathcal{H}^0(\mathbb{R}^d)$ is

$$\left\{g \in \mathcal{S}' : \int_{\mathbb{R}^d} \widehat{g}(\eta, \delta)\sqrt{||\eta|^2 + \delta|}d\eta d\delta < \infty\right\}$$

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More generally, $\mathcal{H}^s$ is

$$\left\{g\in\mathcal{S}':\ \int_{\mathbb{R}^d}\widehat{g}(\eta,\delta)\sqrt{||\eta|^2+\delta|}(|\eta|^2+|\delta|)^s d\eta d\delta<\infty\right\}$$

For the amateurs, $\mathcal{H}^s$ is related to Bourgain spaces, $\mathcal{H}^s=X^s_{1/4}\cap X^0_{s/2+1/4}$, it corresponds to functions such that $e^{-it\Delta'}g\in H^{1/4}_tH^s_x\cap H^{s/2+1/4}_tL^2_x$.

Various properties of $\mathcal{H}^s$ spaces are easily obtained :

- $\mathcal{H}^s(I \times \mathbb{R}^{d-1})$, $\mathcal{H}_0^s(I \times \mathbb{R}^{d-1})$ are defined the usual way,
- There exists explicit extension operators,
- The trace at fixed t is continuous $\mathcal{H}^s(\mathbb{R}_t \times \mathbb{R}^{d-1}) \to H^{s-1/2}(\mathbb{R}^{d-1})$,
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- There holds $[\mathcal{H}^{s_1},\mathcal{H}^{s_2}]_\theta=\mathcal{H}^{\theta s_2+(1-\theta)s_1}$,
- But the critical case $[\mathcal{H}^0_0,\mathcal{H}^1]_{1/2}$ is (as always) more subtle, it defines the $\mathcal{H}^{1/2}_{00}$ space, which involves the condition (when $I=\mathbb{R}^+$)

$$\int_0^\infty\int_{\mathbb{R}^{d-1}}\frac{|e^{-it\Delta'}g|^2}{t}dxdt<\infty.$$

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The linear problem is simply solved by superposition, solving the Cauchy problem (with trace estimates), then the boundary value problem:

$$\begin{cases} i\partial_t v + \Delta v = f, \\ v|_{t=0} = u_0, \end{cases} \quad (x,y) \in \mathbb{R}^d, \begin{cases} i\partial_t w + \Delta w = 0, \\ w|_{t=0} = 0, \\ w|_{\partial\Omega} = g - v|_{\partial\Omega}. \end{cases} \quad (x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}^+.$$

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For general (smooth) u_0, g, we do not have $g - v|_{\partial\Omega} \notin \mathcal{H}^s_0$ as soon as $s \geq 1/2$, it requires the

First order compatibility condition

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Leads to (at best) solutions in $C_t H^s$, $s < 2$.

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The $s = 1/2$ sharp compatibility condition is somehwat more surprising.

Proposition

If $(u_0, g) \in H^{1/2} \times \mathcal{H}^{1/2}$, *and*

$$\iint_{R^{d-1}\times\mathbb{R}^+} \frac{|u_0(x,\sqrt{t}) - e^{-it\Delta'}g(x,t)|^2}{t} dxdt < \infty,$$

then the solution of the IBVP is $C_t H^{1/2}$.

The $\sqrt{t}$ is classical, and due to the anisotropy of the equation.
Higher order compatibility conditions would involve f, for example the second order compatibility condition $\partial_t u|_{y=0}|_{t=0} = \partial_t u|_{t=0}|_{y=0}$ leads to

$$\partial_t g|_{t=0} = i(-f|_{t=0} + \Delta u_0)|_{y=0}.$$

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Consequences for nonlinear problems:

Theoreme

The problem

$$\begin{cases} i\partial_t u + \Delta u = |u|^a u, \\ u|_{t=0} = u_0 \in H^1, \\ u|_{y=0} = g \in \mathcal{H}^1, \end{cases}$$

has a unique maximal solution if $a < 4/(d-2)$.

Less standard:

Theoreme

If moreover the data are small, the solution is global and scatters in the following senses:

- *There exists* $\varphi \in H_0^1$ *such that* $\|u(t) - e^{it\Delta_D}\varphi\|_{H^1} \to 0$
- *There exists* $\varphi' \in H^1$ *such that* $\|u(t) - \Phi(0, t, g, \varphi')\|_{H^1} \to 0$, *where* Φ *is the linear flow associated to the data* φ', g.