

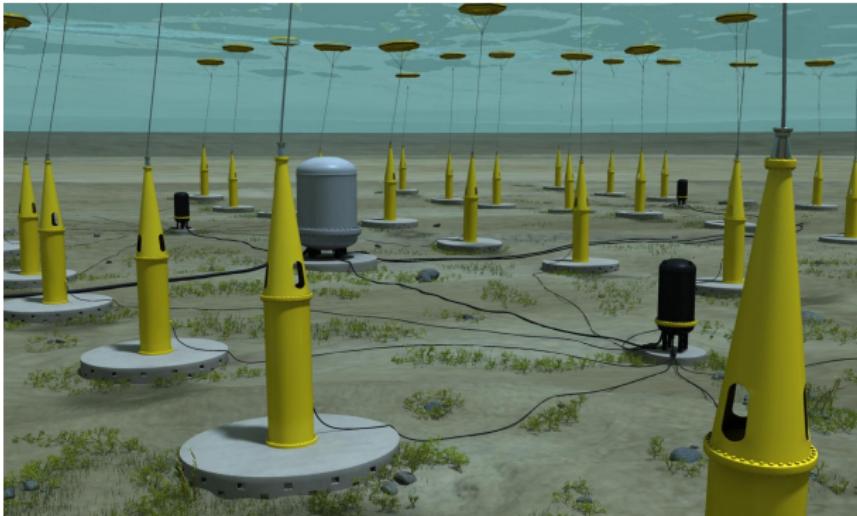


WAVES-FLOATING STRUCTURES INTERACTIONS IN BOUSSINESQ REGIME

Toulouse, NABUCO | G. Beck, D. Lannes L. Weynans | October 2022

WEC : WAVE ENERGY CONVERTER

Green washing slide



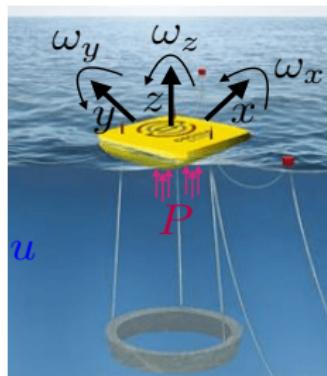
Full Euler system (u, P)

with free boundary.

+ Newton equations

$$Z := (x, y, z, \omega_x, \omega_y, \omega_z) \in \mathbb{R}^6$$

- Too costly



WEC : WAVE ENERGY CONVERTER

Engineering washing slide : Full Euler vs. Cummins

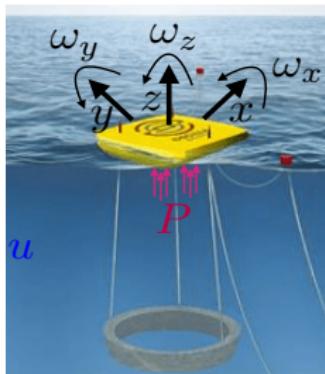
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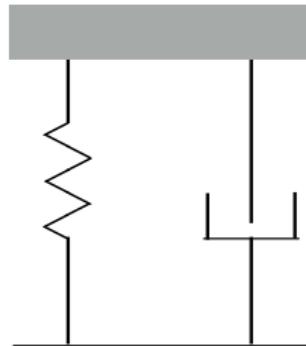


Cummins' Equation

$$M\ddot{Z} + AZ + K *_t Z = 0$$

Linear integro-diff ODE

■ Meaning ?



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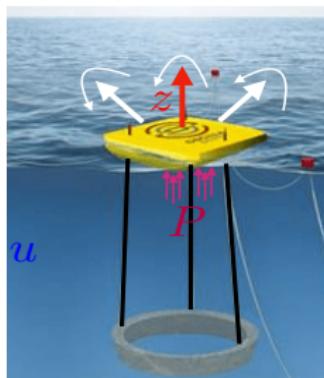
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■ Heave Motion $\delta = z - z_{\text{eq}}$

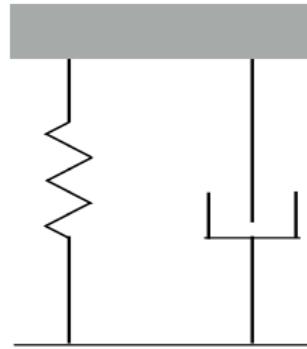
$$m\ddot{\delta} = -mg + \int(P - P_{\text{atm}})$$

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WEC : WAVE ENERGY CONVERTER

Math washing slide : Boussinesq model

- Dimensionless **Shallowness** and **Nonlinearity** parameters

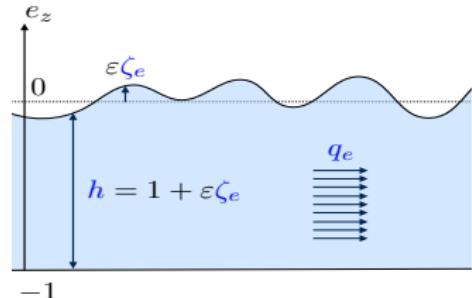
$$\kappa = \frac{(\text{depth})}{(\text{transversal scale})} \quad \text{and} \quad \varepsilon = \frac{(\text{waves amplitude})}{(\text{depth})}.$$

- Weakly dispersive $\kappa \ll 1$,

Weakly non-linear $\varepsilon = O(\kappa^2)$

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{1}{h} q^2 \right) + h \partial_x \zeta = 0 \\ h = 1 + \varepsilon \zeta \end{cases}$$

Precision : $O(\varepsilon \kappa^2, \kappa^4)$



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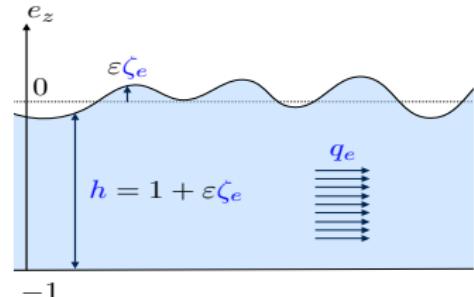
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- Non-linear Shallow Water (NSW)

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \varepsilon \partial_x \left(\frac{1}{h} q^2 \right) + h \partial_x \zeta = 0 \end{cases}$$

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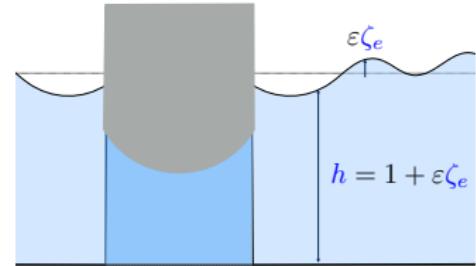
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Modeling : [John], [Lannes]

Fixed body : NSW [Bocchi, He, Vergara-Hermosilla], Boussinesq [Bresch, Lannes, Métivier]

Vertical walls, NSW with viscosity [Maity, Takahashi, Tucsnak]

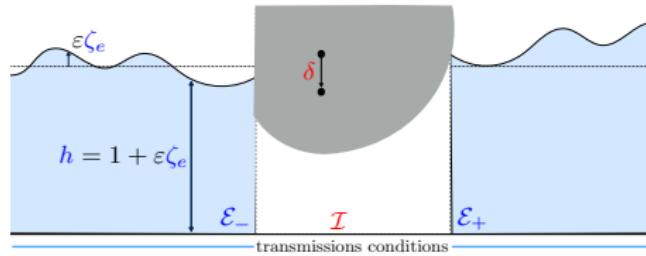
Congested NSW : [Godlewski, Parisot, Sainte-Marie, Wahl]

- Linear Dispersive Waves

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \partial_x \zeta = 0 \end{cases}$$

① Modelisation of the vertical movement of a floating solid

$$\ddot{\delta} = F(\delta, \dot{\delta}, \text{waves variables})$$



$$[\![q]\!] := q_+ - q_- = f_{\text{jump}}(\delta, \dot{\delta})$$

$$\langle q \rangle := \frac{1}{2}(q_+ + q_-) = f_{\text{av}}(\delta, \dot{\delta})$$

② Augmented formulation

- Dispersive boundary layer
- Hidden equation
- Added mass-effect

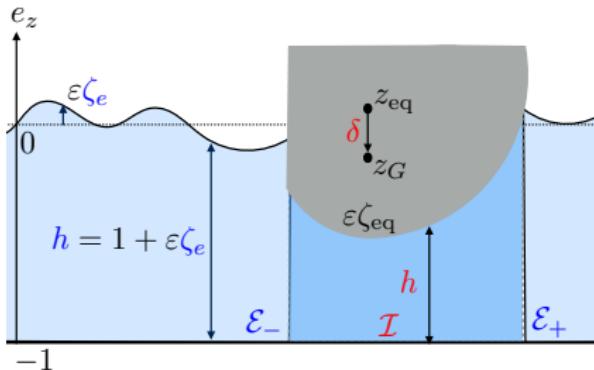
③ Numeric

- Finite volume for non-local conservation law
- ODE schemes

④ Return to equilibrium case (if time)

VERTICAL MOVEMENT OF A FLOATING SOLID

Coupling water-waves and solid dynamic



■ Exterior domain \mathcal{E}

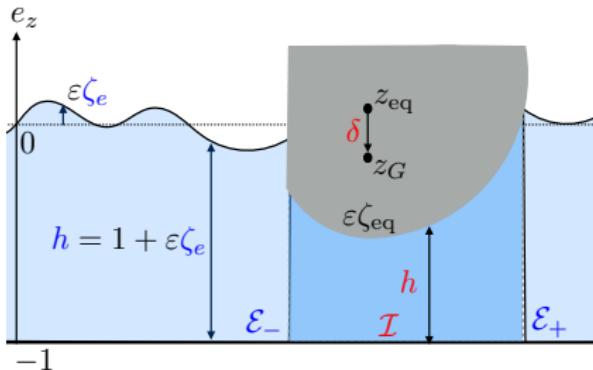
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■ Constraint in \mathcal{E}

$$P_e = P_{\text{atm}}$$

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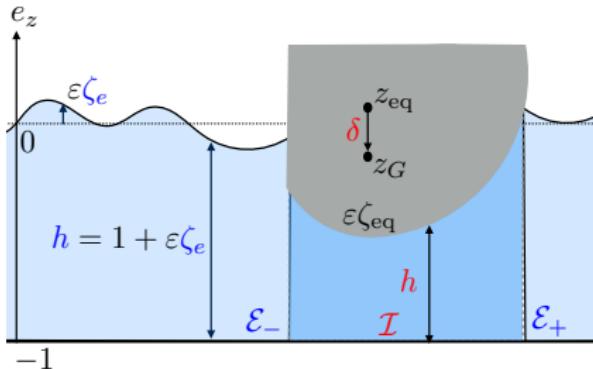
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$$\zeta(x, t) = \zeta_{\text{eq}}(x) + \delta(t)$$

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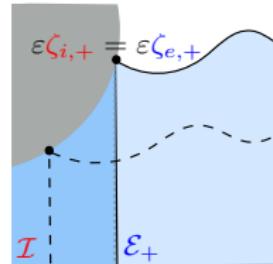
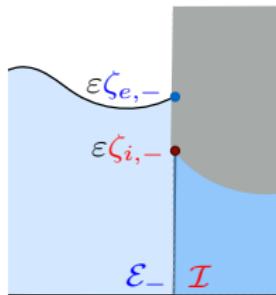
Transmission conditions at the interfaces

- Conservation of the volume \Rightarrow continuity of q : $\underline{q}_{\pm} = \underline{q}_{\pm}$

VERTICAL MOVEMENT OF A FLOATING SOLID

Transmission conditions at the interfaces

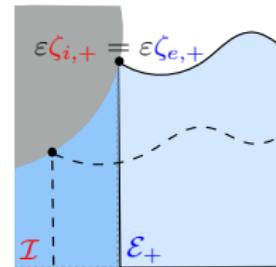
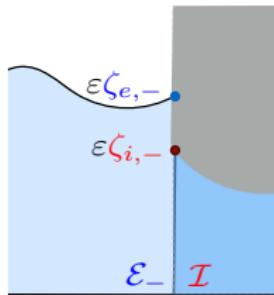
- Conservation of the volume \Rightarrow continuity of q : $\underline{q}_\pm = \underline{q}_\pm$
- Vertical walls
 - discontinuity of ζ
 - fixed interfaces
- Non-vertical walls
 - continuity of ζ (and P)
 - free interfaces



VERTICAL MOVEMENT OF A FLOATING SOLID

Transmission conditions at the interfaces

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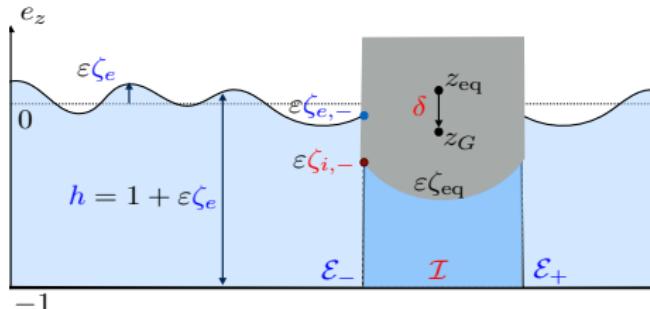


- Boundaries conditions on the pressure (\Rightarrow approximate conservation of energy)

$$\frac{1}{\varepsilon} \underline{P}_\pm = (\underline{\zeta} - \underline{\zeta})_\pm + \frac{\varepsilon}{2} \left(\frac{\underline{q}^2}{h^2} - \frac{\underline{q}^2}{\bar{h}^2} \right)_\pm + \kappa^2 \left(\frac{\ddot{\zeta}}{h} - \frac{\ddot{\zeta}}{\bar{h}} \right)_\pm$$

TRANSMISSION PROBLEM

Coupling water-waves and solid dynamic



■ Newton's equation

$$\tilde{m} \ddot{\delta} + \tilde{g} = \frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varepsilon^{-1} (P_i - P_{\text{atm}})$$

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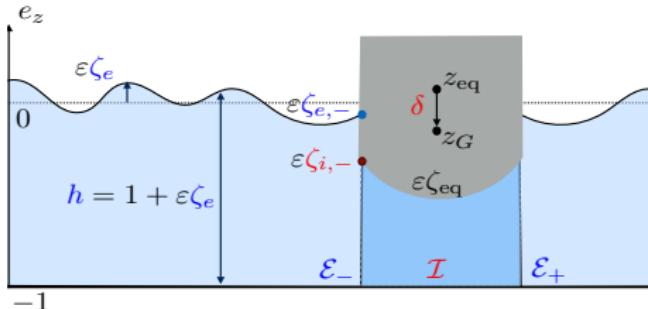
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TRANSMISSION PROBLEM

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Interior domain \mathcal{I}

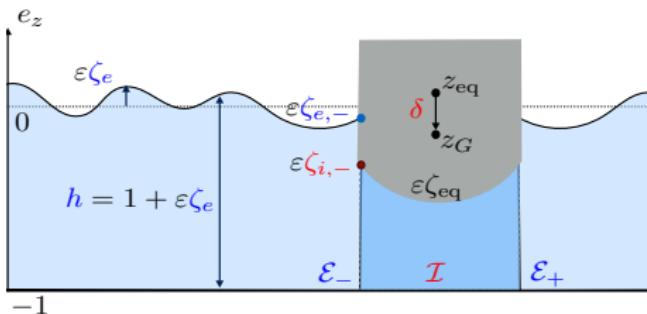
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Injecting the pressure

\Rightarrow EDO's for $\delta, \langle q \rangle := \frac{q_+ + q_-}{2}$

TRANSMISSION PROBLEM

wave-structure equations



■ Added masses

$$\mathfrak{m}(\varepsilon \delta) = \tilde{m} + \frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \frac{x^2}{h_i} + \langle \frac{\kappa^2}{h} \rangle$$

$$h := 1 + \varepsilon(\zeta_{eq} + \delta)$$

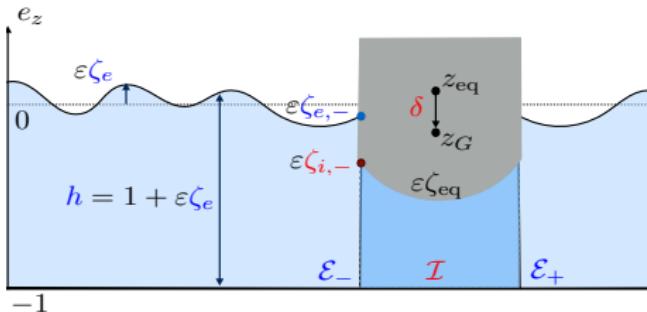
■ Newton

$$\begin{cases} \mathfrak{m}(\varepsilon \delta) \ddot{\delta} + \delta + \varepsilon Q_1(\varepsilon \delta; \dot{\delta}, \langle \dot{q} \rangle) - \langle \underline{\zeta} + \frac{\varepsilon}{2} \frac{q^2}{h^2} + \frac{\kappa^2}{h} \ddot{\zeta} \rangle = 0 \\ \alpha(\varepsilon \delta) \langle \dot{q} \rangle + \varepsilon Q_2(\varepsilon \delta; \dot{\delta}, \langle \dot{q} \rangle) + \frac{1}{|\mathcal{I}|} [\underline{\zeta} + \frac{\varepsilon}{2} \frac{q^2}{h^2} + \frac{\kappa^2}{h} \ddot{\zeta}] = 0 \end{cases}$$

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■ Boussinesq-Abbott

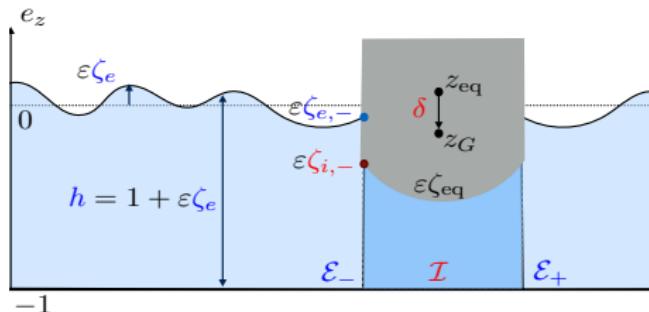
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■ Transmission conditions

$$\underline{q}_{\pm} = \underline{q}_{\pm} \Rightarrow \begin{cases} [\![q]\!] = -\dot{\delta} |\mathcal{I}|, \\ \langle q \rangle = \langle q \rangle, \end{cases}$$

TRANSMISSION PROBLEM

wave-structure equations



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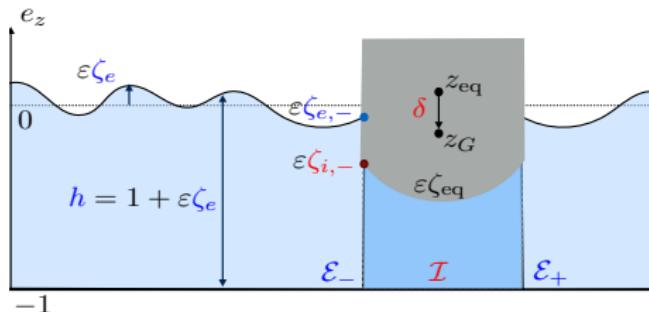
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■ $\kappa = 0$ **hyperbolic system**, compatibility conditions

■ $\kappa \neq 0$ **dispersive perturbation** of hyperbolic system

TRANSMISSION PROBLEM

wave-structure equations



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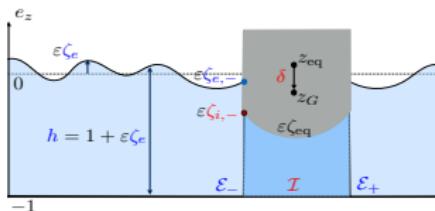
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■ $\kappa = 0$ **hyperbolic system**, compatibility conditions $\underline{\zeta} \in \cap_{j=0}^n C_t^j H_x^{n-j}$

■ $\kappa \neq 0$ **dispersive perturbation** of hyperbolic system $\underline{\zeta} \in C_t^\infty H_x^n$

TRANSMISSION PROBLEM

Augmented added masses



Added masses

$$\begin{pmatrix} \mathfrak{m}(\varepsilon\delta) & 0 & -\frac{1}{2}\frac{\kappa^2}{h_+} & -\frac{1}{2}\frac{\kappa^2}{h_-} \\ 0 & \alpha(\varepsilon\delta) & \frac{1}{|\mathcal{I}|}\frac{\kappa^2}{h_+} & -\frac{1}{|\mathcal{I}|}\frac{\kappa^2}{h_-} \\ \kappa\frac{|\mathcal{I}|}{2} & -\kappa & & \kappa^2 \text{Id}_{2 \times 2} \\ \kappa\frac{|\mathcal{I}|}{2} & \kappa & & \end{pmatrix} \begin{pmatrix} \ddot{\delta} \\ \langle \dot{q} \rangle \\ \underline{\zeta}_+ \\ \underline{\zeta}_- \end{pmatrix}$$

Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon Q_1(\varepsilon\delta; \dot{\delta}, \langle q \rangle) - \langle \underline{\zeta} + \frac{\varepsilon}{2}\frac{q^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta} \rangle = 0 \\ \alpha(\varepsilon\delta)\langle \dot{q} \rangle + \varepsilon Q_2(\varepsilon\delta; \dot{\delta}, \langle q \rangle) + \frac{1}{|\mathcal{I}|}[\underline{\zeta} + \frac{\varepsilon}{2}\frac{q^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta}] = 0 \\ \kappa^2\underline{\zeta}_{\pm}'' \mp \kappa\dot{q}_{\pm} + \underline{\zeta}_{\pm} + \varepsilon Q_3(\underline{\zeta}_{\pm}, \dot{\delta}, \langle q \rangle) = \mathfrak{f}_{\pm} \end{cases} \quad \underline{q}_{\pm} = \mp \frac{|\mathcal{I}|}{2}\dot{\delta} + \langle \dot{q} \rangle$$

Boussinesq-Abbott

$$\begin{cases} \partial_t \underline{\zeta} + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \partial_x \left(\underline{\zeta} + \varepsilon \frac{q^2}{h} + \frac{\varepsilon}{2} \underline{\zeta}^2 \right) = 0 \end{cases}$$

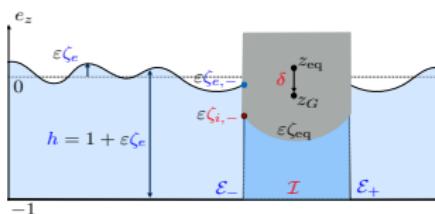
Transmission conditions

$$\begin{cases} [\![q]\!] = -\dot{\delta} |\mathcal{I}|, \\ \langle q \rangle = \langle \dot{q} \rangle, \end{cases}$$

$$\text{Non-local flux } (1 - \kappa^2 \partial_x^2) \mathfrak{f} = \underline{\zeta} + \varepsilon \frac{q^2}{h} + \frac{\varepsilon}{2} \underline{\zeta}^2 \quad \underline{\partial_x f}_{\pm} = 0$$

TRANSMISSION PROBLEM

Augmented added masses



Added masses

$$\begin{pmatrix} \mathfrak{m}(\varepsilon\delta) & 0 & -\frac{1}{2}\frac{\kappa^2}{h_+} & -\frac{1}{2}\frac{\kappa^2}{h_-} \\ 0 & \alpha(\varepsilon\delta) & \frac{1}{|I|}\frac{\kappa^2}{h_+} & -\frac{1}{|I|}\frac{\kappa^2}{h_-} \\ \kappa\frac{|I|}{2} & -\kappa & & \kappa^2 \text{Id}_{2 \times 2} \\ \kappa\frac{|I|}{2} & \kappa & & \end{pmatrix} \begin{pmatrix} \ddot{\delta} \\ \langle \dot{q} \rangle \\ \underline{\zeta}_+ \\ \underline{\zeta}_- \end{pmatrix}$$

Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon Q_1(\varepsilon\delta; \dot{\delta}, \langle q \rangle) - \langle \underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta} \rangle = 0 \\ \alpha(\varepsilon\delta)\langle \dot{q} \rangle + \varepsilon Q_2(\varepsilon\delta; \dot{\delta}, \langle q \rangle) + \frac{1}{|I|}[\underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta}] = 0 \\ \kappa^2 \ddot{\underline{\zeta}}_{\pm} \mp \kappa \dot{\underline{q}}_{\pm} + \underline{\zeta}_{\pm} + \varepsilon Q_3(\underline{\zeta}_{\pm}, \dot{\delta}, \langle q \rangle) = \underline{f}_{\pm} \end{cases} \quad \underline{q}_{\pm} = \mp \frac{|I|}{2} \dot{\delta} + \langle \dot{q} \rangle$$

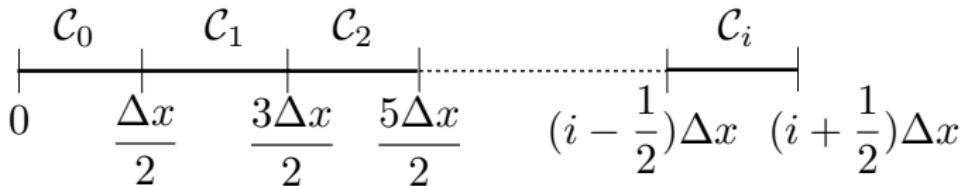
Boussinesq-Abbott

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x f = \underline{q}_{\pm} \exp\left(-\frac{|x-x_{\pm}|}{\kappa}\right) \end{cases}$$

$$\text{Non-local flux } (1 - \kappa^2 \partial_x^2) f = \zeta + \varepsilon \frac{q^2}{h} + \frac{\varepsilon}{2} \zeta^2 \quad \underline{\partial_x f}_{\pm} = 0$$

Transmission conditions

$$\begin{cases} [\![q]\!] = -\dot{\delta} [\![x]\!], \\ \langle q \rangle = \langle \dot{q} \rangle, \end{cases}$$



■ Newton (ODE Scheme)

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \dot{\delta} + \varepsilon Q_1(\varepsilon\delta; \dot{\delta}, \langle q \rangle) - \langle \underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta} \rangle = 0 \\ \alpha(\varepsilon\delta)\dot{\langle q \rangle} + \varepsilon Q_2(\varepsilon\delta; \dot{\delta}, \langle q \rangle) + \frac{1}{|\mathcal{I}|}[\underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta}] = 0 \\ \kappa^2 \ddot{\underline{\zeta}}_{\pm} \mp \kappa \dot{q}_{\pm} + \underline{\zeta}_{\pm} + \varepsilon Q_3(\underline{\zeta}_{\pm}, \dot{\delta}, \langle q \rangle) = \underline{f}_{\pm} \end{cases} \quad \underline{q}_{\pm} = \mp \frac{|\mathcal{I}|}{2} \dot{\delta} + \langle q \rangle$$

■ Boussinesq-Abbott (Finite Volume)

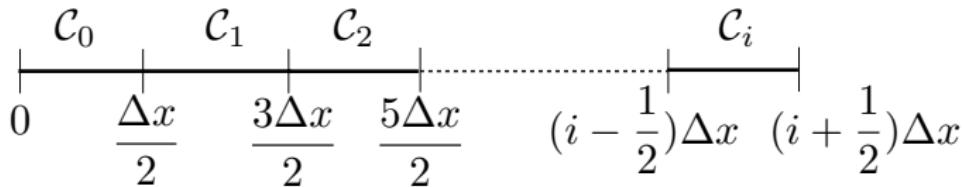
$$\begin{cases} \partial_t \underline{\zeta} + \partial_x \underline{q} = 0 \\ \partial_t \underline{q} + \partial_x \underline{f} = \underline{q}_{\pm} \exp\left(-\frac{|x-x_{\pm}|}{\kappa}\right) \end{cases}$$

■ Non-local flux (Finite Difference)

$$(1 - \kappa^2 \partial_x^2) \underline{f} = \underline{\zeta} + \varepsilon \frac{\dot{q}^2}{h} + \frac{\varepsilon}{2} \underline{\zeta}^2 \quad \underline{\partial_x f}_{\pm} = 0$$

■ Transmission conditions

$$\begin{cases} [\underline{q}] = -\dot{\delta} [x], \\ \langle \underline{q} \rangle = \langle q \rangle, \end{cases}$$



■ Newton (ODE Scheme)

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \dot{\delta} + \varepsilon Q_1(\varepsilon\delta; \dot{\delta}, \langle q \rangle) - \langle \underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta} \rangle = 0 \\ \alpha(\varepsilon\delta)\dot{\langle q \rangle} + \varepsilon Q_2(\varepsilon\delta; \dot{\delta}, \langle q \rangle) + \frac{1}{|\mathcal{I}|}[\underline{\zeta} + \frac{\varepsilon}{2}\frac{\dot{q}^2}{h^2} + \frac{\kappa^2}{h}\ddot{\zeta}] = 0 \\ \kappa^2 \ddot{\underline{\zeta}}_{\pm} \mp \kappa \dot{\underline{q}}_{\pm} + \underline{\zeta}_{\pm} + \varepsilon Q_3(\underline{\zeta}_{\pm}, \dot{\delta}, \langle q \rangle) = \underline{f}_{\pm} \end{cases}$$

$$\underline{q}_{\pm} = \mp \frac{|\mathcal{I}|}{2} \dot{\delta} + \langle q \rangle$$

■ Boussinesq-Abbott (Finite Volume)

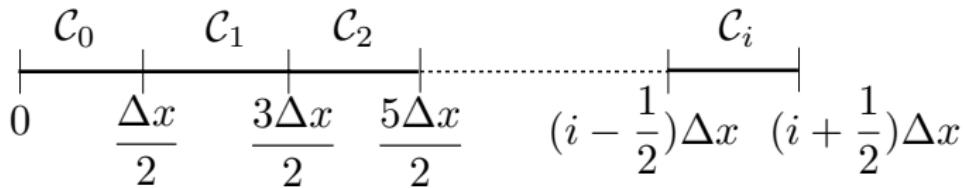
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x f = \underline{q}_{\pm} \exp\left(-\frac{|x-x_{\pm}|}{\kappa}\right) \end{cases}$$

■ Transmission conditions

$$\begin{cases} [\![q]\!] = -\dot{\delta} [\![x]\!], \\ \langle q \rangle = \langle q \rangle, \end{cases}$$

■ Non-local flux (Finite Difference)

$$\underline{f}_i^n - \kappa^2 \frac{\underline{f}_{i+1}^n - 2\underline{f}_i^n + \underline{f}_{i-1}^n}{\Delta x^2} = \underline{\zeta}_i^n + \varepsilon \frac{(q_i^n)^2}{h_i^n} + \varepsilon \frac{(\zeta_i^n)^2}{2}$$



■ Newton (ODE Scheme)

unknowns : $\theta := (\delta, \dot{\delta}, \langle q \rangle, \underline{\zeta}_{\pm}, \dot{\underline{\zeta}}_{\pm})$ Euler scheme : $\frac{\theta^{n+1} - \theta^n}{\Delta t} = \mathcal{G}(\theta^n, \underline{f}_{\pm}^n)$

■ Boussinesq-Abbott (Finite Volume)

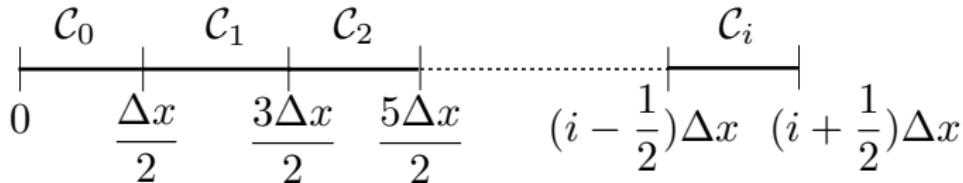
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x f = \dot{q}_{\pm} \exp\left(-\frac{|x-x_{\pm}|}{\kappa}\right) \end{cases}$$

■ Transmission conditions

$$\begin{cases} [\![q]\!] = -\dot{\delta}[\![x]\!], \\ \langle q \rangle = \langle q \rangle, \end{cases}$$

■ Non-local flux (Finite Difference)

$$f_i^n - \kappa^2 \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2} = \zeta_i^n + \varepsilon \frac{(q_i^n)^2}{h_i^n} + \varepsilon \frac{(\zeta_i^n)^2}{2}$$



■ Newton (ODE Scheme)

unknowns : $\theta := (\delta, \dot{\delta}, \langle q \rangle, \underline{\zeta}_{\pm}, \dot{\underline{\zeta}}_{\pm})$ Euler scheme : $\frac{\theta^{n+1} - \theta^n}{\Delta t} = \mathcal{G}(\theta^n, \underline{f}_{\pm}^n)$

■ Boussinesq-Abbott (Finite Volume)

unknowns : $u := (\zeta, q)$ non-local flux : $\mathfrak{F} := (q, f)$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{2\Delta x} \left[\mathfrak{F}_{\kappa, i+1}^n - \mathfrak{F}_{\kappa, i-1}^n - \frac{\Delta x}{\Delta t} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right] = \mathcal{S}_i(\theta^n, \underline{f}_{\pm}^n)$$

■ Non-local flux (Finite Difference)

$$f_i^n - \kappa^2 \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2} = \zeta_i^n + \varepsilon \frac{(q_i^n)^2}{h_i^n} + \varepsilon \frac{(\zeta_i^n)^2}{2}$$

NUMERICAL SCHEMES

Second order prediction-correction

■ Newton (Heun scheme)

Heun - prediction step $\frac{\theta^{n,*} - \theta^n}{\Delta t} = \mathcal{G}\left(\theta^n, \underline{f}_\pm^n\right)$

Heun - correction step $\frac{\theta^{n+1} - \theta^n}{\Delta t} = \frac{\mathcal{G}\left(\theta^n, \underline{f}_\pm^n\right) + \mathcal{G}\left(\theta^{n,*}, \underline{f}_\pm^{n,*}\right)}{2}$

■ Boussinesq-Abbott (Finite Volume)

unknowns : $\boldsymbol{u} := (\zeta, \boldsymbol{q})$ non-local flux : $\boldsymbol{\mathfrak{F}} := (\boldsymbol{q}, \underline{f})$

Mac-Cormack - prediction step $i > 1$

$$\frac{\boldsymbol{u}_i^{n,*} - \boldsymbol{u}_i^n}{\Delta t} + \frac{1}{\Delta x} [\mathfrak{F}_{\kappa,i}^n - \mathfrak{F}_{\kappa,i-1}^n] = \mathcal{S}_i\left(\theta^n, \underline{f}_\pm^n\right)$$

Mac-Cormack - correction step $i \geq 1$

$$\frac{\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n}{\Delta t} + \frac{1}{2\Delta x} [\mathfrak{F}_{\kappa,i}^n - \mathfrak{F}_{\kappa,i-1}^n + \mathfrak{F}_{\kappa,i+1}^{n,*} - \mathfrak{F}_{\kappa,i}^{n,*}] = \frac{\mathcal{S}_i\left(\theta^n, \underline{f}_\pm^n\right) + \mathcal{S}_i\left(\theta^{n,*}, \underline{f}_\pm^n\right)}{2}$$

■ Non-local flux (Finite Difference)

$$\underline{f}_i^n - \kappa^2 \frac{\underline{f}_{i+1}^n - 2\underline{f}_i^n + \underline{f}_{i-1}^n}{\Delta x^2} = \zeta_i^n + \varepsilon \frac{(\boldsymbol{q}_i^n)^2}{h_i^n} + \varepsilon \frac{(\zeta_i^n)^2}{2}$$

■ Newton (Heun scheme)

$$\text{Heun - prediction step } \frac{\theta^{n,*} - \theta^n}{\Delta t} = \mathcal{G}\left(\theta^n, \underline{f}_{\pm}^n\right)$$

$$\text{Heun - correction step } \frac{\theta^{n+1} - \theta^n}{\Delta t} = \frac{\mathcal{G}\left(\theta^n, \underline{f}_{\pm}^n\right) + \mathcal{G}\left(\theta^{n,*}, \underline{f}_{\pm}^{n,*}\right)}{2}$$

■ Boussinesq-Abbott (Finite Volume)

$$\text{unknowns : } \underline{u} := (\zeta, q) \quad \text{non-local flux : } \mathfrak{F} := (q, f)$$

Mac-Cormack - prediction step $i > 1$

$$\frac{\underline{u}_i^{n,*} - \underline{u}_i^n}{\Delta t} + \frac{1}{\Delta x} [\mathfrak{F}_{\kappa,i}^n - \mathfrak{F}_{\kappa,i-1}^n] = \mathcal{S}_i\left(\theta^n, \underline{f}_{\pm}^n\right)$$

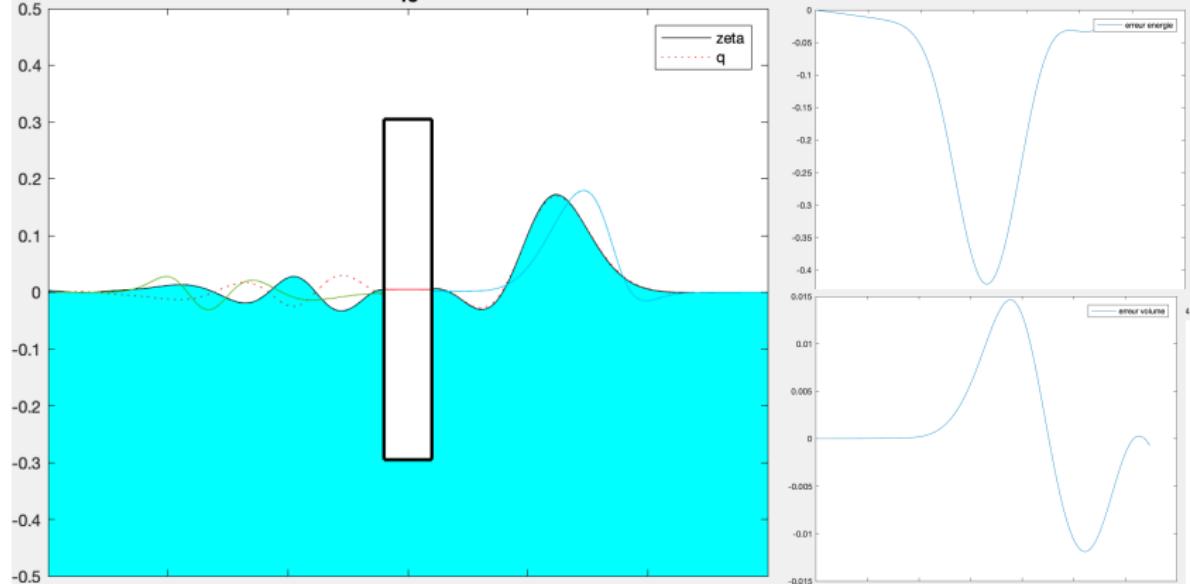
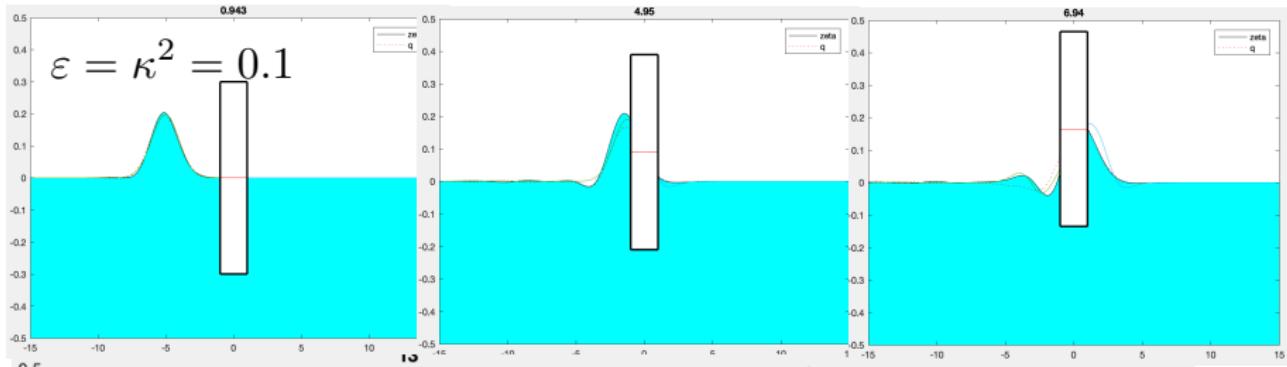
Mac-Cormack - correction step $i \geq 1$

$$\frac{\underline{u}_i^{n+1} - \underline{u}_i^n}{\Delta t} + \frac{1}{2\Delta x} [\mathfrak{F}_{\kappa,i}^n - \mathfrak{F}_{\kappa,i-1}^n + \mathfrak{F}_{\kappa,i+1}^{n,*} - \mathfrak{F}_{\kappa,i}^{n,*}] = \frac{\mathcal{S}_i\left(\theta^n, \underline{f}_{\pm}^n\right) + \mathcal{S}_i\left(\theta^{n,*}, \underline{f}_{\pm}^{n,*}\right)}{2}$$

■ Non-local flux (Finite Difference)

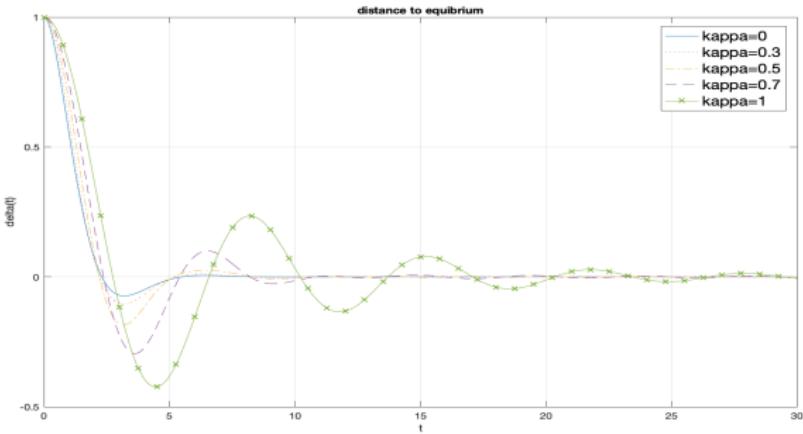
$$\underline{f}_i^{n,*} - \kappa^2 \frac{\underline{f}_{i+1}^{n,*} - 2\underline{f}_i^{n,*} + \underline{f}_{i-1}^{n,*}}{\Delta x^2} = \zeta_i^{n,*} + \varepsilon \frac{(\underline{q}_i^{n,*})^2}{\underline{h}_i^{n,*}} + \varepsilon \frac{(\zeta_i^{n,*})^2}{2}$$

Simulation : <https://geoffreybeckpoems.wixsite.com/math>



RETURN TO EQUILIBRIUM - LINEAR CASE

Long time behaviour



- **Linear case** $\varepsilon = 0$ with $(\zeta_0 = 0, q_0 = 0, \boxed{\dot{\delta}_0 \neq 0}, \ddot{\delta}_0 = 0)$

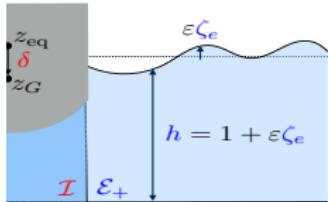
$$\left[m + m_{\text{added,SW}} + \kappa(\ell + \kappa h_{\text{eq}}^{-1}) \right] \ddot{\delta} + \dot{\delta} + \ell \left(\frac{1}{t} J_1 \left(\frac{t}{\kappa} \right) \right) *_{\text{t}} \dot{\delta} = 0$$

- $\kappa = 0 : |\delta| \sim e^{-\alpha t}$

$\zeta, q \in C_{x,t}^0$ admit singularities at $\{x = t\}$ (transport equation)

- $\kappa > 0 : |\delta| > ct^{-3/2}$

$\zeta, q \in C_x^n H_{t,\text{loc}}^1$ (**non-local** transport equation)



■ Return to the equilibrium

$$\zeta|_{t=0} = q|_{t=0} = \dot{\delta}|_{t=0} = 0 \quad \delta|_{t=0} \neq 0$$

■ Added masses

$$\begin{pmatrix} \mathfrak{m}(\varepsilon\delta) & -\frac{1}{2}\frac{\kappa^2}{h} \\ \kappa & \kappa^2 \end{pmatrix}$$

■ Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon Q(\varepsilon\delta)\dot{\delta}^2 - \left(\underline{\zeta} + \frac{\varepsilon}{2}\frac{q^2}{h^2} + \frac{\kappa^2}{h}\ddot{\underline{\zeta}} \right) = 0 \\ \kappa^2\underline{\zeta}'' + \kappa\frac{|I|}{2}\ddot{\delta} + \underline{\zeta} + \varepsilon Q_3(\underline{\zeta}, \dot{\delta}) = \underline{(1 - \kappa^2\partial_x^2)^{-1}}(\zeta + \varepsilon\frac{q^2}{h^2} + \frac{\varepsilon}{2}\zeta^2) \end{cases}$$

■ Boussinesq-Abbott

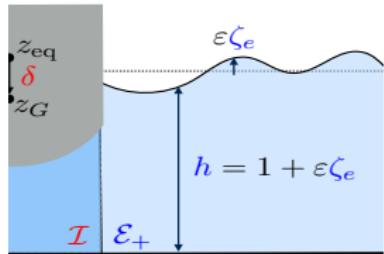
$$\begin{cases} \partial_t \underline{\zeta} + \partial_x q = 0 \\ (1 - \kappa^2\partial_x^2)\partial_t q + \varepsilon\partial_x \left(\frac{q^2}{h} \right) + h\partial_x \underline{\zeta} = 0 \\ \text{B.C : } \underline{q} = -\dot{\delta} \end{cases}$$

■ $\kappa = 0$ **hyperbolic system**, compatibility conditions

■ $\kappa \neq 0$ **dispersive perturbation** of hyperbolic system

WELL-POSEDNESS

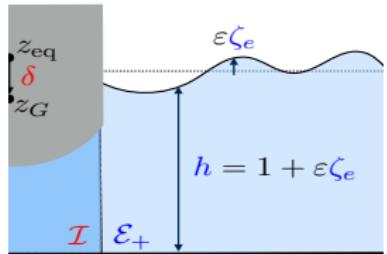
Dispersive boundary layer



$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } \underline{q} = -\dot{\delta} \end{cases}$$

WELL-POSEDNESS

Dispersive boundary layer

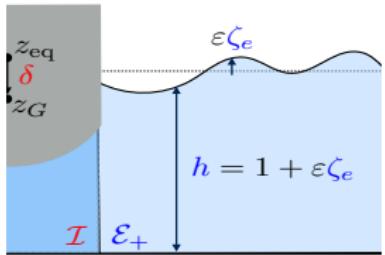


$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } \underline{q} = -\dot{\delta} \end{cases}$$

$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$

WELL-POSEDNESS

Dispersive boundary layer



$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } q = -\dot{\delta} \end{cases}$$

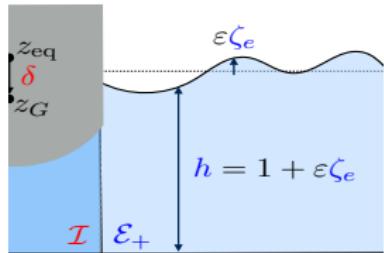
$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$

$$(1 - \kappa^2 \partial_x^2) \left[\partial_t q - \underline{q} \exp \left(-\frac{x}{\kappa} \right) \right] + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0$$

$$\partial_t q = -(1 - \kappa^2 \partial_x^2)_D^{-1} \partial_x \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) - \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

WELL-POSEDNESS

Dispersive boundary layer



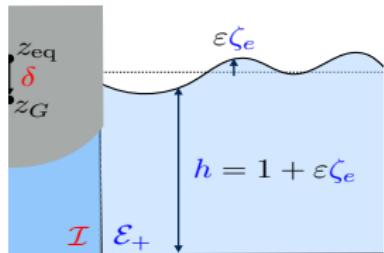
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } \underline{q} = -\dot{\delta} \end{cases}$$

$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$
 $(1 - \kappa^2 \partial_x^2)_N^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Neumann BC at $x = 0$

$$\partial_t q = -\partial_x (1 - \kappa^2 \partial_x^2)_N^{-1} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) - \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

WELL-POSEDNESS

Dispersive boundary layer



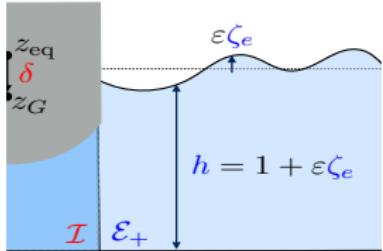
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } \underline{q} = -\dot{\delta} \end{cases}$$

$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$
 $(1 - \kappa^2 \partial_x^2)_N^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Neumann BC at $x = 0$

$$\partial_t q = -\partial_x (1 - \kappa^2 \partial_x^2)_N^{-1} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) - \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

WELL-POSEDNESS

Dispersive boundary layer



$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } q = -\dot{\delta} \end{cases}$$

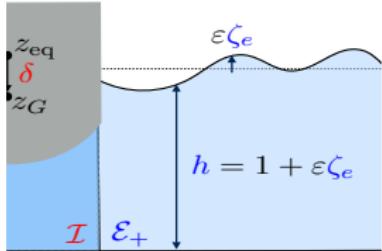
$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$
 $(1 - \kappa^2 \partial_x^2)_N^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Neumann BC at $x = 0$

$$\partial_t q = -\partial_x (1 - \kappa^2 \partial_x^2)_N^{-1} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) - \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

$$\kappa^2 \partial_x \partial_t q = \underbrace{-\kappa^2 \partial_x^2 (1 - \kappa^2 \partial_x^2)_N^{-1}}_{Id - (1 - \kappa^2 \partial_x^2)^{-1}} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) + \kappa \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

WELL-POSEDNESS

Dispersive boundary layer



$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } q = -\dot{\delta} \end{cases}$$

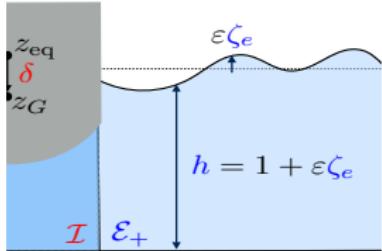
$(1 - \kappa^2 \partial_x^2)_D^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Dirichlet BC at $x = 0$
 $(1 - \kappa^2 \partial_x^2)_N^{-1}$ inverse of $(1 - \kappa^2 \partial_x^2)$, homogeneous Neumann BC at $x = 0$

$$\partial_t q = -\partial_x (1 - \kappa^2 \partial_x^2)_N^{-1} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) - \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

$$-\kappa^2 \ddot{\zeta} = \kappa^2 \partial_x \partial_t q = \underbrace{-\kappa^2 \partial_x^2 (1 - \kappa^2 \partial_x^2)_N^{-1}}_{Id - (1 - \kappa^2 \partial_x^2)_N^{-1}} \left(\varepsilon \frac{q^2}{h} + \zeta + \varepsilon \frac{\zeta^2}{2} \right) + \kappa \ddot{\delta} \exp \left(-\frac{x}{\kappa} \right)$$

WELL-POSEDNESS

Dispersive boundary layer



$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2) \partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h} \right) + h \partial_x \zeta = 0 \\ \text{B.C : } q = -\dot{\delta} \end{cases}$$

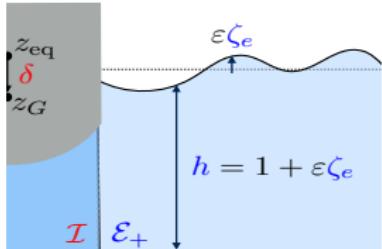
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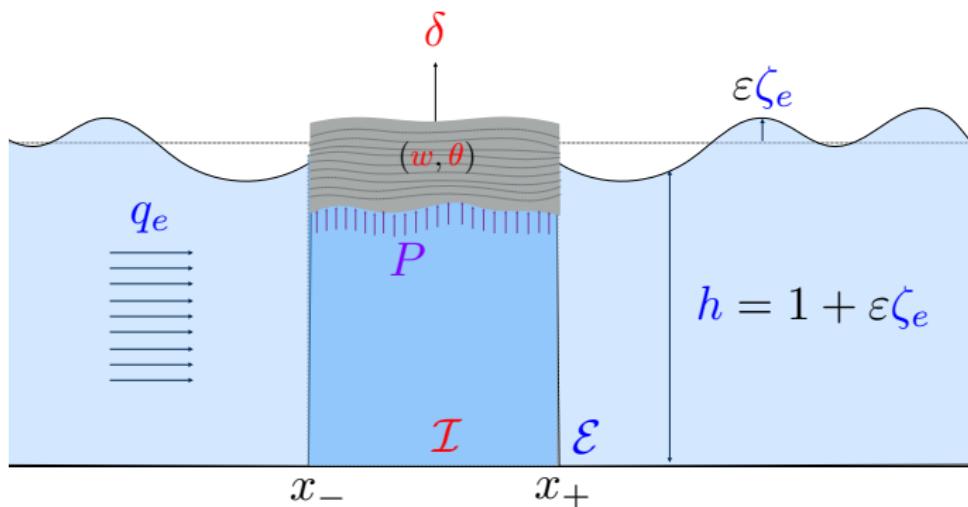
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Dispersion regularization :

$$(\zeta, q)(t) \in H^1 \times H^2 \Rightarrow \begin{cases} (\partial_t \zeta, \partial_t q)(t) \in H^1 \times H^2 \\ \zeta, \dot{\zeta}, \ddot{\zeta} \text{ well-defined} \end{cases}$$

- Marginal Ice Zone : homogenization of small bodies (with Matthieu Hillairet)
- La Banquise Insoumise : water-waves / elastic rod interactions (with Loïc Le Marrec)
- Generic Drug : almost-sure well-posedness with random forcing (with Ricardo Grande)



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Augmented energy $E = \|\zeta, \frac{q}{\sqrt{h}}, \frac{\kappa}{\sqrt{h}} \partial_x q\|_2^2 + |\delta, \dot{\delta}|^2 + \varepsilon \kappa |\kappa \underline{\zeta}, \kappa^2 \dot{\underline{\zeta}}|^2$

Goal : Lifespan $T^* = O(\varepsilon^{-1})$?

Conditional uniform estimate $M[\mathbf{u}] := \|\zeta, q\|_{L_t^\infty W_x^{1,\infty}}$

$$E(t) \leq C(M[\mathbf{u}], E_0)$$

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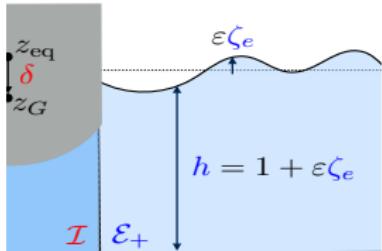
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RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



■ Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta = \underline{\zeta}$$

■ Linear waves

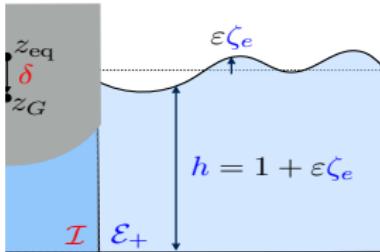
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \zeta = 0 \end{cases}$$

■ Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



■ Explicit solution

$$\underline{q} = \underline{\zeta} = -\frac{|\mathcal{I}|}{2} \dot{\delta}(t-x) \mathbb{I}_{t \geq x},$$

■ Newton

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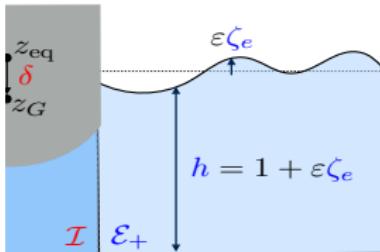
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Linear waves

$$\begin{cases} \partial_t \underline{\zeta} + \partial_x \underline{q} = 0 \\ \partial_t \underline{q} + \partial_x \underline{\zeta} = 0 \end{cases}$$

Decay test

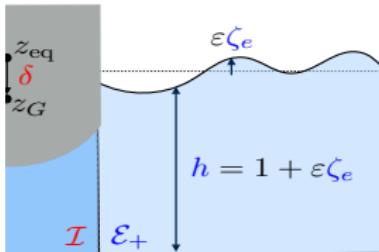
$$\delta \underset{t \rightarrow \infty}{\sim} e^{-\alpha t}$$

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- Regularity $C_{x,t}^0$

- Singularities at $\{x = t\}$ since $\ddot{\delta}(0) = -\delta_0 \mathfrak{m}(0)^{-1}$

- Newton

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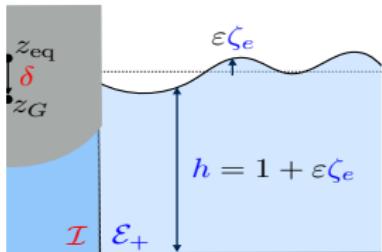
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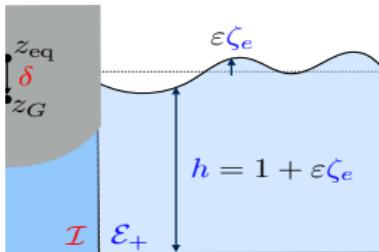
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$$\blacksquare \text{ Laplace } \mathcal{L} : q(t) \mapsto \widehat{q}(s) := \int q(t) e^{-st} dt \quad \Re(s) > 0$$

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Non-dispersive linear waves



■ Explicit solution in Laplace domain

$$\begin{cases} \hat{q}(s, x) = \underline{\hat{q}}(s) e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}} x} \\ \hat{\zeta}(s, x) = \frac{\underline{\hat{q}}(s, x)}{\sqrt{1+\kappa^2 s^2}} \end{cases}$$

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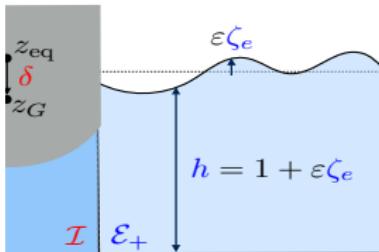
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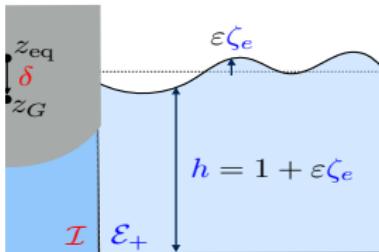
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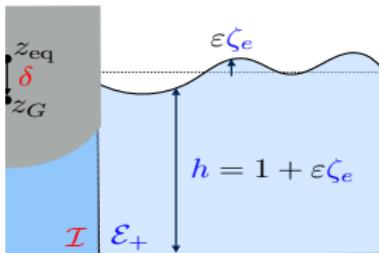
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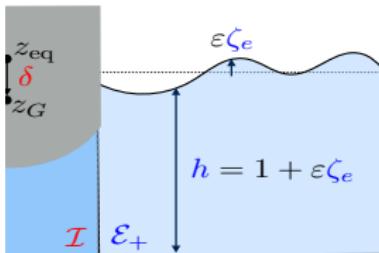
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$$q \in C_x^0 H_t^1 \cap C_x^n H_{t,\alpha}^1 \quad \text{where } L_{t,\alpha}^2 := L^2(e^{-\alpha t} dt)$$

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Transmission conditions at the interfaces

- Conservation of the volume \Rightarrow continuity of q : $q_{\pm} = \underline{q}_{\pm}$

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$$\dot{E}_{\text{tot}} = -[\![q \left(\zeta + \frac{\varepsilon q^2}{2 h^2} + \frac{\kappa^2}{h} \ddot{\zeta} \right) - q \left(\zeta + \frac{\varepsilon q^2}{2 h^2} + \frac{\kappa^2}{h} \ddot{\zeta} \right)]\!] - \frac{1}{\varepsilon} q \underline{P} + O(\varepsilon \kappa^2).$$

- jump between x_+ and x_- : $\llbracket u \rrbracket = u_+ - u_-$

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WELL-POSEDNESS

Local theory : Dispersive Vs. Hyperbolic

$$\begin{cases} \boldsymbol{u} := (\zeta, q) \\ \theta := (\delta, \dot{\delta}, \kappa \underline{\zeta}, \kappa^2 \dot{\underline{\zeta}}) \end{cases} \quad \mathfrak{f}_\kappa := " (1 - \kappa^2 \partial_x^2)^{-1} " (\zeta + \varepsilon \frac{q^2}{h^2} + \frac{\varepsilon}{2} \zeta^2) \quad \mathfrak{F}_\kappa := \begin{pmatrix} q \\ \mathfrak{f}_\kappa \end{pmatrix}$$

$$\begin{cases} \partial_t \boldsymbol{u} + \partial_x [\mathfrak{F}_\kappa(\boldsymbol{u})] = - \begin{pmatrix} 0 \\ \ddot{\delta} \end{pmatrix} \exp(-\frac{x}{\kappa}) & \mathbb{R}_x^+ \times [0, T] \\ \underline{q} = -\dot{\delta} & \{x = 0\} \times [0, T] \\ \dot{\theta} = F(\theta, \underline{\mathfrak{f}}_\kappa) & [0, T] \end{cases}$$

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WELL-POSEDNESS

Local theory : Dispersive Vs. Hyperbolic

$$\begin{cases} \underline{\boldsymbol{u}} := (\zeta, \underline{\boldsymbol{q}}) \\ \theta := (\delta, \dot{\delta}, \kappa \underline{\zeta}, \kappa^2 \dot{\underline{\zeta}}) \end{cases} \quad \underline{\mathfrak{f}}_\kappa := "((1 - \kappa^2 \partial_x^2)^{-1})(\zeta + \varepsilon \frac{\underline{\boldsymbol{q}}^2}{h^2} + \frac{\varepsilon}{2} \zeta^2)" \quad \underline{\mathfrak{F}}_\kappa := \begin{pmatrix} \underline{\boldsymbol{q}} \\ \underline{\mathfrak{f}}_\kappa \end{pmatrix}$$

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Local theory ($\kappa > 0$) $\dot{U} = \Phi(U)$, $\Phi : H^n \times H^{n+1} \times \mathbb{R}^3 \rightarrow H^n \times H^{n+1} \times \mathbb{R}^3$

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Local theory ($\kappa = 0$) Kreiss' theory of 1D **hyperbolic** system**Energy (boundary dissipation)** $E[\textcolor{blue}{u}, \theta](t) = |\theta(t)|^2 + \|\textcolor{blue}{u}(t)\|_2 + \|\textcolor{violet}{u}\|_{L^2(0,t)}^2$

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- ① Linearized problem
- ② Control of derivatives of Linearized problem (compatibility conditions)
- ③ Quasi-linear iterative scheme

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Return to equilibrium : Compatibility conditions don't work

- Diagonalization with Riemann invariants $\Rightarrow u$ Burgers equation

Unknowns $\begin{cases} \underline{u} := (\zeta, q) \\ \theta := (\delta, \dot{\delta}, \kappa \underline{\zeta}, \kappa^2 \dot{\underline{\zeta}}) \end{cases}$ $\underline{f}_\kappa = \frac{(1 - \kappa^2 \partial_x^2)^{-1}(\zeta + \varepsilon \frac{q^2}{h^2} + \frac{\varepsilon}{2} \zeta^2)}{\kappa}$

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Goal : Lifespan $T^* = O(\varepsilon^{-1})$?

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① **Conditional uniform estimate** $M[\underline{u}] := \|\zeta, q\|_{L_t^\infty W_x^{1,\infty}}$

$$E[\underline{u}, \theta](t) \leq C(M[\underline{u}], E_0)$$

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- ② **Control of time derivatives** $\underline{u}^{(i)} := (\partial_t^i \zeta, \partial_t^i q)$, $\|\underline{u}^{(i)}\|_2 \leq C$ until T^* ?
- ③ **Space derivatives** $\underline{u}^{(i,j)} := (\partial_x^j \partial_t^i \zeta, \partial_x^j \partial_t^i q)$, $\|\underline{u}^{(i,j)}\|_2 \leq C$ until T^* ?
- ④ **Sobolev Embedding** $M[\underline{u}] \leq C$ until T^* ?

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Boussinesq full line [Linares, Pilod, Saut] $O(\varepsilon^{-\frac{1}{2}})$, [Saut, Xu] $O(\varepsilon^{-1})$

Weakly non-linear approximation, fixed object [Bresch, Lannes, Métivier] $O((\varepsilon + \kappa^2)^{-1})$
(compatibility inequalities)