ANR Nabuco: Coastal flows and boundary conditions.

Nonlinear wave interactions with floating structure in shallow-water: a robust high-order DG-ALE formulation

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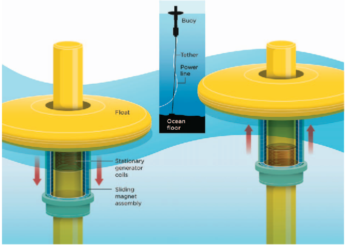
Introduction : Production of energy from water waves

- Waves power presents characteristics that make them an attractive energy source
- Many studies have been developed to study the production of energy from waves generated by the winds blowing on the sea surface

- Several types devices (floating bodies) are available :



goal : Model the water-body interactions for such floating devices

Introduction : Nonlinear shallow-water (NSW) system

- The nonliear Shallow-Water (NSW) equations

$$\partial_t \eta + \partial_x q = 0, \tag{1a}$$

$$\partial_t q + \partial_x \left(uq + \frac{1}{2} g(\eta^2 - 2\,\eta\, b) \right) = -g\eta \partial_x b. \tag{1b}$$

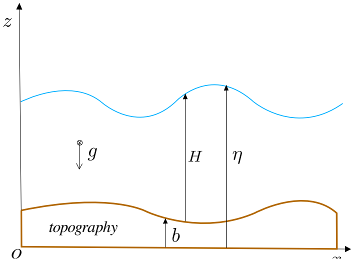


Figure – Free surface flow : main notations

Introduction : Numerical methods

• Great efforts have been made since the sixties in order to produce accurate approximations of weak solutions of the NSW equations.

• large variety of numerical methods have been developed :

- **Finite-Volumes (FV) methods**
- Finite-Elements (FE) methods
- **Discontinuous Galerkin (DG)**
- spectral methods
- residual distribution methods

Arbitrary Lagrangian Eulerian (ALE) method

- Firstly used in aerodynamic simulations to correctly predict and model the conditions at which aeroelastic instabilities occur.
- Combining the positive features of both approaches : material domain (Lagrangian) and spatial domain (Eulerian).
- ALE was later on widely used in several domains as : flows in reciprocating engines, airfoil oscillations, wing flutter, fighter tail buffeting, aircraft maneuvering, and a large class of free-surface flow problems.

Introduction : Some recent works

- In 2019 Mario Ricchiuto et al, have modelled a vertically-walled object, based on Boussinesq-type equations employing a spectral/*hp* finite element method (SEM) is presented. This model was proposed by Jiang (2001), it is about a unified Boussinesq model, decomposing the problem into two domains : free surface and body.
- In 2016, David Lannes analyzed a similar configuration in a nonlinear equation in shallow water, by deducing a exact and semi-analytical solution for moving bodies.

• **Numerically** In those works, only the case of a vertical motion of the body having vertical lateral walls is treated.

Introduction : A more complex situation

Here we consider the more complex and general situation :

- the body is allowed to a vertical, horizontal and rotational motion
- considering body wich walls are not vertical at the contact points (elliptical body), which will lead to a longitudinal displacement of the water-body contact points.
- considering all types of motion : fixed body, prescribed body motion and freely floating body.

• Determining the portion of the solid in contact with the water is a free boundary problem difficult to handle in numerical studies.

Introduction : Solving the problem via a ALE approach

ALE is a good idea to deal with such situation

- In our approach, the change of the contact points position, accordingly to the position of the body, forces a change at the mesh grid level at each time step
- This can be done setting up the problem in ALE framework

Discrete formulation

Let $\Omega_X \subset \mathbb{R}$ denote the initial domain (a segment), such that $\Omega_X = \bigcup_i \omega_i$, where :

- $\omega_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$.
- h_{ω_i} is the length of element ω_i.
- $\mathbb{P}^k(\omega_i)$ the space of polynomials in ω_i.

ALE settings

We define the mapping function $x(.,t)$ at time t by :

$$\begin{array}{cccccc} x(.,t) & : & \Omega_X & \longrightarrow & \Omega(t) \\ & & X & \longrightarrow & x(X,t) = x, \end{array} \tag{2}$$

- $\frac{\mathrm{d}x(X,t)}{\mathrm{d}t} = \mathrm{v_g}(x(X,t),t)$
- $\omega_i(t) = [x_{i-\frac{1}{2}}(t), x_{i+\frac{1}{2}}(t)]$

Well-balanced DG/ALE scheme

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\omega_i(t)} v_h^{\omega_i}\psi \mathrm{d}x - \int_{\omega_i(t)} (F(v_h^{\omega_i}, b_h^{\omega_i}) - v_h^{\omega_i}\mathrm{v_g})\partial_x\psi \mathrm{d}x + \left[\psi(F^* - v^*\mathrm{v_g})\right]_{i-\frac{1}{2}}^{i+\frac{1}{2}} = \int_{\omega_i(t)} \psi B(v_h^{\omega_i}, \partial_x b_h^{\omega_i})\mathrm{d}x, \tag{3}$$

$$F^*_{i+1/2} = F^*\left(v^-_{i+1/2}, v^+_{i+1/2}, b_{i+1/2}\right) \text{ and } v^*_{i+1/2} = v^*\left(v^-_{i+1/2}, v^+_{i+1/2}, b_{i+1/2}\right) \tag{4}$$

where $v^{\pm}_{i+1/2}$ and $b_{i+1/2} = b^{\pm}_{i+1/2}$ are the interpolated left and right values of ω_i interface $x_{i+1/2}$, F^* and v^* are consistent numerical fluxes and $\mathrm{v_g}$ is the grid velocity.

Well-balanced DG/ALE scheme : Basis functions

ψ satisfy :

$$\frac{\mathrm{d}\ \psi(\boldsymbol{x}(\boldsymbol{X},\boldsymbol{t}),\boldsymbol{t})}{\mathrm{dt}} = 0,$$

i.e. the basis function ψ follows the trajectory of $x(X,t)$.

$$\psi^k(x) = \left(\frac{x - x_i}{\Delta x}\right)^k, \ \ on \ \ \omega_i(t)$$

the mapping function $x(X,t)$ is explicitly defined by the following linear expression :

$$x|_{\omega_i(t)} = \frac{\left(X_{i+\frac{1}{2}} - X\right)}{|\omega_i(0)|} x_{i-\frac{1}{2}}(t) + \frac{\left(X - X_{i-\frac{1}{2}}\right)}{|\omega_i(0)|} x_{i+\frac{1}{2}}(t). \tag{5}$$

DG/ALE formulation as a FV scheme on subcells

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(|S_m^{\omega_i}(t)|\overline{v}_m^{\omega_i}\right) = -\left(\widehat{G}_{m+\frac{1}{2}}^{\omega_i} - \widehat{G}_{m-\frac{1}{2}}^{\omega_i}\right) + |S_m^{\omega_i}(t)|\overline{B}_m^{\omega_i}. \tag{6}$$

$$\widehat{G}_{m+\frac{1}{2}}^{\omega_i} = G_h^{\omega_i}\left(\widetilde{x}_{m+\frac{1}{2}}\right) - C_{m+\frac{1}{2}}^{i-\frac{1}{2}}\left(G_h^{\omega_i}\left(x_{i-\frac{1}{2}}\right) - \mathscr{G}_{i-\frac{1}{2}}\right) - C_{m+\frac{1}{2}}^{i+\frac{1}{2}}\left(G_h^{\omega_i}\left(x_{i+\frac{1}{2}}\right) - \mathscr{G}_{i+\frac{1}{2}}\right) \tag{7}$$

$$G_h^{\omega_i} = F_h^{\omega_i} - \mathrm{v_g}\, v_h^{\omega_i} \quad \text{and} \quad \mathscr{G} = F^* - \mathrm{v_g}\, v^*.$$

we refer to [Haidar et al(2021)] and [Vilar(2019)] for more details.

Scheme properties

ALE moving grid DG scheme properties :

- GCL and DGCL properties
- Well-balance

We can add *a postriori* FV local subcell corection :

- ensure the preservation of the water height positivity at the subcell level
- accurately handle strong shocks with no robustness issues

David Lanne's model

- We follow the approach proposed by David Lannes 2017
- Under the body the surface of the fluid coincides with the bottom of the body
- For such configurations, the horizontal projection of the portion of the solid in contact with the water is time dependent, $[x^{-}(t), x^{+}(t)]$
- This can be treated setting up the NSW system in DG/ALE framework

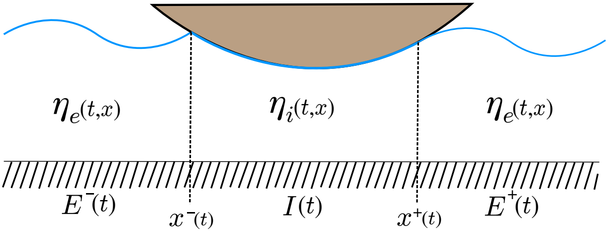


Figure – Water interacting with a floating body.

Shallow water model, interior/exterior region

The shallow water model for the water, in the *exterior* region, has the form :

$$\begin{cases} \partial_t \eta_e + \partial_x q_e = 0, & \text{in} \quad E(t), \\ \partial_t q_e + \partial_x \left(\dfrac{q_e^2}{H_e} + \dfrac{1}{2} g({\eta_e}^2 - 2\eta_e b) \right) = -g\eta_e \partial_x b & \text{in} \quad E(t). \end{cases} \tag{8}$$

while under the object (*interior* region), we have :

$$\begin{cases} \partial_t \eta_i + \partial_x q_i = 0, & \text{in} \quad I(t), \\ \partial_t q_i + \partial_x \left(\dfrac{q_i^2}{H_i} + \dfrac{1}{2} g{H_i}^2 \right) = -gH_i \partial_x b - \dfrac{1}{\rho} H_i \partial_x \underline{P}_i & \text{in} \quad I(t). \end{cases} \tag{9}$$

Inner/outer pression and boundary conditions

$$\underline{P}(t,x) = \begin{cases} P_{atm} & \text{in} \quad E(t), \\ \underline{P}_i(t,x) & \text{in} \quad I(t). \end{cases} \tag{10}$$

The transmission boundary conditions :

$$\eta_e = \eta_i, \;\; q_e = q_i, \;\; \underline{P}_i = P_{atm}, \quad \text{on} \quad E(t) \cap I(t), \tag{11}$$

A case of a fixed body : mesh nodes and contact points velocity

The velocity of the contact points at time t, for the case of a fixed body, is defined by :

$$x^{\pm\prime}(t) = \frac{\partial_x q_e(x^\pm(t),t)}{\partial_x \eta_e(x^\pm(t),t) - \partial_x \eta_i(x^\pm(t),t)}, \tag{12}$$

we define the velocity $\mathrm{v_g}$ at nodes $x_{i+\frac{1}{2}}$ at time t :

$$\mathrm{v_g}_{i+\frac{1}{2}}(t) = \mathrm{v_g}(X_{i+\frac{1}{2}},t) = \begin{cases} \psi\left(\frac{X_{i+\frac{1}{2}} - X_{in}^-}{\varepsilon}\right)\cdot x^{-\prime}(t), & \text{for} \quad X_{i+\frac{1}{2}} \text{ node of } E_{in}^- \\ \psi\left(\frac{X_{i+\frac{1}{2}} - X_{in}^+}{\varepsilon}\right)\cdot x^{+\prime}(t) & \text{for} \quad X_{i+\frac{1}{2}} \text{ node of } E_{in}^+ \end{cases} \tag{13}$$

mesh points velocity and position

The velocity of the mesh points (for the exterior region) are given by :

$$\mathrm{v_g}(X,t) = \frac{(X_{i+\frac{1}{2}} - X)}{|\omega_i(t)|}\mathrm{v_g}_{i-\frac{1}{2}}(t) + \frac{(X - X_{i-\frac{1}{2}})}{|\omega_i(t)|}\mathrm{v_g}_{i+\frac{1}{2}}(t), \quad \text{for} \quad X \in E_{in}^{\pm}. \tag{14}$$

we can calculate the new position of all mesh points, and $x^{\pm}(t)$ particularly :

$$x^{n+1} = x^n + \Delta t \mathrm{v}_{\mathrm{g}}^n. \tag{15}$$

A case of a fixed body : water elevation and discharge under body

Now, the problem under consideration, for *exterior* region, for a DG context is reduced to :

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\left(|S_m(t)|\overline{v}_{e,m}\right) = -\left(\widehat{G}_{e,m+\frac{1}{2}} - \widehat{G}_{e,m-\frac{1}{2}}\right) + |S_m(t)|\overline{B}_m, & \text{in} \quad E(t) \\ \eta_e = \eta_i, \quad q_e = q_i, & \text{on} \quad E(t) \cap I(t), \end{cases}$$

As for the *interior* region the problem is reduced to :

$$\begin{cases} q_i(t,x) = \underline{q}_i(t), \text{ with, } \underline{q}_i' = -\dfrac{1}{\int_{I(t)} \frac{1}{H_i}\, dx}\left([\![\frac{1}{2}\frac{q_\mathrm{i}^2}{H_\mathrm{i}^2} + \mathrm{g}H_\mathrm{i}]\!] + \int_{I(t)} g\partial_x b\, dx \right), & \text{in} \quad I(t), \\ \eta_i(t,x) = \eta_{lid}(x), & \text{in} \quad I(t), \end{cases}$$

A case of a fixed body : Inner pressure

The pression under the body $\underline{P}_{\mathrm{i}}$ is expressed by

$$\underline{P}_{\mathrm{i}}(t,x) = P_{\mathrm{atm}} - \rho \left\{ \underline{q}_i'(t) \int_{x^-(t)}^{x} \frac{\mathrm{d}x'}{H_i(x')} + \frac{1}{2}\underline{q}_i(t)^2 \left(\frac{1}{H_i(x)^2} - \frac{1}{H_i(x^-(t))^2} \right) + g\left(H_i(x) - H_i(x^-(t))\right) + \int_{x^-(t)}^{x} g\partial_x b(x')\mathrm{d}x' \right\}.$$

Time marching algorithm

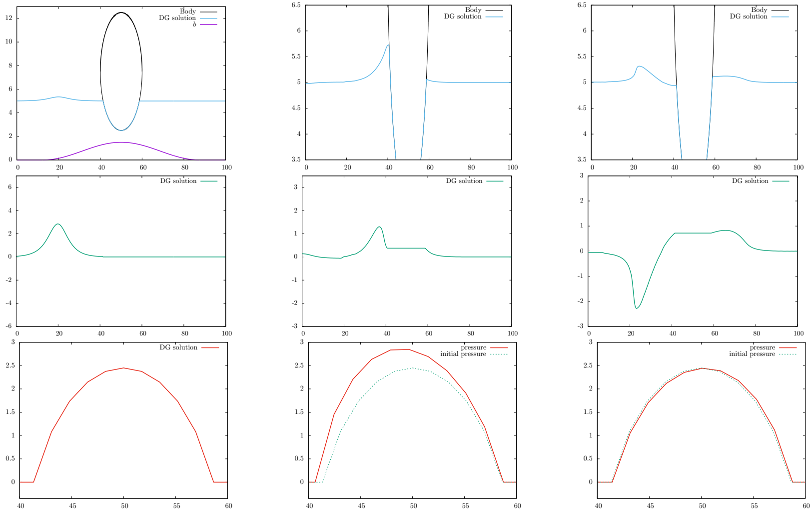
We advance in time steps using the explicit SSP-RK schemes. For instance, writing the semi-discrete NSW equation in the operator form

$$\partial_t v_h + A_h(v_h) = 0,$$

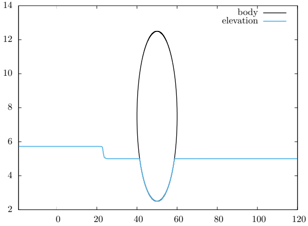
we advance from time level n to $(n+1)$ with the third-order scheme as follows :

$$\begin{aligned}
{v_h}^{n,1} &= {v_h}^{n} - \Delta t^n A_h({v_h}^{n}),\\
{v_h}^{n,2} &= \frac{1}{4}(3{v_h}^{n} + {v_h}^{n,1}) - \frac{1}{4}\Delta t^n A({v_h}^{n,1}),\\
{v_h}^{n+1} &= \frac{1}{3}({v_h}^{n} + 2{v_h}^{n,2}) - \frac{2}{3}\Delta t^n A_h({v_h}^{n,2}).
\end{aligned}$$

Test cases : case of a fixed body



Test cases : A fixed body with a shock wave



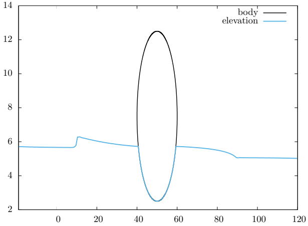


Figure – Test 6 - A fixed body with a shock wave - Free surface elevation computed for different values of time $t = 2.7\,s$ and $14.8\,s$ respectively for $k = 3$, $N_{ex} = 70$ and $N_{in} = 10$.

Test cases : A fixed body with a shock wave

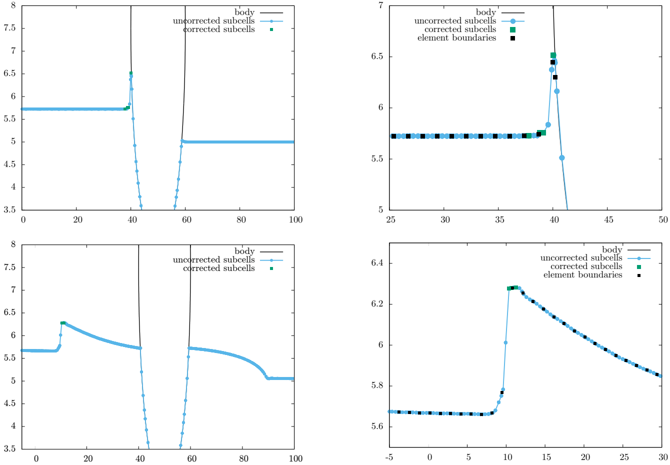


Figure – Test 6 - A fixed body with a shock wave - Free surface elevation computed for different values of time $t = 5.5\,s$ and $14.8\,s$, respectively : corrected and uncorrected subcells are respectively plotted with green squares and blue dots, with a zoom on discontinuity, for $k = 3$, $N_{ex} = 70$ and $N_{in} = 10$.

A case of a prescribed body motion

$$Z = \eta_{lid}(X) \quad \text{and} \quad z = \eta_i(t, x), \tag{16}$$

and that :

$$\begin{pmatrix} x - x_G(t) \\ z - z_G(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix} \begin{pmatrix} X - x_G(0) \\ Z - z_G(0) \end{pmatrix}. \tag{17}$$

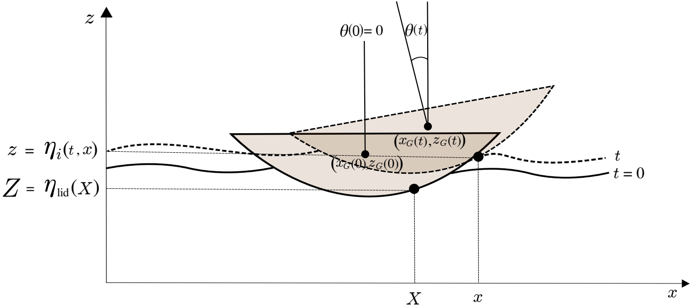


Figure – Translating and rotating body in water.

A case of a prescribed body motion : water elevation under the body

For a given x the water elevation η_i evaluated at x under the body at time t whites :

$$\eta_i(t,x) = z_G(t) + \sin(\theta(t))(X - x_G(0)) + \cos(\theta(t))(\eta_{lid}(X) - z_G(0))$$

X is given by the implicit equation :

$$\frac{x - x_G(t) + \sin(\theta(t))(\eta_{lid}(X) - z_G(0))}{\cos(\theta(t))} + x_G(0) - X = 0. \tag{18}$$

A case of a prescribed body motion : discharge under the body

For a given x the water discharge q_i evaluated at x under the body at time t whites :

$$q_i(t,x) = \begin{pmatrix} \mathrm{U}_G(t) \\ \omega(t) \end{pmatrix} \cdot \mathbf{T}\left(\mathbf{r}_G(t,x)\right) + \underline{q}_i(t),$$

$\underline{q}_i$ satisfy the following EDO :

$$\partial_t \underline{q}_i = -\left(\langle F^{\mathrm{I}} \rangle + \langle F^{\mathrm{II}} \rangle + \langle F^{\mathrm{III}} \rangle\right). \tag{19}$$

We refer to [Iguchi and Lannes(2018)] for more details.

A case of a prescribed body motion : Interior/exterior solution

The problem under consideration, for *exterior* and *interior* region is :

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}(|S_m(t)|\overline{v}_{e,m}) = -\left(\widehat{G}_{e,m+\frac{1}{2}} - \widehat{G}_{e,m-\frac{1}{2}}\right) + |S_m(t)|\overline{B}_m, & \text{in} \quad E(t), \\ \eta_e = \eta_i, \quad q_e = q_i, & \text{on} \quad E(t) \cap I(t), \end{cases}$$

$$\begin{cases} \eta_i(t,x) = z_G(t) + \sin(\theta(t))(X - x_G(0)) + \cos(\theta(t))(\eta_{lid}(X) - z_G(0)), \text{ in } \; I(t), \\ q_i(t,x) = \begin{pmatrix} \mathrm{U}_G(t) \\ \omega(t) \end{pmatrix} \cdot \mathbf{T}(\mathbf{r}_G(t,x)) + \underline{q}_i(t), \text{with, } \; \partial_t \underline{q}_i = -\left(\langle F^{\mathrm{I}} \rangle + \langle F^{\mathrm{II}} \rangle + \langle F^{\mathrm{III}} \rangle\right), \quad \text{in } \; I(t) \end{cases}$$

and the velocity of the contact points at time t, for the case of a prescribed motion body, is defined by :

$$x^{\pm\prime}(t) = \frac{\partial_x q_e(x^{\pm}(t),t) + \partial_t \eta_i(x^{\pm}(t),t)}{\partial_x \eta_e(x^{\pm}(t),t) - \partial_x \eta_i(x^{\pm}(t),t)}.$$

Test cases : case of a prescribed horizontal motion

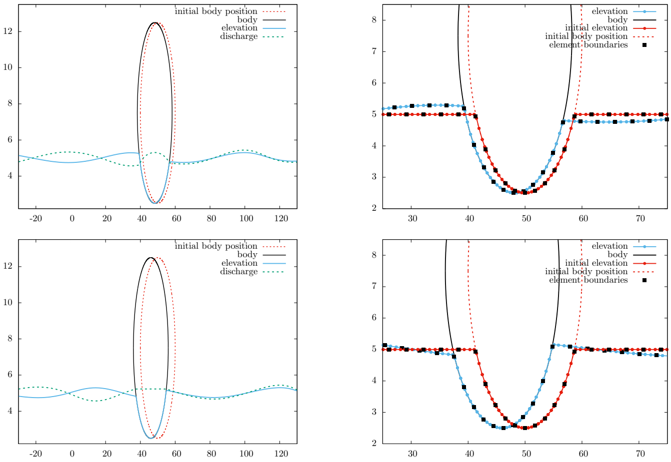


Figure – Test 6 - A floating body with prescribed horizontal motion - Free surface elevation and discharge computed for different values of time $t = 3T + \frac{T}{4}, 3T + \frac{T}{2}$ (left) with a zoom showing the displacement of the mesh nodes near water/body contact points (right), for $k = 3$ and $N_{ex} = 50$ and $N_{in} = 10$.

Test cases : Prescribed vertical motion and wet/dry transition

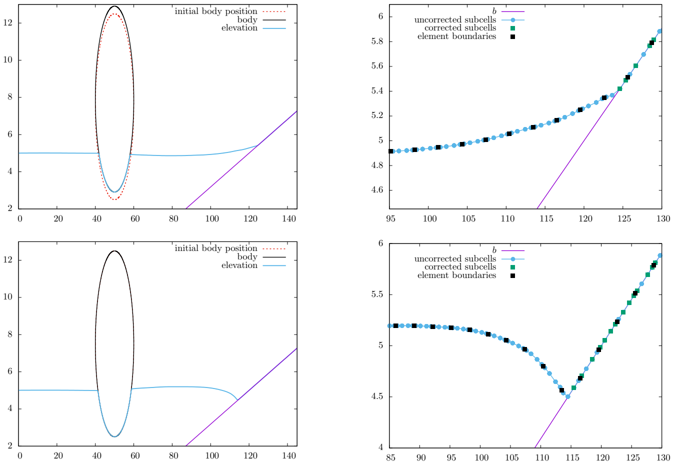


Figure – Test 6 - Test 6 - A floating body with prescribed vertical motion and wet/dry transition - Free surface elevation computed for different values of time 53 s and 60.5 s respectively (left) : corrected and uncorrected subcells are respectively plotted with green squares and blue dots, with a zoom on the shoreline (right), for $k = 3$ and $N_{ex} = 60$ and $N_{in} = 10$.

Conclusion

- Develop DG/ALE formulation for nonlinear Shallow water equations
  - model and implement all types of floating body motion (vertical, horizontal and rotational motion) in all contexts (fixed body, prescribed body motion and freely floating body)
  - track the position of the water-body contact points and redefine the resulting mesh grid at each time step following the ALE approch
- Respect the GCL and DGCL property
- It preserves the class of motionless steady states (well-balancing)
- *a posteriori* FV subcell correction
  - assures water the preservation height positivity at the subcell level
  - accuratly handle strong shocks and deal with spurious oscillations
  - It retains the highly accurate subcell resolution of DG schemes

Future works

- Use other system of equations to model this water-body inteaction, as Boussinesq equations
- Extend to a general 2D case

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