Sharp stability results for finite difference approximations of hyperbolic equations

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Joint work with J-F. Coulombel

An illustrative example

We consider the modified Lax-Friedrichs numerical scheme which reads

$$u_j^{n+1} = u_j^n + \frac{D}{2} \left(u_{j-1}^n - 2\, u_j^n + u_{j+1}^n \right) - \frac{\lambda\, a}{2} \left(u_{j+1}^n - u_{j-1}^n \right) , \quad j \geq 1 ,$$

where $D > 0$ and $\lambda\, a > 0$, along with some specific boundary condition at $j = 0$ which we shall specify later.

Using our formalism, we have $p = r = 1$ and

$$a_{-1} = \frac{D + \lambda\, a}{2} , \quad a_0 = 1 - D , \quad \text{and} \quad a_1 = \frac{D - \lambda\, a}{2} .$$

Our assumption **(A1)** is verified since

$$a_{-1} + a_0 + a_1 = 1, \text{ and } -a_{-1} + a_1 = -\lambda a.$$

Dissipativity condition

If we denote

$$F(\theta) := \sum_{\ell=-1}^{1} a_\ell \, \mathrm{e}^{\mathbf{i}\,\theta\,\ell}, \quad \theta \in [-\pi, \pi],$$

then we have

$$F(\theta) = 1 - D + D\cos(\theta) - \mathbf{i}\,\lambda\, a \sin(\theta).$$

As a consequence, provided that $0 < \lambda\, a < 1$ and $(\lambda\, a)^2 < D < 1$, we get

$$\forall\, \theta \in [-\pi, \pi] \setminus \{0\}, \quad |\, F(\theta)\,| < 1.$$

Next, we compute that

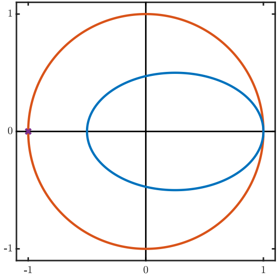
$$F(\theta) = 1 - \mathbf{i}\,\lambda\, a\, \theta - \frac{D}{2}\,\theta^2 + O(\theta^3),$$

as θ tends to 0, and we have

$$\mu := 1, \quad \text{and} \quad \beta := \frac{D - (\lambda\, a)^2}{2} > 0.$$

Assumption **(A2)** is thus satisfied provided that we have $0 < \lambda\, a < 1$ and $(\lambda\, a)^2 < D < 1$. We also assume from now on $D \neq \lambda\, a$ so that the coefficient a_1 is nonzero.

Spectral configuration



We use a boundary condition of the form:

$$u_0^n = b\, u_1^n\,, \quad n \geq 1\,,$$

where $b \in \mathbb{R}$ is a constant. In order to ensure that $\ker \mathscr{B} \cap \mathbb{E}^s(\underline{z}) \neq \emptyset$ is satisfied, we impose that

$$1 = b\, \kappa_s(\underline{z})\,,$$

where $\kappa_s(\underline{z})$ refers to the (unique) stable eigenvalue of $\mathbb{M}(\underline{z})$. Finally, we select $\underline{z} = -1$. This is the only value on the unit circle, apart from $z = 1$, which ensures that $\kappa_s(\underline{z})$ is real. Our actual boundary condition is thus

$$u_0^n = \frac{1}{\kappa_s(-1)}\, u_1^n, \quad n \geq 1.$$

Numerical illustration