

Large time stability issues for finite difference schemes

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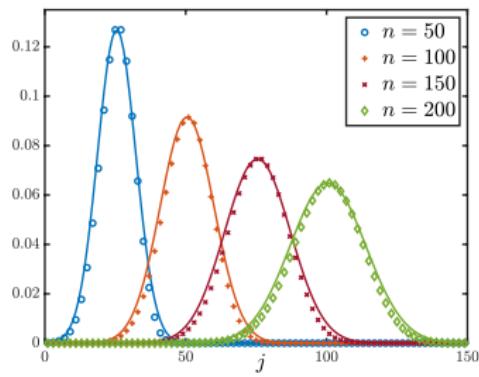
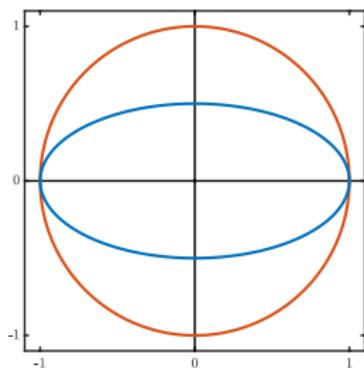


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Toulouse, October 24th 2022

Example 1 : the Lax-Friedrichs scheme

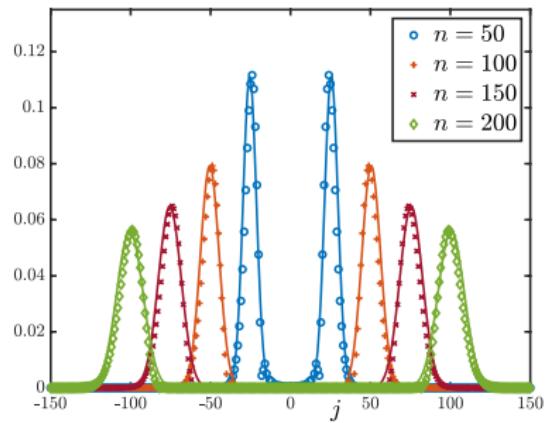
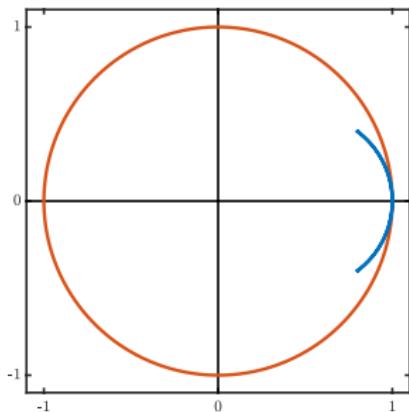
Numerical scheme : $u_j^{n+1} = \frac{1 - \alpha}{2} u_{j+1}^n + \frac{1 + \alpha}{2} u_{j-1}^n, \quad |\alpha| \leq 1.$



Gaussian bound (CLT) : $0 \leq g_j^n \leq \frac{C}{\sqrt{n}} \exp\left(-c \frac{(j - \alpha n)^2}{n}\right).$

Example 2 : an implicit scheme

Numerical scheme : $u_j^{n+1} + \frac{\alpha}{2} (u_{j+1}^{n+1} - u_{j-1}^{n+1}) = u_j^n.$



Example 2 : an implicit scheme

Gaussian bound (with two waves) :

$$|\mathcal{G}_j^n| \leq \frac{C}{\sqrt{n}} \left(\exp \left(-c \frac{(j - \alpha n)^2}{n} \right) + \exp \left(-c \frac{(j + \alpha n)^2}{n} \right) \right),$$

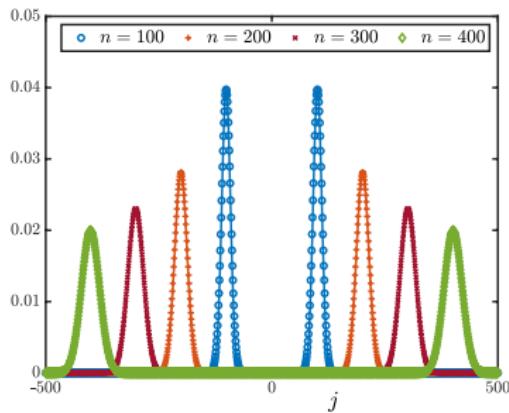
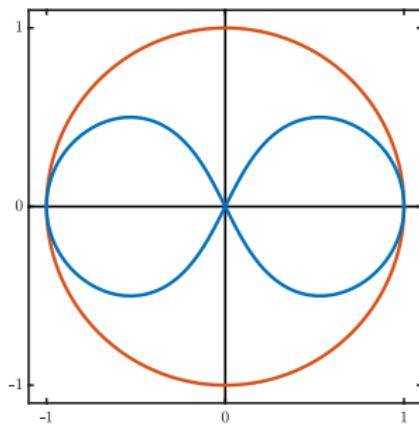
for $|j| \lesssim n$, and exponential bound otherwise :

$$|\mathcal{G}_j^n| \leq C \exp \left(-c n - c |j| \right).$$

Example 3 : an explicit scheme with two waves

Numerical scheme :

$$u_j^{n+1} = \frac{7}{16} (u_{j-1}^n + u_{j+1}^n) - \frac{1}{4} (u_{j+2}^n - u_{j-2}^n) + \frac{1}{16} (u_{j-3}^n + u_{j+3}^n).$$



Example 3 : an explicit scheme with two waves

Gaussian bound (with two waves) :

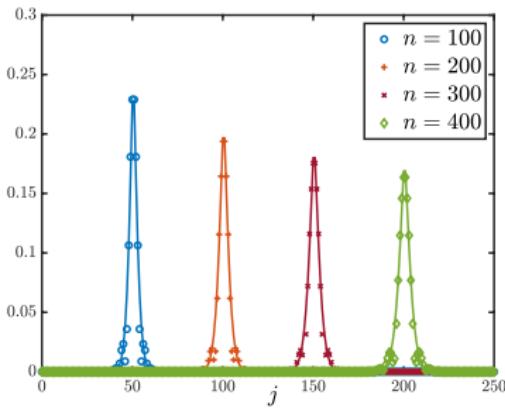
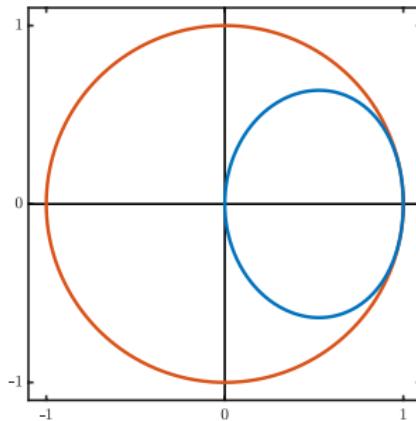
$$|\mathcal{G}_j^n| \leq \frac{C}{\sqrt{n}} \left(\exp \left(-c \frac{(j-n)^2}{n} \right) + \exp \left(-c \frac{(j+n)^2}{n} \right) \right).$$

Example 4 : the O3 scheme

Numerical scheme :

$$u_j^{n+1} = -\frac{\alpha(1-\alpha^2)}{6} u_{j-2}^n + \frac{\alpha(2-\alpha)(1+\alpha)}{2} u_{j-1}^n \\ + \frac{(2-\alpha)(1-\alpha^2)}{2} u_j^n + \frac{\alpha(2-\alpha)(\alpha-1)}{6} u_{j+1}^n.$$

(Formally) **third order** approximation of the transport equation.



Example 4 : the $O3$ scheme

Generalized Gaussian bound :

$$|\mathcal{G}_j^n| \leq \frac{C}{n^{\frac{1}{4}}} \exp\left(-c \frac{|j - \alpha n|^{\frac{4}{3}}}{n^{\frac{1}{3}}}\right).$$

The pointwise decay of the green's function is much slower than for the heat kernel.