Where is that quantum? Where is that particle?

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19 mars 2015

IHES , Le Monde Quantique

Where is that particle?

I. Non-relativistic quantum mechanics. There is a standard answer.

- $\psi \in \mathcal{H} \simeq L^2(\mathbb{R}^3, \mathrm{d}x), \|\psi\| = 1 \Rightarrow \int_B |\psi|^2(x)\mathrm{d}x =$ probability it is in B.
- Projection-valued measure:

$$B \subset \mathbb{R}^3 \to P_B; \quad P_B\psi(x) = \chi_B(x)\psi(x); \quad X\psi(x) = x\psi(x).$$

- Set of states perfectly localized in $B \subset \mathbb{R}^3$ is $L^2(B, \mathrm{d}x) = P_B\mathcal{H} \subset \mathcal{H}$.

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II. Relativistic quantum theory. Trickier: no standard answer?

• Relativistic theories have an infinite number of degrees of freedom. They are field theories.

• The quanta of these fields are thought of as particles.

$\Rightarrow$ WHERE IS THAT QUANTUM?

• Can one associate a position operator to them? And use the associated projection-valued measure to answer the yes-no question: "Is the particle inside $B \subset \mathbb{R}^3$?" Is the set of such states a vector subspace of the one-particle sector of the field theory?

Where is that quantum/particle? Is there a good position operator?

- Attitude number 1. "Yes": the Newton-Wigner position operator.

  It seems to us that the above postulates are a reasonable expression for the localization of the system to the extent that one would naturally call a system unlocalizable if it should prove to be impossible to satisfy these requirements.

  T. D. Newton & E.P. Wigner, 1949

  One either accepts the Newton-Wigner position operator when it exists, or abandons his axioms. We believe the first alternative is well worth investigation and adopt it here.

  S. S. Schweber & A. S. Wightman, 1955

  I venture to say that any notion of localizability in three-dimensional space which does not satisfy [the axioms] will represent a radical departure from present physical ideas.

  A. Wightman, 1962

Where is that quantum/particle?
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By a single particle state we mean an entity of mass m and spin 0 *which has the property that the events caused by it are localized in space.*

S. S. Schweber, 1961

To merit the term 'particle', however, such excitations [of the quantum field] should be localisable.

G. Sterman, 1993

- Attitude number 2. "No": the NW position operator breaks causality.

  One either has to accept this [referring to non-causality] or deny the possibility of measuring position precisely or even giving significance to this concept: a very difficult choice. . . . Finally, we had to recognize, every attempt to provide a precise definition of a position coordinate stands in direct contradiction to relativity.

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- Attitude number 3. That is not really true. So “Yes!”
- Attitude number 4. Relativistic particles can *obviously* not be localized in regions smaller than their Compton wavelength because of the uncertainty principle and pair creation.
- Attitude number 5. This question is of no interest. Or ill-posed.

  The position operator is only for . . . people interested in the sex of the angles, this kind of people you find among mathematical physicists, even among the brightest ones such as Schrödinger and Wigner.

  H. Bacry, 1988

⋮

- Attitude number n. . . .

⋮

Where is that quantum/particle? Is there a good position operator?

What I suggest is that

- the NW operator is ill-suited to determine the localization of quanta;
- this has nothing to do with relativity or causality;
- this is not a departure from standard physical ideas.

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To corroborate these claims, I will argue

- that this situation is familiar in nonrelativistic theories as well, even in ones with a finite number of degrees of freedom;
- that this does not at all mean one abandons all reasonable notions of localizability;
- that, on the contrary, this means one remembers that quanta are excitations of a quantum field, for which a natural notion of "localization" exists (J. M. Knight, 1961);
- that one can readily show that such quanta cannot be perfectly localized in bounded regions.
- that this could/should be explained in every graduate course in quantum mechanics.

Classical harmonic systems: definition

Equation of motion:

$$\ddot{q}(t) = -\Omega^2 q(t), \quad \mathbf{\Omega^2 \geq 0},\ \mathrm{Ker}\ \Omega^2 = \{0\}, \quad \mathcal{D}(\Omega^2) \subset \mathcal{K} = \mathbf{L^2}(\mathbf{K}, \mathbf{d}\mu; \mathbb{R}).$$

Hamiltonian description: $\mathcal{K}_\lambda = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_\lambda}$, $\|\cdot\|_\lambda = \|\Omega^\lambda \cdot\|_{\mathcal{K}}$ $(\lambda \in \mathbb{R})$.

$$H(X) = \frac{1}{2} p \cdot p + \frac{1}{2} q \cdot \Omega^2 q,\ X = (q, p) \in \mathcal{H} = \mathcal{K}_{1/2} \oplus \mathcal{K}_{-1/2}$$

Symplectic structure on classical phase space $\mathcal{H}$:

$$s(X, X') = q \cdot p' - q' \cdot p \quad \Rightarrow \quad \dot{q} = p, \dot{p} = -\Omega q$$

Hamiltonian flow Φ_t:

$$\Phi_t = \cos \Omega t I_2 - \sin \Omega t J, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\Omega^{-1} \\ \Omega & 0 \end{pmatrix}.$$

Classical harmonic systems: examples

(i) Finite dimensional systems of coupled oscillators. Non-relativistic.
$\mathcal{K} = \mathbb{R}^n = L^2(\mathbb{Z}/n\mathbb{Z})$; $\Omega^2 =$ a positive definite matrix ;
(ii) Lattices of coupled oscillators. Non-relativistic.
$q \in \mathcal{K} = \ell^2(\mathbb{Z}^d, \mathbb{R})$, $q : j \in \mathbb{Z}^d \mapsto q(j) \in \mathbb{R}$, and, for all $j \in \mathbb{Z}^d$,

$$(\Omega^2 q)(j) = \omega_{\mathrm{w},j}^2 q(j) - \omega_{\mathrm{n}}^2 \sum_{i \in \mathrm{nn}(j)} (q(i) - q(j)).$$

(iii) The wave or Klein-Gordon equation. Relativistic.
$q \in \mathcal{K} = L^2(\mathbb{R}^d, \mathbb{R})$, $q : x \in \mathbb{R}^d \mapsto q(x) \in \mathbb{R}$, and for all $x \in \mathbb{R}^d$

$$\Omega^2 q(x) = -\Delta q(x) + m^2 q(x).$$

Generalization: $K \subset \mathbb{R}^d$, $-\Delta+$boundary conditions; NOT relativistic.
Note: $\phi(x) = q(x)$, usually.

Classical harmonic systems:observables, local observables, local perturbations of the ground state

Definition A local structure for the oscillator field determined by Ω and $\mathcal{K} = L^2(K, d\mu : \mathbb{R})$ is a subspace $\mathcal{S}$ of $\mathcal{K}$ with the following properties:

1. $\mathcal{S} \subset \mathcal{K}_{1/2} \cap \mathcal{K}_{-1/2}$;
2. Let B be a Borel subset of K, then $\mathcal{S}_B \equiv \mathcal{S} \cap L^2(B, d\mu : \mathbb{R})$ is dense in $L^2(B, d\mu : \mathbb{R})$.

Examples: (i) Oscillator lattices: $\mathcal{S} =$ sequences of finite support.
(ii) Klein-Gordon equations ($d \geq 2$), $\mathcal{S} = C_0^\infty(\mathbb{R}^d)$.

Definition (a) Observables are "all" functions on $\mathcal{H}$, that is all functions of q and p. In particular linear ones: $\eta \cdot q, \eta \cdot p$. And polynomial ones. A local observable in B is a function of the $\eta \cdot q, \eta \cdot p$ with $\eta \in \mathcal{S}_B$.
(b) A local perturbation/excitation of the ground state ($X = 0$) in $B \subset K$ is an initial condition with support in B, by which we mean an element of $\mathcal{H}(B, \Omega) \stackrel{\text{def}}{=} \mathcal{S}_B \times \mathcal{S}_B \subset \mathcal{H}$.

Classical harmonic systems:towards the quantum treatment

- To prepare for the quantum treatment, it is convenient to introduce:

$$z_\Omega : X = (q,p) \in \mathcal{H} \mapsto z_\Omega(X) = \frac{1}{\sqrt{2}}(\sqrt{\Omega}q + i\frac{1}{\sqrt{\Omega}}p) \in \mathcal{K}^{\mathbb{C}} = L^2(K, d\mu, \mathbb{C}).$$

This identifies the real phase space $\mathcal{H}$ with a complex Hilbert space. Then

$$\forall \xi \in \mathcal{K}^{\mathbb{C}}, \quad a_{\mathrm{c}}(\xi) : X \in \mathcal{H} \mapsto \bar{\xi} \cdot z_\Omega(X) \in \mathbb{C}, \quad a_{\mathrm{c}}^\dagger(\xi) = \overline{a_{\mathrm{c}}(\xi)}$$

$$\eta \cdot q = \frac{1}{\sqrt{2}}(a_{\mathrm{c}}(\Omega^{-1/2}\overline{\eta}) + a_{\mathrm{c}}^\dagger(\Omega^{-1/2}\eta)), \quad \eta \in \mathcal{K}^{\mathbb{C}}_{-1/2},$$

$$\eta \cdot p = \frac{i}{\sqrt{2}}(a_{\mathrm{c}}^\dagger(\Omega^{1/2}\eta) - a_{\mathrm{c}}(\Omega^{1/2}\overline{\eta})), \quad \eta \in \mathcal{K}^{\mathbb{C}}_{1/2}.$$

- $H = \sum_i \omega_i a_{\mathrm{c}}^\dagger(\eta_i) a_{\mathrm{c}}(\eta_i), \text{ where } \Omega\eta_i = \omega_i \eta_i.$
- Points $X = (q,p)$ in the classical phase space $\mathcal{H}$ have an immediate physical interpretation, independent of Ω.
- Because the position and momentum variables, viewed as functions on $\mathcal{K}^{\mathbb{C}}$, depend on Ω, the physical interpretation of the points in $\mathcal{K}^{\mathbb{C}}$ DOES depend on the dynamics via Ω.
- Nobody in his right mind would use this formalism to study the classical dynamics of such harmonic systems. . .

Quantum harmonic systems: definition

Quantum Hilbert space = symmetric Fock space $\mathcal{F}(\mathcal{K}^{\mathbb{C}})$ over $\mathcal{K}^{\mathbb{C}}$. In terms of the usual creation and annihiliation operators $a^\dagger(\xi), a(\xi), \xi \in \mathcal{K}^{\mathbb{C}}$, the Hamiltonian is

$$H = d\Gamma(\Omega) = \sum \omega_i a^\dagger(\eta_i) a(\eta_i).$$

and the field and conjugate field are

$$\eta \cdot Q = \frac{1}{\sqrt{2}}(a(\Omega^{-1/2}\overline{\eta}) + a^\dagger(\Omega^{-1/2}\eta)), \quad \eta \in \mathcal{K}^{\mathbb{C}}_{-1/2},$$

$$\eta \cdot P = \frac{i}{\sqrt{2}}(a^\dagger(\Omega^{1/2}\eta) - a(\Omega^{1/2}\overline{\eta})), \quad \eta \in \mathcal{K}^{\mathbb{C}}_{1/2}.$$

Quantum harmonic systems:observables, local observables

Definition Let $\big(\mathcal{K} = L^2(K, d\mu : \mathbb{R}), \Omega, \mathcal{S}\big)$ be as above and let B be a Borel subset of K. The algebra of local observables over B is the algebra

$$\mathrm{CCR}_0(\mathcal{H}(B,\Omega)) = \mathrm{span}\,\{W_{\mathrm{F}}(z_\Omega(Y)) \mid Y = (\eta_1, \eta_2) \in \mathcal{S}_B \times \mathcal{S}_B\},$$

where, $\forall \xi \in \mathcal{K}^{\mathbb{C}}$,

$$W_{\mathrm{F}}(\xi) = \exp(a^\dagger(\xi) - a(\xi)), \quad \text{so that } W_{\mathrm{F}}(z_\Omega(Y)) = \exp -i(\eta_1 \cdot P - \eta_2 \cdot Q).$$

Example: for a lattice, the local algebra associated to the site $n \in \mathbb{Z}^d$ is simply the algebra generated by $\exp -i(aP_n - bQ_n)$, for all $a, b \in \mathbb{R}$.
For the Klein-Gordon equation, the local algebra over $B \subset \mathbb{R}^d$ is (roughly speaking) the algebra generated by all $Q(x), P(x)$, with $x \in B$:

$$\begin{aligned} W_{\mathrm{F}}(z_\Omega(Y)) &= \exp -i(\eta_1 \cdot P - \eta_2 \cdot Q) \\ &= \exp -i \left(\int_B (\eta_1(x)P(x) - \eta_2(x)Q(x))dx \right). \end{aligned}$$

Quantum harmonic systems: local perturbations of the ground state

Definition A strictly local excitation/perturbation of the vacuum with support in $B \subset K$ is a normalized vector $\psi \in \mathcal{F}(\mathcal{K}^{\mathbb{C}})$, different from the vacuum itself, which is indistinguishable from the vacuum outside of B. In other words, for all $Y = (q, p) \in \mathcal{H}(B^{c}, \Omega)$,

$$\langle \psi | W_{\mathrm{F}}(z_{\Omega}(Y)) | \psi \rangle = \langle 0 | W_{\mathrm{F}}(z_{\Omega}(Y)) | 0 \rangle.$$

This is the straightforward quantum equivalent of the classical notion of an initial condition with support in B (Knight 1961).

Questions. Are there such states? How can you characterize all such states? Are there states with a finite number of quanta with this property ?

Definition Ω is said to be strongly non-local on B if there does not exist a non-vanishing $h \in \mathcal{K}_{1/2}$ for which both h and Ωh vanish outside B. Intuitively, a strongly non-local operator is one that does not leave the support of any function h invariant.

Theorem (DB 03) Let B be a Borel subset of K. Then the following are equivalent:
(i) Ω is strongly non-local on B;
(ii) There do *not* exist states in $\mathcal{F}^0(\mathcal{K}^{\mathbb{C}})$ which are strictly local excitations of the vacuum with support in $B \subset K$;
(iii) The vacuum is a cyclic vector for the algebra of local observables $\mathrm{CCR}_0(\mathcal{H}(B^{\mathrm{c}}, \Omega))$ over the *complement of B*.

This generalizes a theorem of Knight ('61) and links it to the Reeh-Schlieder theorem ('59).

QUANTA CANNOT BE (PERFECTLY) LOCALIZED

Examples. The non-locality of Ω is a general feature of many models.

- It holds in translationally invariant lattices.
- It holds for the Klein-Gordon equation (plus potential) for example, for bounded (open) sets B.
- It is true for a system of two coupled harmonic oscillators.

Consequently, in none of those models can states with a finite number of quanta be localized excitations of the vacuum.

Should we be surprised? This result may be construed as being counterintuitive. Indeed, the Fock space structure of the Hilbert space of states of the field invites a particle interpretation of the field states and this is heavily used in physics. Now, since the notion of a particle evokes an entity that is localized in space, it may seem paradoxical that the perfectly reasonable notion of "strictly localized excitation of the vacuum" (Knight) is incompatible with the existence of only a finite number of particles in the state. What is going on?

Resolving the apparent contradiction:

- The 'particles' under discussion here are just *excited states of an extended system* $\neq$ the point particles of elementary nonrelativistic classical or quantum mechanics courses.
- Calling those excitations particles = an occasionally confusing abuse of language: such field quanta may carry momentum and energy, but they cannot be perfectly localized states of the field.
- This is not surprising, nor counterintuitive, nor a departure from physical practice, since it is true in a system with two oscillators, and in oscillator lattices. It is therefore also not particularly linked to relativity or causality.
- One should therefore not hope to associate a position operator with those quanta, with all the usual properties familiar from the description of point particles in ordinary Schrödinger quantum mechanics. In particular, the Newton-Wigner position operator IZNOGOOD.

So, to sum it all up, you could put it this way. To the question

Why is there no sharp position observable for particles?

the answer is

It is the non-locality of Ω, stupid!

Question. Can you characterize all strictly local excitations of the vacuum? (Licht '63, DB '03)

Theorem Consider an oscillator field determined by Ω, $\mathcal{K} = L^2(K, d\mu : \mathbb{R})$, and with a local structure $\mathcal{S}$. Let $B \subset K$ and suppose Ω is strongly non-local over B. Let $\psi \in \mathcal{F}(\mathcal{K}^{\mathbb{C}})$. Then the following are equivalent:
(i) $\psi \in \mathcal{F}(\mathcal{K}^{\mathbb{C}})$ is a strictly local excitation of the vacuum inside B;
(ii) There exists a partial isometry U, belonging to the commutant of $\mathrm{CCR}_0(\mathcal{H}(B^{\mathrm{c}}, \Omega))$ so that

$$\psi = U|0\rangle.$$

Example: $U = W_F(z_\Omega(Y))$, with $Y \in \mathcal{H}(B, \Omega)$ COHERENT STATE!!

More surprises?

- Since the sum of two unitaries is not in general a unitary, the strictly local excitations of the vacuum do NOT form a vector space: there is in particular no projector P_B on Fock space so that the probability of finding the system in the set B is given by $P_B\psi$!!
- This is the one axiom of the Newton-Wigner-Wightman approach that is violated by the notion of "local excitation of the vacuum" of Knight.
- But it is NOT counterintuitive, nor a departure from physical practice, since it is true for a system of two coupled oscillators. It is therefore also not particularly linked to relativity or causality.
- If P is some projector belonging to $\mathrm{CCR}_{\mathrm{w}}(\mathcal{H}(B,\Omega))$, the algebra of local observables over B, then $P|0\rangle$ is a state obtained by a local measurement inside B, but it is nevertheless NOT a local excitation of the vacuum! But how can the vacuum outside B be affected immediately by a measurement inside B? Does that not violate causality in the relativistic context?

One final quote on the subject

> *On the other hand, one of the essentially particle-like properties of the electron is that its position is an observable, there is no such thing as the position of the photon. . . .*
> *If we work at relativistic energies, the electron shows the same disease. So in this region, the electron is as bad a particle as the photon.*
>
> R.E. Peierls, 1972