REMARKS ON TACHYON FIELDS IN DE SITTER SPACE-TIME

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$(d+1)-$MINKOWSKI SPACE-TIME
d-DE SITTER SPACE-TIME

$$M_{d+1} = \mathbf{R}^{d+1}, \quad M^{(c)}_{d+1} = \mathbf{C}^{d+1},$$

Scalar product:

$$x \cdot y = x^0 y^0 - x^1 y^1 - \ldots - x^d y^d = x^0 y^0 - \vec{x} \cdot \vec{y} \,.$$

d-DIMENSIONAL DE SITTER SPACE-TIME

$$X_d = \{x \in M_{d+1} \; : \; x \cdot x = -R^2\}$$

$$X^{(c)}_d = \{x \in M^{(c)}_{d+1} \; : \; x \cdot x = -R^2\}$$

FUTURE LIGHT CONE IN MINKOWSKI

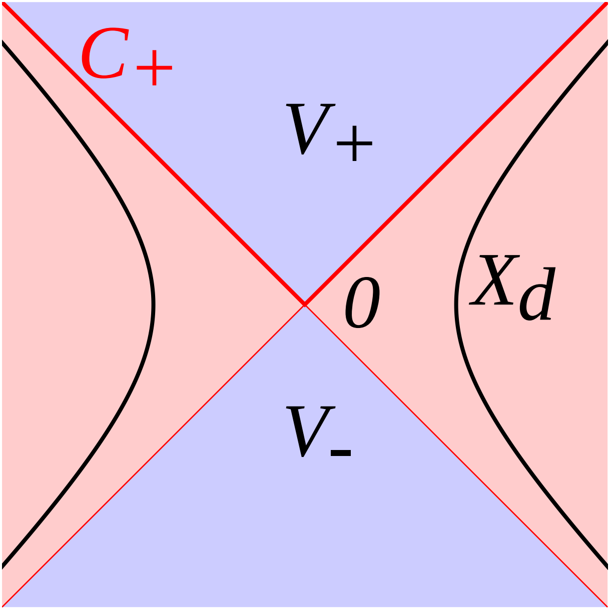
$$C_+ = -C_- = \{x \in \mathbf{R}^{d+1} \; : \; x^{(0)} > 0, \;\; x \cdot x = 0\}$$

$$V_+ = -V_- = \{x \in \mathbf{R}^{d+1} \; : \; x^{(0)} > 0, \;\; x \cdot x > 0\}$$

FUTURE TUBE

$$\mathrm{T}_+ = \{x + iy \in \mathbf{C}^{d+1} \; : \; y \in V_+\} = -\mathrm{T}_-$$

View of de Sitter space-time



Free fields in Minkowski and de Sitter

Scalar neutral free fields : labeled by mass $m > 0$.
Fully characterized by 2-point function:

$$(\Omega, \ \phi(x)\phi(y)\Omega) = \mathcal{W}_m(x, \ y)$$

Wish list

$$\left.\begin{array}{l}\text{Invariance}\\ \text{Analyticity}\end{array}\right\} \Longrightarrow \text{Locality}$$

Klein-Gordon in both variables

Canonical commutation $\Longrightarrow$ Locality

Positivity

All requirements can be satisfied for $m^2 > 0$.
Case $m^2 < 0$: tachyons ???

Analyticity of 2-point function

$\exists$ a function $W_m(z_1, z_2)$ analytic for $z_1 \in \mathrm{T}_-$, $z_2 \in \mathrm{T}_+$, such that

$$\mathcal{W}_m(x_1, x_2) = \lim_{\substack{z_1 \in \mathrm{T}_-,\ z_2 \in \mathrm{T}_+ \\ z_1 \to x_1,\ z_2 \to x_2}} W_m(z_1, z_2)\ .$$

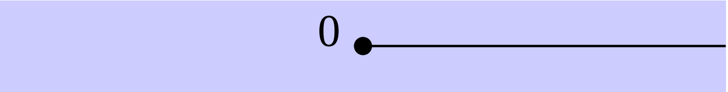
This implies the existence of Wick powers

Analyticity + Invariance $\Longrightarrow$

$$W_m(z_1, z_2) = w_m((z_1 - z_2)^2),$$

w_m holomorphic in

$$\mathbf{C} \setminus \mathbf{R}_+$$



Complex masses in Minkowski case

Explicit 2-point function on M_d :
For $d > 2$, $\mathrm{Im}\, t > 0$

$$w_m(t^2) = (2\pi)^{-d/2} m^{d-2} (-imt)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(-imt)$$
$$= \frac{\pi}{2} i^{d-1} (2\pi)^{-d/2} m^{d-2} (mt)^{1-\frac{d}{2}} H^{(1)}_{\frac{d}{2}-1}(mt)$$

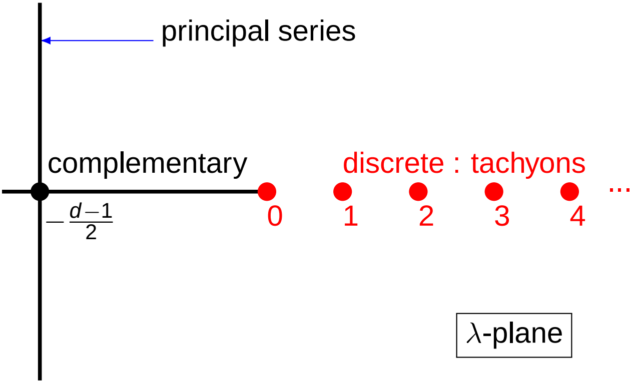
Can be continued to complex masses, e.g. to $m \in i\mathbf{R}$ keeping invariance, analyticity, canonical commutation, **loosing positivity.**

In Minkowski space, it is not possible to satisfy all requirements for $m^2 < 0$

Complex masses in de Sitter space

Parameter $= \lambda$

$$m_\lambda^2 = -\lambda(\lambda + d - 1)$$



Explicit W_λ for de Sitter

$$z_1,\ z_2 \in X_d^{(c)}, \quad (z_1 - z_2)^2 \notin \mathbf{R}_+, \quad \zeta = z_1 \cdot z_2$$

$$W_\lambda(z_1,\ z_2) = \frac{\Gamma(-\lambda)\Gamma(\lambda+d-1)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} F\left(-\lambda,\ \lambda+d-1\ ;\ \frac{d}{2}\ ;\ \frac{1-\zeta}{2}\right)\ .$$

$$\mathcal{W}_\lambda(x_1,\ x_2) = \lim_{\substack{z_1 \in \mathrm{T}_-,\ z_2 \in \mathrm{T}_+ \\ z_1 \to x_1,\ z_2 \to x_2}} W_\lambda(z_1,\ z_2)\ .$$

Satisfies all requirements when $m_\lambda^2 > 0$

Commutator

$$\mathcal{W}_\lambda(x,\ x') - \mathcal{W}_\lambda(x',\ x) = C_\lambda(x,\ x')\ ,$$

$$(\Box_x + m_\lambda^2)C_\lambda(x,\ x') = (\Box_{x'} + m_\lambda^2)C_\lambda(x,\ x') = 0$$

$$C_\lambda(t, \mathrm{x}, t', \mathrm{x}')|_{t=t'} = 0,$$

$$\frac{\partial}{\partial t'} C_\lambda(t, \mathrm{x}, t', \mathrm{x}')|_{t=t'} = i(\mathrm{ch}\, t')^{d-1}\, \delta(\mathrm{x}, \mathrm{x}')$$

Unique, Lorentz invariant solution, entire in λ.
Vanishes at space-like separations

Expansion of $\mathcal{W}_\lambda(x,\ x')$ into hyperspherical harmonics

$z = (t,\ \mathrm{x}),\ z' = (t',\ \mathrm{x}'),\ 0 < -\mathrm{Im}\, t < \pi,\ 0 < \mathrm{Im}\, t' < \pi$

$$W_\lambda(z,\ z') = \sum_{l,M} W_{\lambda l M}(z,\ z')$$

$$W_{\lambda l M}(z,\ z') = \int_{S^{d-1}} W_\lambda(t, \mathrm{x}'',\ z')\, Y_{lM}(\mathrm{x}'') Y_{lM}(\mathrm{x})\, d\mathrm{x}''$$

$$= \gamma_l(\lambda)\, \Xi_{\lambda l M}(z)\, \Xi_{\lambda l M}(z')$$

$$\gamma_l(\lambda) = \frac{1}{2}\Gamma(l-\lambda)\Gamma(1+\lambda+l+2\kappa)$$

$$\lambda \text{ real} \Longrightarrow \Xi_{\lambda l M}(z) = \overline{\Xi_{\lambda l M}(\bar{z})}$$

$$\Xi_{\lambda l M}(z) = e^{\mp\frac{i\pi}{2}(\lambda-l)}(\mathrm{ch}\, t)^{-\kappa}\mathbf{P}^{-l-\kappa}_{\lambda+\kappa}(\pm i\,\mathrm{sh}\, t) Y_{lM}(\mathrm{x}),\quad 0 < \pm\mathrm{Im}\, t < \pi$$

$\lambda = n$: tachyons

FIX $n \geq 0$ INTEGER

$$W_\lambda(z,\ z') = \Gamma(-\lambda)\, G_\lambda(z,\ z')$$

As $\lambda \to n$, $\Gamma(-\lambda) \sim (-1)^{n+1}/n!(\lambda - n)$

$$G_n(z,\ z') = \frac{\Gamma(n+1)\Gamma(d-1)}{(4\pi)^{d/2}} C_n^{\frac{d-1}{2}}(\zeta), \quad \zeta = z \cdot z'$$

Define

$$\widehat{W}_\lambda(z,\ z') = \Gamma(-\lambda)\big(G_\lambda(z,z') - G_n(z,z')\big)\big|_{\lambda=n}$$
$$= \frac{(-1)^{n+1}}{n!}\frac{d}{d\lambda}G_\lambda(z,z')\big|_{\lambda=n}$$

Pole removal procedure

If $f(\lambda) = \frac{a}{\lambda - n} + h(\lambda)$, h analytic at $\lambda = n$,

$$\widehat{f}(n) = \frac{(-1)^{n+1}}{n!}\frac{d}{d\lambda}\left(\frac{f(\lambda)}{\Gamma(-\lambda)}\right)\bigg|_{\lambda=n} = a\psi(1+n) + h(n)$$

Properties of $\widehat{W}_n(z\,,\ z')$

Analyticity
Invariance
Locality
Canonical commutator :

$$\widehat{\mathcal{W}}_n(x\,,\ x') - \widehat{\mathcal{W}}_n(x'\,,\ x) = C_n(x\,,\ x')$$

Not positive-definite
Not K-G :

$$(\Box_{z,z'} - n(n+d-1))\,\widehat{W}_n(z,z') = \frac{(-1)^{n+1}}{n!}(2n+d-1)\,G_n(z,z')$$

The space $\mathcal{E}_n$

Equivalent definitions :

$$\mathcal{E}_n = \{f \in \mathcal{D}(X_d) \ : \ \int_{X_d} f(x) p(x) \, dx = 0\}$$

$p(x)$: any homogeneous polynomial of degree n on M_{d+1} with $\Box_{\mathrm{Mink}} p = 0$.

$$\mathcal{E}_n = \{f \in \mathcal{D}(X_d) \ : \ \int_{X_d} G_n(x, \ x') f(x') \, dx' = 0 \quad \forall x \in X_d\}$$

$$\mathcal{E}_n = \{f \in \mathcal{D}(X_d) \ : \ \int_{X_d} (\xi \cdot x)^n f(x) = 0 \quad \forall \xi \in C_+\}$$

$$\mathcal{E}_n = \{f \in \mathcal{D}(X_d) \ : \ \int_{X_d} \Xi_{nlM}(x) f(x) \, dx = 0 \quad \forall l \leq n \quad \forall M \ \}$$

Main property of $\mathcal{E}_n$

For any $f \in \mathcal{E}_n$,

$$\int_{X_d \times X_d} f(x)\widehat{\mathcal{W}_n}(x,\ x')\overline{f(x')}\, dx\, dx' \geq 0$$

Free tachyonic field

In the Fock space constructed with $\widehat{\mathcal{W}_n}(x,\ x')$, $\mathcal{E}_n$ generates an invariant (physical) subspace with positive metric.

$$(\Box + m_n^2)\phi(x) = Q_n(x),$$

$$Q_n^- |\mathrm{Phys}\rangle = 0$$

Other tachyonic fields

Other possible 2-point functions

$$F_n(x,\ x') = \sum_{l,M} F_{nlM}(x,\ x')$$

$$F_{nlM}(x,\ x') = \widehat{\mathcal{W}}_{nlM}(x,\ x') = \gamma_l(n)\Xi_{nlM}(x)\Xi_{nlM}(x') \quad \text{for } l > n$$

For $l \leq n$,

$$F_{nlM}((t,\mathrm{x}),(t',\mathrm{x}')) = \frac{\Gamma(1+l+n+2\kappa)}{2(n-l)!}(\mathrm{ch}\,t)^{-\kappa}\,(\mathrm{ch}\,t')^{-\kappa}\times$$

$$\times\left\{\left[(A+iB)\mathbf{P}^{-l-\kappa}_{n+\kappa}(i\,\mathrm{sh}\,t) - \frac{(-1)^{n-l}}{A}\mathbf{Q}^{-l-\kappa}_{n+\kappa}(i\,\mathrm{sh}\,t)\right]\times\right.$$

$$\times\left[(A-iB)\mathbf{P}^{-l-\kappa}_{n+\kappa}(-i\,\mathrm{sh}\,t') - \frac{(-1)^{n-l}}{A}\mathbf{Q}^{-l-\kappa}_{n+\kappa}(-i\,\mathrm{sh}\,t')\right]$$

$$\left.+\,C\mathbf{P}^{-l-\kappa}_{n+\kappa}(i\,\mathrm{sh}\,t)\mathbf{P}^{-l-\kappa}_{n+\kappa}(-i\,\mathrm{sh}\,t')\right\} Y_{lM}(\mathrm{x})\,Y_{lM}(\mathrm{x}').$$

Constants $A > 0$, $B \in \mathbf{R}$, $C \geq 0$ may depend on l and M.

Properties of the F_n

Satisfy K-G
Canonical commutation
Positivity
Not invariant
Do not coincide with $\widehat{W}_n$ on $\mathcal{E}_n$
Coincide with $\widehat{W}_n$ on

$$\mathcal{F}_n = \{f \in \mathcal{E}_n \ : \ f(x) = (-1)^n f(-x)\}$$