Optimal Permutation estimation in crowdsourcing problems

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Ranking Problems

Crowdsourcing Problems = Aggregation of Experts' opinion

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Question / Expert								
True answer	Edible	Toxic	Toxic	Edible	Edible	Edible	Toxic	Edible
Bob	Toxic 0	Toxic 1	Edible 0	Toxic 0	Edible 1	Toxic 0	Toxic 1	Edible 1
Alice	Edible 1	Edible 0	Toxic 1	Edible 1	Edible 1	Edible 1	Toxic 1	Edible 1

1 : Correct answer 0 : Wrong answer

Our Goal

Ranking $\mathbf{n}$ experts according to their ability on $\mathbf{d}$ questions

Statistical Model

n experts and d questions

Observation Model

$Y = M + E \quad \in \mathbb{R}^{n \times d}$

- $(E_{i,k})$ independent and Subgaussian (e.g. Bernoulli)
- $M_{i,k} \in [0, 1]$ for all i, k

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- *Questions equally difficult* $\rightsquigarrow M_{ij} = a_i \qquad \approx$ [Dawid and Skene, 1979]

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Non-Parametric Models for M ≈ [Mao et al., 2018]

- Increasing columns **up to permutation π^* of rows** : $M_{\pi^{*-1}(i),k} \leq M_{\pi^{*-1}(i+1),k}$

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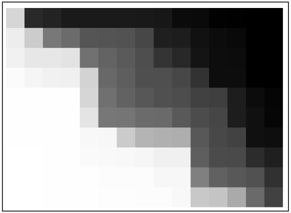
Estimation of π^*.

Partial observation of Y discussed later.



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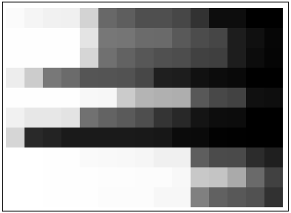

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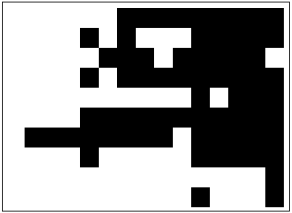

Loss functions

Permutation loss for $\hat{\pi}$

$$l(\hat{\pi}, \pi^*) := \|M_{\hat{\pi}^{-1}} - M_{\pi^{*-1}}\|_F^2 = \sum_{i=1}^{n} \sum_{k=1}^{d} (M_{\pi^{-1}(i),k} - M_{\pi^{*-1}(i),k})^2$$

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Remark :

- Estimation of π^* is "less demanding" than estimation of M.
- Estimating a bi-isotonic matrix computationally simple.

Interpretation of permutation Loss

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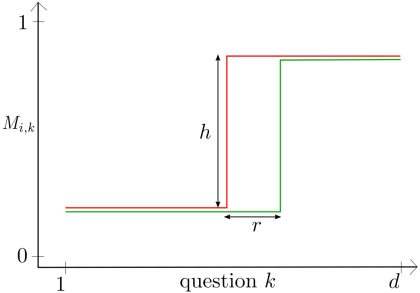

If green and red misclassified : Perm-Loss = $2rh^2$.

Minimax Risk

$$\mathcal{R}^*_{perm}[n,d] = \inf_{\hat{\pi}} \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}^{-1}} - M_{\pi^{*-1}}\|_F^2]$$

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Recovering π^* is easier then estimating M

$$\mathcal{R}^*_{perm}[n,d] \lesssim \mathcal{R}^*_{est}[n,d]$$

Related Rectangular Problems :

- **Two permutations** [Mao et al., 2018, Shah et al., 2019] :
  M is bi-isotonic up to permutations π_1^* and π_2^* of rows and columns.
  Objective : ranking the experts and the questions.

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Short story :

- No computational gap for *parametric models* (BLT, noisy sorting)
- mostly unknown for *non-parametric* models computational gaps are conjectured

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Our Results

- Control of $\mathcal{R}^*_{perm}(n, d)$
- A polynomial-time procedure achieves $\mathcal{R}^*_{perm}(n, d)$

Analysis of least-squares estimator [Shah et al., 2016]

- $\mathbf{\Pi}_n$ collection of all permutations of $[n]$
- Biso collection all bi-isotonic matrices in $[0,1]$

Least-square estimator

$$(\hat{M}^{\mathrm{LS}}, \hat{\pi}^{\mathrm{LS}}) = \underset{\widetilde{M} \in \mathrm{Biso}, \widetilde{\pi} \in \mathbf{\Pi}_n}{\arg\min} \left(\|\widetilde{M}_{\widetilde{\pi}} - Y\|_F^2\right)$$

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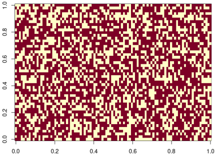

Matrix Y.

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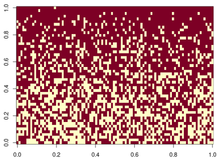

Matrix $Y_{\hat{\pi}^{\mathrm{LS}},.}$

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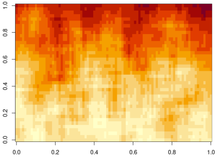

Matrix $\hat{M}^{\mathrm{LS}}_{\hat{\pi}^{\mathrm{LS}},.}$.

Proposition (e.g.[Shah et al., 2016])

$$\mathbb{E}[\|\widehat{M} - M\|_F^2] \lesssim n + (\sqrt{nd} \wedge nd^{1/3})$$

In this presentation, $\asymp$, $\lesssim$, $\gtrsim$ is up to polylogarithms

Minimax Estimation Rates

Remarks :

- $\hat{M}^{\mathrm{LS}}$ is minimax for the estimation loss

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$\mathcal{R}^*_{perm}$	? ?	? ?	n
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But the algorithms are not polynomial time.

Global Average Comparison

e.g. [Pananjady and Samworth, 2020, Shah et al., 2019]

A simple ranking method :

- For each expert i, average performances on **all** questions :

$$\overline{Y}_i = \frac{1}{d}\sum_{k=1}^{d} Y_{i,k}$$

- Rank experts according to their average : $\hat{\pi}^{\mathrm{av}}$

Performances and failures of Global Average

Perfect expert on easy questions VS random expert :

$$M_{1,.} = (0.5, 0.5 \ldots 0.5, 0.5, \underbrace{0.9, 0.9, 0.9, 0.9}_{\asymp\sqrt{d}}) \quad ; \qquad M_{2,.} = (0.5, 0.5, \ldots, 0.5, 0.5)$$

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Sup-optimality of Global average :

- comparisons are not localized (similar phenomenon in tournament problems)
- Furthermore, one-to-one comparisons are not sufficient...

Improvements in [Mao et al., 2018] using local averages on bins.

Localization and Hierarchical clustering [Liu and Moitra, 2020]

[Liu and Moitra, 2020] **consider only** $d = n$, and provide a **poly. time** estimator $\hat{\pi}^{(LM)}$

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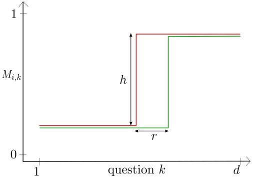
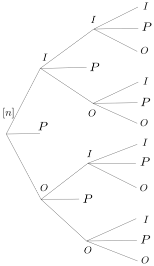


Localization through change-point detection.



Hierarchical sorting.

Summary

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Global average (UB)	$n\sqrt{d}$	$n\sqrt{d}$	$n\sqrt{d}$
Ext. of [Liu and Moitra, 2020] (UB)	d	d	n

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Remarks :

- Poly. time method of [Liu and Moitra, 2020] minimax for $d = n$
- Known UB for rates in $\mathcal{R}^*_{est}$ and $\mathcal{R}^*_{perm}$ not in polynomial time.

Minimax risks and polynomial time algorithm

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There exists a estimator $\hat{\pi}$ of π^* which is **poly. time** and **minimax optimal**

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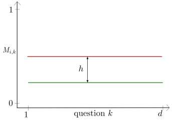
Consequence : Optimal estimation rate of M achievable in polynomial time.

From global to local averages



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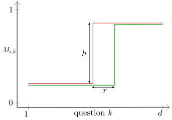

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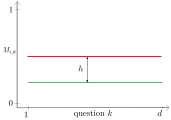

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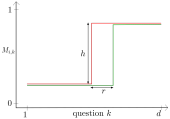

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Idea :

- Local difference between experts $\rightsquigarrow$ a high-variation signature
- Variation signatures detectable at larger scale

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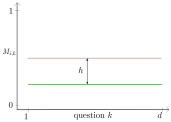

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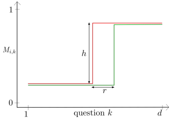

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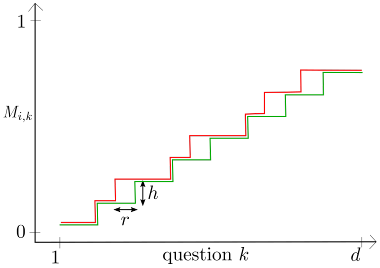
undistinguishable if $\|M_{1,.} - M_{2,.}\|_2^2 \lesssim \sqrt{d}$.

Idea :

- Local difference between experts
  $\rightsquigarrow$ a high-variation signature
- Variation signatures detectable at larger scale

Procedure

- Localize areas where any of the two experts varies by more than h...
- ... and compute local averages.



CUSUM Statistic :
$C_{l,r} = \frac{1}{r}\left(\sum_{k=l}^{l+r-1} Y_{1,k} - \sum_{k=l-r}^{l-1} Y_{1,k}\right)$

Pick height $h > 0$ and scale $r > 0$:

Step 1 High-Variation Detection $C_{l,r} \gtrsim h$

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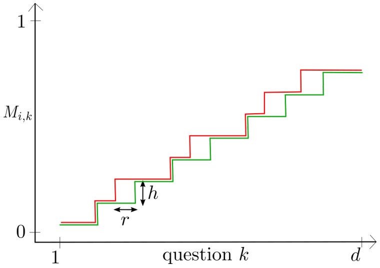
$\Psi(S_{r,h}) = \frac{1}{\sqrt{|S_{r,h}|}} \sum_{k \in S_{r,h}} (Y_{2,k} - Y_{1,k})$



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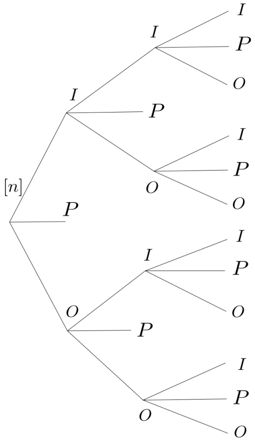
Proposition

Whp valid comparison if $\|M_{1,.} - M_{2,.}\|_2^2 \gtrsim d^{1/6}$

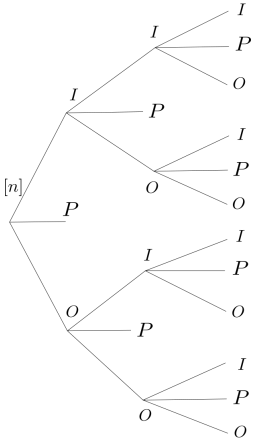
$\rightsquigarrow$ Conversely, optimal for $n = 2$.

Summary

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
$\mathcal{R}^*_{perm}$	$nd^{1/6}$	$n^{3/4}d^{1/4}$	n
$\mathcal{R}^*_{est}$	$nd^{1/3}$	$\sqrt{nd}$	n
Global average (UB)	$n\sqrt{d}$	$n\sqrt{d}$	$n\sqrt{d}$
Ext. of [Liu and Moitra, 2020] (UB)	d	d	n



Start from the **complete set** $[n]$ of experts

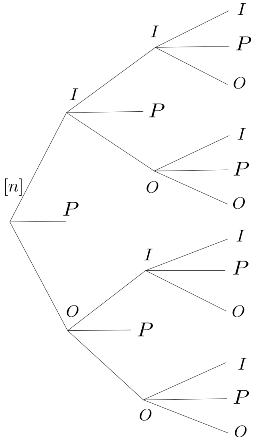


Start from the **complete set** $[n]$ of experts

Build a **Trisection** (O, P, I) of this set where :

1. Experts in O are whp among the $n/2$ worst
2. Experts in I are whp among the $n/2$ best
3. Experts in P are undecided

Hierarchical Sorting Tree



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⇝ Random partition $\widehat{\pi}$

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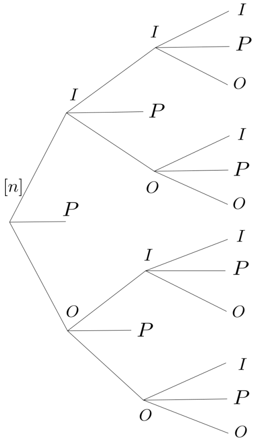

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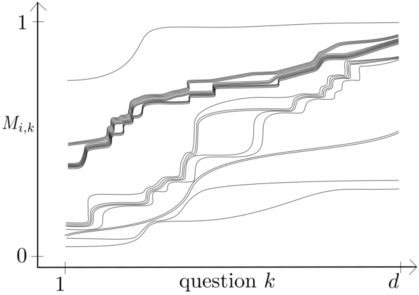
$\rightsquigarrow$ **ordered partition** of $[n]$
  $\rightsquigarrow$ Random partition $\widehat{\pi}$

Lemma

$$l(\widehat{\pi}, \pi^*) \lesssim \sum_{\overline{P}} \|M(\overline{P}) - \overline{M}(\overline{P})\|_F^2 \ ,$$

where $\overline{P} \supset P$ (slighty larger set)

Partitionning a group G into three blocks



General Strategy :

$O = \varnothing$; $I = \varnothing$
For all heights h, scales r.

1. **Dimension Reduction**
   $\rightsquigarrow$ high-variation regions h of mean expert at scale r

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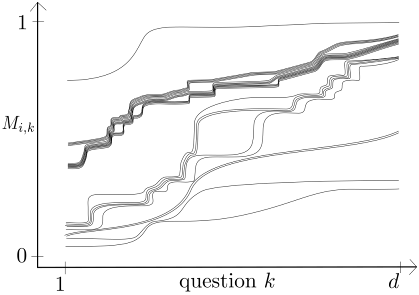

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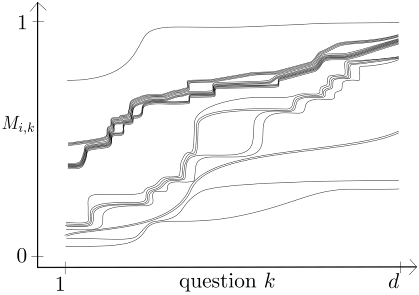

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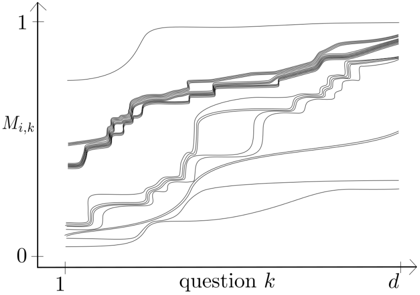
   $G \leftarrow G \setminus (U \cup L)$; $O \leftarrow O \cup L$; $I \leftarrow I \cup U$



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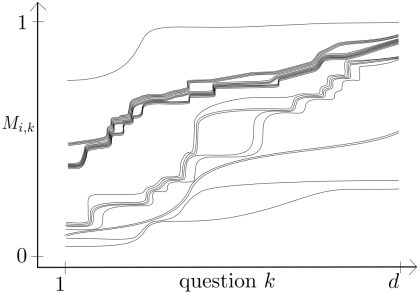

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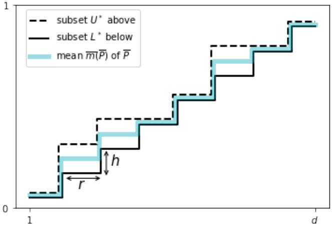
Toy example (with two pure subgroups)



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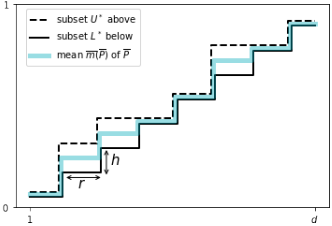

Variation detection :

$\rightsquigarrow$ keeping windows of size r with variation h

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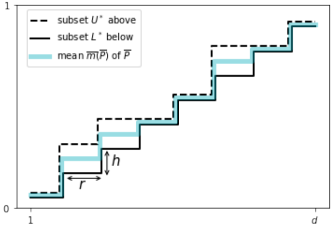

Variation detection :
⇝ keeping windows of size r with variation h
Aggregation :
⇝ rescaled sum of observations on each window :

$$\frac{1}{2}\sqrt{r}h \times \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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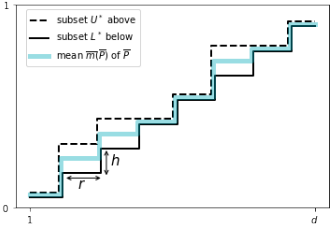

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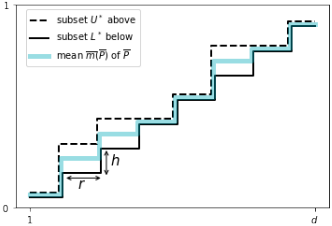

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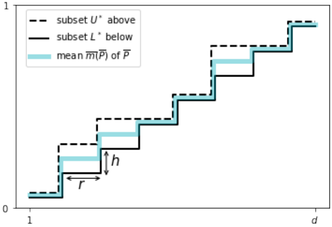

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Direction ω **selection** : ~~right singular vector~~
left singular vector + image thresholding + correction
($\neq$ [Liu and Moitra, 2020])

Suboptimality of the procedure

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
$\mathcal{R}^*_{perm}$	$nd^{1/6}$	$n^{3/4}d^{1/4}$	n
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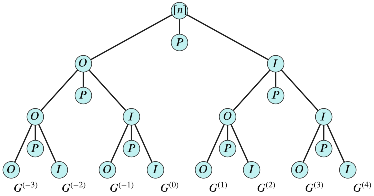
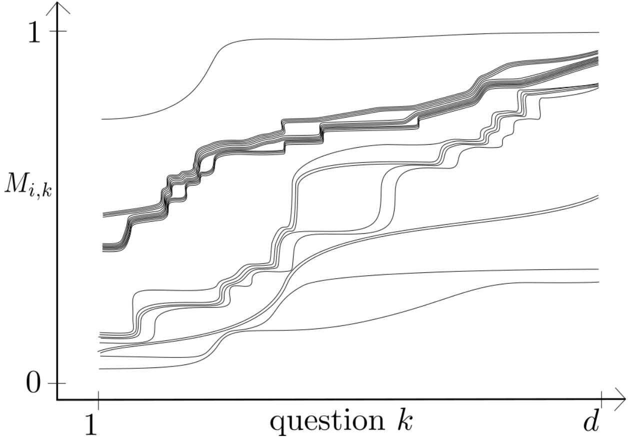
- . . . but **oblivious** of previous structure found in the data



⇝ Hierarchical Sorting with Memory which is optimal.

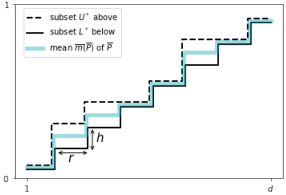
Bi-isotonic M



Each line $M_{i,\cdot}$ represents an expert i

Which information is brought by the tree ?

Our vanilla dimension reduction techniques :
Detection of variations of the mean expert in G



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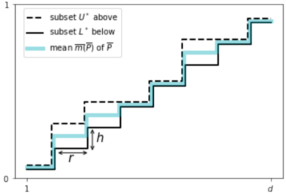

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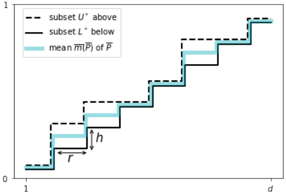
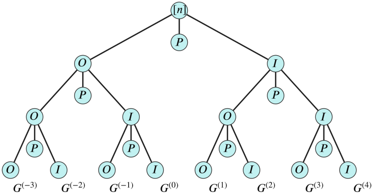




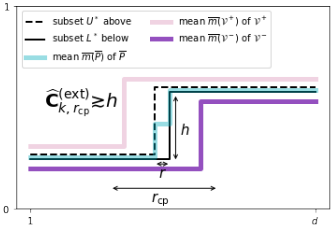
Idea :
Using the partial **ordering** to :

- decrease the variance of the CUSUM
(with $\mathcal{V} \supset G$ experts)
- Estimate the width Δ of G
$\Delta_k = \max_{i \in G} M_{i,k} - \max_{i \in G} M_{i,k}$ of G
by comparing mean experts in groups above and below G.

Fix a height h, and a scale r (possibly too small for G).
Consider expert sets $\mathcal{V}^+$ above G and $\mathcal{V}^-$ below G

Simultaneously check :

1. If **variations** at scale r higher than h
$\widehat{\mathbf{C}}^{(ext)}_{k,r} = \frac{1}{r} \sum_{l=k+1}^{k+r} \overline{y}_l(\mathcal{V}^+ \cup \mathcal{V}^-) - \sum_{l=k+1}^{k+r} \overline{y}_l(\mathcal{V}^+ \cup \mathcal{V}^-)$



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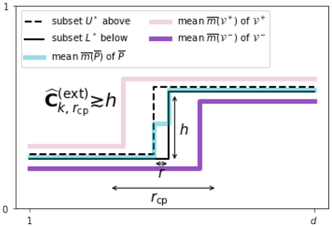
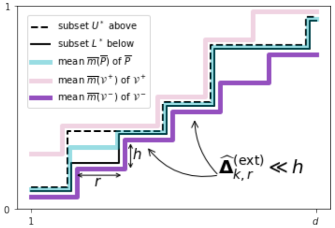
2. If the **width** of G at scale $\frac{r}{2}$ higher than h.
$\widehat{\Delta}^{(ext)}_{k,r} = \frac{1}{r}\sum_{l=k-r}^{k+r} \overline{y}_l(\mathcal{V}^+) - \overline{y}_l(\mathcal{V}^-)$





Main result

Estimator $\widehat{\pi}^{WM}$ with this new **dimension reduction** step

Theorem

$$\textit{Max-Perm}(\hat{\pi})^{WM} \lesssim \left[nd^{1/6} \wedge (n^{3/4} d^{1/4}) \right] + n \asymp \textit{MiniMax-Perm}$$

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$\rightsquigarrow$ As a corollary, minimax polynomial-time estimator of M.

Conclusion

- No **computational gap** for this ranking (and estimation) problem
- In comparison to $n = d$, rectangular setting requires **new ideas** :
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For **two permutations**, existence of a computational gap is not clear.

References I

Bradley, R. A. and Terry, M. E. (1952).
Rank analysis of incomplete block designs : I. the method of paired comparisons.
Biometrika, 39(3/4) :324–345.

Braverman, M. and Mossel, E. (2008).
Noisy sorting without resampling.
In Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms, pages 268–276.

Chen, P., Gao, C., and Zhang, A. Y. (2020).
Partial recovery for top-k ranking : Optimality of mle and sub-optimality of spectral method.
arXiv preprint arXiv :2006.16485.

Chen, Y., Fan, J., Ma, C., and Wang, K. (2019).
Spectral method and regularized mle are both optimal for top-k ranking.
Annals of statistics, 47(4) :2204.

Dawid, A. P. and Skene, A. M. (1979).
Maximum likelihood estimation of observer error-rates using the em algorithm.
Journal of the Royal Statistical Society : Series C (Applied Statistics), 28(1) :20–28.

Flammarion, N., Mao, C., and Rigollet, P. (2019).
Optimal rates of statistical seriation.
Bernoulli, 25(1) :623–653.

Liu, A. and Moitra, A. (2020).
Better algorithms for estimating non-parametric models in crowd-sourcing and rank aggregation.
In Abernethy, J. and Agarwal, S., editors, Proceedings of Thirty Third Conference on Learning Theory, volume 125 of Proceedings of Machine Learning Research, pages 2780–2829. PMLR.

Mao, C., Pananjady, A., and Wainwright, M. J. (2018).
Breaking the $1/\sqrt{n}$ barrier : Faster rates for permutation-based models in polynomial time.
In Conference On Learning Theory, pages 2037–2042. PMLR.

Pananjady, A. and Samworth, R. J. (2020).
Isotonic regression with unknown permutations : Statistics, computation, and adaptation.
arXiv preprint arXiv :2009.02609.

References III

Shah, N., Balakrishnan, S., Guntuboyina, A., and Wainwright, M. (2016).
Stochastically transitive models for pairwise comparisons : Statistical and computational issues.
In International Conference on Machine Learning, pages 11–20. PMLR.

Shah, N. B., Balakrishnan, S., and Wainwright, M. J. (2019).
Feeling the bern : Adaptive estimators for bernoulli probabilities of pairwise comparisons.
IEEE Transactions on Information Theory, 65(8) :4854–4874.