1. The Euclidean conformal algebra for a 3 dimensional CFT is given by

$$[K_p, P_q] = 2M_{pq} - 2\delta_{pq}D \qquad [D, P_p] = P_p \qquad [D, K_q] = -K_q$$

$$[D, M_{pq}] = 0 \qquad [M_{pq}, K_r] = \delta_{qr}K_p - \delta_{pr}K_q \qquad [M_{pq}, P_r] = \delta_{qr}P_p - \delta_{pr}P_q$$

$$[M_{pq}, M_{rs}] = \delta_{qr}M_{ps} - \delta_{pr}M_{qs} - \delta_{qs}M_{pr} + \delta_{ps}M_{qr}$$

Further the † acts as follows

$$K_p^\dagger = P_p \qquad D^\dagger = D \qquad M_{pq}^\dagger = M_{pq}$$

Introduce the state $|\frac{1}{2}, 0\rangle$ which is the lowest dimension state (also called the primary state) of dimension $\frac{1}{2}$ and spin 0 so that

$$D|\frac{1}{2}, 0\rangle = \frac{1}{2}|\frac{1}{2}, 0\rangle \qquad M_{pq}|\frac{1}{2}, 0\rangle = 0 \qquad K_p|\frac{1}{2}, 0\rangle = 0 \tag{1}$$

The fact that this state is a primary is the statement that it is annihilated by K_p. Assume that this state is normalized

$$\langle\frac{1}{2}, 0|\frac{1}{2}, 0\rangle = 1 \tag{2}$$

The complete conformal multiplet built on this state is obtained by taking $|\frac{1}{2}, 0\rangle$ together with all descendents (allow any number of P_q's to act)

$$P_{p_1} \cdots P_{p_k}|\frac{1}{2}, 0\rangle \tag{3}$$

Not all of these should states are retained. To see this, argue that the dimension $\frac{5}{2}$ and spin 0 state given by (sum over i with $i = 1, 2, 3$)

$$|\frac{5}{2}, 0\rangle = P_i P_i|\frac{1}{2}, 0\rangle \tag{4}$$

is both primary (it is annihilated by K_p) and null ($\langle\frac{5}{2}, 0|\frac{5}{2}, 0\rangle = 0$). Convince yourself that descendants of this null primary are also null. This null state actually corresponds to the massless Klein Gordon equation in Euclidean spacetime

$$\partial_i\partial_i\phi = 0 \tag{5}$$

for a free massless scalar field.

2. Repeat the above exercise, but now in Minkowski space. Thus, indices now run over $0, 1, 2$ and the Kronecker delta is replaced by the Minkowski metric.

3. Consider the state $|\Delta, 0\rangle$. Compute the norm of the state $P_i P_i|\Delta, 0\rangle$. For what values of Δ is this norm negative? You should find that $\Delta < \frac{1}{2}$. For these values of Δ the conformal field theory is not unitary. You should also find that for $\Delta > \frac{1}{2}$ the norm of the state $P_i P_i|\Delta, 0\rangle$ is positive. If the dimension is lowered to the point that some descendants are null, then the dimension of the primary can't be lowered further without spoiling unitarity.

4. Repeat the above exercise, but now in Minkowski space.

5. In the conformal field theory we considered, there were conserved currents of spin j and dimension $\Delta = j + 1$. A current of spin j has j indices. For one index (i.e. spin 1) denote the state as $|\Delta, 1; i\rangle$ where i is the index of the current. The action of M_{pq} is as follows

$$M_{pq}|\Delta, 1; r\rangle = \delta_{qr}|\Delta, 1; p\rangle - \delta_{pr}|\Delta, 1; q\rangle \tag{6}$$

The extension to more than one index is the usual (obvious) thing. For example, for spin two (i.e. two indices) we have

$$M_{pq}|\Delta, 2; r_1, r_2\rangle = \delta_{qr_1}|\Delta, 2; p, r_2\rangle - \delta_{pr_1}|\Delta, 2; q, r_2\rangle + \delta_{qr_2}|\Delta, 2; r_1, p\rangle - \delta_{pr_2}|\Delta, 2; r_1, q\rangle \tag{7}$$

Starting from the primary state

$$D|j + 1, j; r_1, r_2, \cdots, r_j\rangle = (j + 1)|j + 1, j; r_1, r_2, \cdots, r_j\rangle \tag{8}$$

$$K_p|j + 1, j; r_1, r_2, \cdots, r_j\rangle = 0 \tag{9}$$

Prove that the state

$$\sum_p P_r|j + 1, j; r, r_2, \cdots, r_j\rangle \tag{10}$$

is null and primary. This state has dimension $j + 2$ and spin $j - 1$. What is the character for the representation constructed on the primary $|j + 1, j; r_1, r_2, \cdots, r_j\rangle$?

6. Constructing the conserved spinning currents: In this question you will get some idea about how the spinning currents are constructed. Introduce

$$P = P_1 + iP_2 \qquad K = K_1 - iK_2 \tag{11}$$

Try the following ansatz for the spinning current

$$|j + 1, j\rangle = \sum_{n=0}^{j} c_n P^n|\frac{1}{2}, 0\rangle \otimes P^{j-n}|\frac{1}{2}, 0\rangle \tag{12}$$

Show that our choice for P ensures that the RHS, when written as a rank j tensor, is traceless. The primary condition

$$K|j+1,j\rangle = 0 \tag{13}$$

now fixes the coefficients c_n. Compare the answer you get to the known expression for the spinning currents, given by (α in this formula is just a book keeping device)

$$\begin{aligned} J_s(t,\vec{x},\alpha) &= J_{i_1 i_2 \cdots i_s}(t,\vec{x})\alpha^{i_1}\alpha^{i_2}\cdots\alpha^{i_s} \\ &= \sum_{a=1}^{N}\sum_{k=0}^{s} \frac{(-1)^k \ :(\alpha\cdot\partial)^{s-k}\phi^a(t,\vec{x})\ (\alpha\cdot\partial)^k\phi^a(t,\vec{x}):}{k!(s-k)!\Gamma(k+\frac{1}{2})\Gamma(s-k+\frac{1}{2})} \end{aligned} \tag{14}$$

When performing this computation you must use

$$K|j+1,j\rangle = \sum_{n=0}^{j} c_n K P^n|\frac{1}{2},0\rangle \otimes P^{j-n}|\frac{1}{2},0\rangle + \sum_{n=0}^{j} c_n P^n|\frac{1}{2},0\rangle \otimes K P^{j-n}|\frac{1}{2},0\rangle \tag{15}$$

as usual.