Lattice Gravity

Jan Ambjørn

Lectures

Quantum gravity, random geometry and holography
09/01/2023 - 13/01/2023

Institut Henri Poincaré, Paris

Lecture 1: basic terminology

These lecture will discuss the quantization of geometries using the path integral, i.e. we will sum over geometries using suitable classical actions for the geometries. The path integral will be regularized by using certain piecewise linear geometries, and the path integral can be performed by counting the piecewise linear geometries. The “continuum” limit of the path integrals will be obtained by letting the link-length of the piecewise linear geometries, the so-called cut off, go to zero. The Wilsonian aspect of this program will be emphasized: the continuum limit will to a large extent be independent of the details of piecewise linear geometries used for the regularization of the path integral.

A reminder of the non-relativistic path integral

$$H(x,p) = \frac{p^2}{2m} + V(x) \qquad \dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

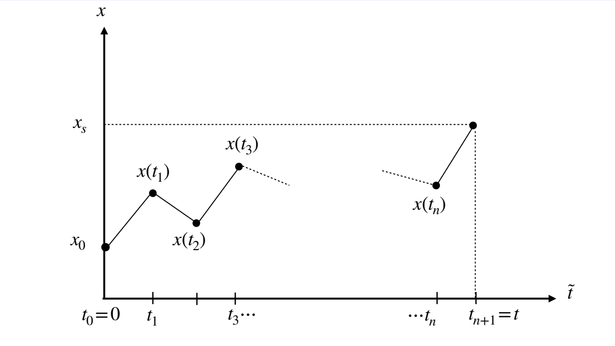
$$\dot{x}\, p - H(x,p) = \frac{1}{2} m\, \dot{x}^2 - V(x) \equiv L(x, \dot{x})$$

$$S[x] = \int_0^t dt'\; L(x(t'), \dot{x}(t')) = \int_0^t dt'\; \left[\frac{m}{2}\left(\frac{dx}{dt'}\right)^2 - V(x(t'))\right].$$

$$(\hat{x}\,\psi)(x) = x\psi(x), \qquad \hat{p}: \;\; (\hat{p}\,\psi)(x) = \frac{\hbar}{i}\,\frac{d\psi}{dx}$$

$$\hat{H} = \frac{1}{2m}\,\hat{p}^2 + V(\hat{x}) = -\frac{\hbar^2}{2m}\,\frac{d^2}{dx^2} + V(\hat{x}).$$

$$\boxed{\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle = \int_{\substack{x(0)=x_0 \\ x(t)=x_t}} \mathcal{D}x(\tilde{t})\; e^{\frac{i}{\hbar} S[x(\tilde{t})]}}$$



With $\varepsilon = t/(n+1)$, the path integral can be defined as

$$\lim_{n\to\infty} \left(\frac{m}{2\pi i \varepsilon \hbar}\right)^{\frac{n+1}{2}} \int \prod_{i=1}^{n} dx_i \exp\left(\frac{i}{\hbar}\sum_{i=0}^{n} \varepsilon \left[\frac{m}{2}\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2 - V(x_i)\right]\right)$$

$$(x_{i+1} - x_i)^2/\varepsilon \approx O(1)$$

$$\frac{|x_{i+1} - x_i|}{\varepsilon} \approx \frac{1}{\sqrt{\varepsilon}}\sqrt{\frac{2\hbar}{m}}, \quad \text{i.e.} \quad \frac{|x_{i+1} - x_i|}{\varepsilon} \to \infty.$$

In the limit $\varepsilon \to 0$ the derivative of the curve diverges everywhere and it should be viewed as a continuous curve which is nowhere differentiable. We can also estimate the length of such a curve when $\varepsilon \to 0$ (recall $\varepsilon = t/(n+1)$):

$$\ell(x(\tilde{t})) = \sum_{i=0}^{n} |x_{i+1} - x_i| \approx \sqrt{\frac{2\hbar}{m}}\, n\sqrt{\varepsilon} \approx \sqrt{\frac{2\hbar t}{m}}\, \sqrt{n}.$$

The fact that the length $\ell(x(\tilde{t})) \propto \sqrt{n}$ for large n signifies that the curve $x(\tilde{t})$ is not "nice", but actually fractal with a so-called Hausdorff dimension equal to 2.

Does this fractal structure of the paths in the path integral have any physical interpretation ?

In quantum mechanics a path of a particle is not a physical observable in the usual sense since

$$[\hat{x}, \hat{p}] = i\hbar, \qquad \Delta\hat{x}\Delta\hat{p} \geq \hbar/2$$

The concept of a path where we at all time know the position (and thus the momentum) is not valid in QM. However, we can try to measure the position at times t_i, but when we do it we will affect the momentum in an uncontrolled way, more the better we try to determine the position, due to the uncertainty relation. Choosing some compromise and taking the number of measurements to infinity, connecting the data points with straight lines, one can argue that one ends with a fractal piecewise linear path with Hausdorff dimension 2.

$$A(x_1, x_2; t) = \langle x_2 | e^{-it\hat{H}/\hbar} | x_1 \rangle = \int_{\substack{x(0)=x_1 \\ x(t)=x_2}} \mathcal{D}x(t')\, e^{iS[x(t')]/\hbar}$$

It is convenient to rotate to Euclidean spacetime,

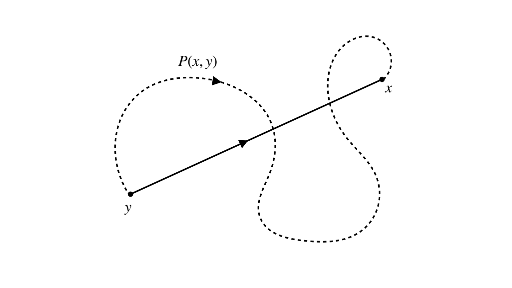
$$t \to -it_e, \quad ds^2 = -dt^2 + dx^2 \to dt_e^2 + dx^2$$

$$A_e(x_1, x_2; t) = \langle x_2 | e^{-t_e\hat{H}/\hbar} | x_1 \rangle = \int_{\substack{x(0)=x_1 \\ x(t_e)=x_2}} \mathcal{D}x(t'_e)\, e^{-S_e[x(t'_e)]/\hbar}$$

$$S_e[x(t_e)] = \int dt_e L^{(e)}(x, \dot{x}) = \int dt_e \left(\frac{m}{2}\, \dot{x}^2 + V(x) \right).$$

$$L \to L^{(e)}, \qquad -V(x) \;\to\; +V(x)$$

The free (Euclidean) relativistic particle



There exists a beautiful geometric action for the classical free relativistic particle (x, y are spacetime points)

$$S[P(x,y)] = m_0 \, \ell[P(x,y)].$$

$$G(x-y) = \int \mathcal{D}P(x,y) \; e^{-S_E[P(x,y)]} = \int \mathcal{D}P(x,y) \; e^{-m_0 \ell[P(x,y)]}.$$

Remainder:

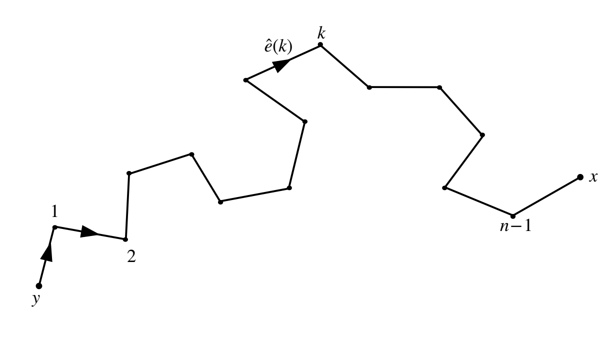
$$\hat{G}(p) = \int d^D x \, e^{-ip(x-y)} \, G(x-y) = \frac{1}{p^2 + m_{ph}^2}$$

We can write

$$G(x-y) = \int_0^\infty d\ell \, e^{-m_0 \ell} \int\limits_{\ell[P(x,y)]=\ell} \mathcal{D}P(x,y) \cdot 1 = \int_0^\infty d\ell \, e^{-m_0 \ell} \, \mathcal{N}_{x,y}(\ell),$$

where $\mathcal{N}_{x,y}(\ell)$ denotes the number of paths of length ℓ between x and y. Thus the propagator of the free particle is entirely determined by the entropy of paths.

Of course $\mathcal{N}_{x,y}(\ell) = \infty$. We need a regularization, a cut-off, to start calculation $\mathcal{N}_{x,y}(\ell)$.



$$\ell[P_n] = n \cdot a, \qquad S[P_n] = m_0 \ell[P_n] = m_0\, a\, n,$$

$$\begin{aligned} G_a(x-y) &= \sum_{n=1}^{\infty} \mathrm{e}^{-m_0\, a\, n} \int_{\{P_n\}} \mathcal{D}P_n \cdot 1 \\ &= \sum_{n=1}^{\infty} \mathrm{e}^{-m_0\, a\, n} \int \prod_{j=1}^{n} d\hat{e}(j)\; \delta\Big(a \sum_{j=1}^{n} \hat{e}(j) - (x-y)\Big), \end{aligned}$$

$$\hat{G}_a(p) = \int d^D x \, \mathrm{e}^{-ip_i(x_i-y_i)} \, G_a(x-y) = \sum_{n=1}^{\infty} \mathrm{e}^{-m_0 a n} \int \prod_{j=1}^{n} d\hat{e}(j) \, \mathrm{e}^{-ia\, p_i \hat{e}_i(j)},$$

$$\int d\hat{e} \, \mathrm{e}^{-ia\, p_i \hat{e}_i} = f(|p|a) = f(0)\Big(1 - \frac{1}{2}\sigma^2 a^2 p^2 + \mathcal{O}(a^3)\Big).$$

This last result is universal and will be true even if we had a different probability distribution $\mathcal{P}(|x-y|)$ than $\delta(|x-y|-a)$. The reason is the central limit theorem of probability theory: repeat convolutions of $\mathcal{P}(x-y)$ result in a limiting Gaussian distribution, the Fourier transform of which is also a Gaussian which when expanded, and correctly normalized, has the form $1-\frac{1}{2}\sigma^2 a^2 p^2 + \mathcal{O}(a^3)$.

$$\hat{G}_a(p) = \sum_{n=1}^{\infty} \left(e^{-m_0 a} f(|p|a) \right)^n = \frac{e^{-m_0 a} f(|p|a)}{1 - e^{-m_0 a} f(|p|a)}.$$

We are interested in the limit $a \to 0$. If we can arrange

$$e^{-m_0 a} f(|p|a) \to 1 - \frac{1}{2} a^2 \sigma^2 (p^2 + m_{\rm ph}^2) + \mathcal{O}(a^3)$$

$$\hat{G}_a(p) \quad \to \quad \frac{2}{a^2\sigma^2} \frac{1}{p^2 + m_{\rm ph}^2} = \frac{2}{a^2\sigma^2} G_{\rm cont}(p),$$

provided

$$m_0 = \frac{\ln(f(0))}{a} + \frac{1}{2} a \sigma^2 m_{ph}^2, \qquad \text{mass renormalization.}$$

Here we treat m_0 as an adjustable parameter not directly related to the physical mass of our particle. The divergent factor relating the lhs and rhs is a wave function renormalization.

Let us formulate the insight in terms of dimensionless variables:

$$\hat{G}(k,\mu) \propto \sum_{n=0}^{\infty} e^{-(\mu-\mu_c)n} \hat{\mathcal{P}}^n(k) = \frac{1}{1 - e^{-(\mu-\mu_c)}\hat{\mathcal{P}}(k)}.$$

where $\hat{\mathcal{P}}(k) = 1 - \frac{1}{2}\sigma^2 k^2 + \cdots$ and is the Fourier transform of $\mathcal{P}(|x|)$, the probability of performing one step of length $|x|$ in our piecewise linear RW. Thus

$$\hat{G}(k,\mu) \propto \frac{1+\cdots}{m^2(\mu) + k^2 + \cdots}, \qquad m^2(\mu) = \frac{2}{\sigma^2}\,(\mu - \mu_c).$$

where $m_0 a = \mu$, $\mu_c = \ln f(0)$ and $m(\mu) = m_{ph} a$.

By inverse Fourier transformation we obtain

$$G(x,\mu) \propto e^{-m(\mu)|x| + \cdots}, \quad m(\mu) \propto (\mu - \mu_c)^{\nu}, \quad \nu = \frac{1}{2}.$$

We can change between dimensionless parameters and parameters with dimension by introducing an scaling parameter $a(\mu)$ such that

$$m_{ph} a(\mu) = m(\mu) = c(\mu - \mu_c)^{\nu}, \quad x_{ph} = x\, a(\mu), \quad m_{ph} x_{ph} = m(\mu) x.$$

This ensures that the exponential fall off of the propagator survives in the limit $\mu \to \mu_c$ when expressed in terms of "physical" distances x_{ph} and a "physical" mass m_{ph}.

We can view $G(x,\mu)$ as the partition function for the ensemble of piecewise linear path in $\mathbb{R}^D$ defined by $\mathcal{P}(x)$. Recall

$$G(x,\mu) = \sum_{n=0}^{\infty} e^{-\mu\, n} \int_{\{P_n(0 \to x)\}} \mathcal{D}P_n \cdot 1$$

Expectation value of an"observable" O are defined by

$$\langle O \rangle_{\mu} = \frac{1}{G(x,\mu)} \sum_{n=0}^{\infty} e^{-\mu\, n} \int_{\{P_n(0 \to x)\}} \mathcal{D}P_n \cdot O(P_n),$$

We now use as an observable $O[P(x)]$ the dimensionless length $\ell[P(x)]$ of a curve $P(x)$ from 0 to x (i.e. when we before wrote $\ell[P_n) = n \cdot a$ we now just write $\ell[P_n] = n$), and we define the Hausdorff dimension d_H of the ensemble of paths by:

$$\langle \ell[(x)] \rangle = \langle n(x) \rangle \propto |x|^{d_H} \quad \text{for} \quad \mu \to \mu_c, \quad m(\mu)\,|x| = \text{const.}$$

$$\langle n(x) \rangle_\mu = -\frac{1}{G(x,\mu)} \frac{\partial G(x,\mu)}{\partial \mu} = -\frac{\partial \ln G(x,\mu)}{\partial \mu} \approx m'(\mu)|x|,$$

where $m'(\mu)$ denotes the derivative of $m(\mu)$ wrt μ.

Since $m(\mu) \propto (\mu - \mu_c)^\nu$ and $m(\mu)\,|x| = m_{ph}|x_{ph}|$ independent of μ:

$$m'(\mu)|x| = \frac{\nu}{\mu - \mu_c}\, m(\mu)|x| = \frac{\nu m_{ph} x_{ph}}{\mu - \mu_c} \propto m_{ph} x_{ph}\, \frac{|x|^{\frac{1}{\nu}}}{(m_{ph} x_{ph})^{\frac{1}{\nu}}}.$$

We thus conclude

$$\langle \ell(x) \rangle_\mu = \langle n(x) \rangle_\mu \propto |x|^{\frac{1}{\nu}} \qquad \text{i.e.} \quad \boxed{d_H = \frac{1}{\nu}}$$

In the case of our RWs we have $\nu = 1/2$ and thus $d_H = 2$.

Lecture 2: 2d Euclidean quantum gravity

The E-H action in D-dimensional (Euclidean) spacetime is:

$$S[g_{ij}, G, \Lambda] = -\frac{1}{16\pi G} \int d^D x \sqrt{g} \Big(R - 2\Lambda \Big)$$

However, for $D = 2$ the curvature term is topological:

$$\int d^2 x \sqrt{g}\, R(x) = 2\pi\chi.$$

It is independent of the geometry, defined by g_{ij}, and a constant as long as the topology of the 2d manifold is not changed. We will only consider that situation and restrict ourselves to the manifolds with the topology of the 2-sphere with n boundaries. We write

$$S[g_{ij}, \Lambda] = \Lambda \int d^2 x \sqrt{g} = \Lambda\, V[g] \to \Lambda\, V[g] + \sum_{i=1}^{n} Z_i L_i[g]$$

Here Z_i denote boundary cosmological constants, $V[g]$ the 2d volume of spacetime and $L_i[g]$ the length of the ith boundary, calculated in the geometry defined by the metric g_{ij}. The partition function for 2d gravity is thus

$$\begin{aligned} W(\Lambda; Z_1, \ldots, Z_n) &= \int \mathcal{D}[g] \, \mathrm{e}^{-S[g,\Lambda,Z_1,\ldots,Z_n]}, \\ W(V; L_1, \ldots, L_n) &= \int \mathcal{D}[g] \, \delta(V - V[g]) \prod_{i=1}^{n} \delta(L_i - L_i[g]), \end{aligned}$$

$$W(\Lambda; Z_1, \ldots, Z_n) = \int_0^\infty dV \, \mathrm{e}^{-\Lambda V} \int_0^\infty \prod_{i=1}^{n} dL_i \, \mathrm{e}^{-Z_i L_i} \; W(V; L_1, \ldots, L_n)$$

In these formulas we have to sum over geometries. Note that

$$W(V; L_1, \ldots, L_n) = \# \text{ of geometries with fixed } V \text{ and } L_i$$

It follows that these partition functions of two-dimensional quantum gravity are completely determined if we can count the number of geometries with volume V and boundary lengths L_i and that these partition functions in that sense are entirely entropic. We can perform this counting and find

$$\boxed{W(V; L_1, \ldots, L_n) \propto V^{n-7/2} \sqrt{L_1 \cdots L_n} \, \exp\Big(-\frac{(L_1 + \cdots + L_n)^2}{4V}\Big)}$$

Note that the action associated with $W(V; L_1, \ldots, L_n)$ is zero, which is the formal limit of $S[g]/\hbar$, for $\hbar \to \infty$. Thus 2d Euclidean quantum gravity is a pure quantum theory.

To perform this counting we first need a regularization of the geometries. Inspired by the relativistic particle case we will use geometric building blocks. It is natural to use piecewise linear geometries constructed by gluing together equilateral triangles (this regularization is denoted dynamical triangulation (DT)):

$$\int \mathcal{D}[g] \to \sum_{T \in \mathcal{T}} .$$

where $\mathcal{T}$ denotes a suitable class of equilateral triangulations. If we have a triangulation T with $|T|$ triangles and boundaries with l_i links and an assignment of length ε to the links, we can write

$$V \propto |T|\,\varepsilon^2, \qquad L_i \propto \ell_i\,\varepsilon$$

and we will take a limit where $\varepsilon \to 0$ while $|T|$ and ℓ_i go to infinity in such a way that V and L_i stay fixed.

The corresponding discretized expressions are

$$
\begin{aligned}
S_T(\mu, \lambda_1, \ldots, \lambda_n) &= \mu\, k + \sum_{i=1}^{n} \lambda_i l_i \\
w(\mu, \lambda_1, \ldots, \lambda_n) &= \sum_{T \in \mathcal{T}(n)} \mathrm{e}^{-S_T(\mu, \lambda_1 \ldots, \lambda_n)} \\
&= \sum_k \mathrm{e}^{-\mu k} \sum_{l_1, \ldots, l_n} \mathrm{e}^{-\sum_i \lambda_i l_i}\; w_{k, l_1, \ldots, l_n}
\end{aligned}
$$

Assume now that $w_{k,l_1,\ldots,l_n}$ grows exponentially with k and l_i:

$$
w_{k,l_1,\ldots,l_n} = e^{\mu_c k}\, \mathrm{e}^{\lambda_c (l_1 + \cdots + l_n)}\; \tilde{w}_{k,l_1,\ldots,l_n}
$$

$$
w(\mu, \lambda_1, \ldots, \lambda_n) = \sum_k \mathrm{e}^{-(\mu - \mu_c) k} \sum_{l_1, \ldots, l_n} \mathrm{e}^{-\sum_i (\lambda_i - \lambda_c) l_i}\; \tilde{w}_{k,l_1,\ldots,l_n}
$$

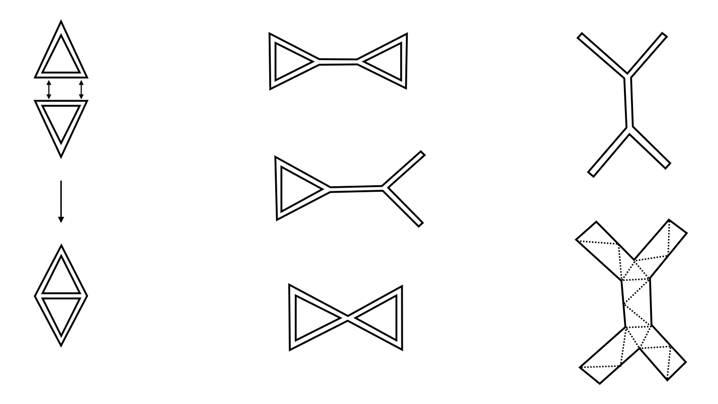
We thus obtain the formal continuum expressions by making the identifications

$$V = k\varepsilon^2, \quad \mu - \mu_c = \varepsilon^2 \Lambda, \qquad L_i = \varepsilon l_i, \quad \lambda_i - \lambda_c = Z_i \varepsilon$$

and would expect

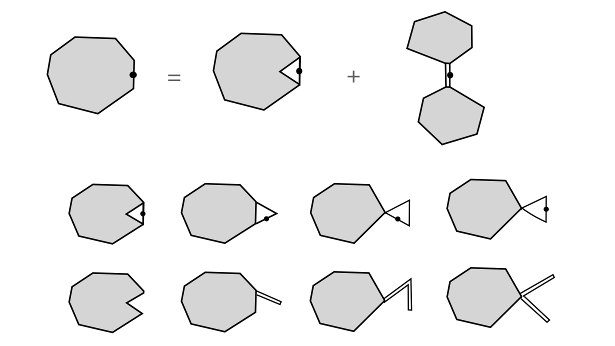
$$w(\mu, \lambda_1, \ldots, \lambda_n) \propto \varepsilon^{-\alpha} W(\Lambda, Z_1, \ldots, L_n), \qquad (\alpha = 7n/2 - 5, \;\; n \geq 2)$$

This formula will be correct and universal, i.e. independent of the detailed choice of building blocks (triangulations) that we use, for $n \geq 2$ where the pre-factor is divergent for $\varepsilon \to 0$. For $n = 0, 1$ there is a non-universal part involving small k, l.



We use a larger class of triangulations, which allows double links, since they are easier to count and from the point of view of universality it should not matter.

Counting triangulations with disk topology



Let $w_{k,l}$ denote the number of triangulations with k triangles and l boundary links, where one link is marked. Let $z\,w(g,z)$ denote the generating function for these numbers:

$$z\,w(g,z) = \sum_{k,l} g^k z^{-l}\, w_{k,l}, \quad w_l(g) = \sum_k g^k\, w_{k,l}, \quad w_0(g) = 1.$$

$$w(\mu,\lambda) = zw(z,g), \qquad g = e^{-\mu}, \qquad z = e^{\lambda},$$

and thus

$$g - g_c \propto \varepsilon^2 \Lambda, \quad z - z_c \propto \varepsilon Z, \qquad g_c = e^{-\mu_c}, \;\; z_c = e^{\lambda_c}$$

We can now find the generating function.

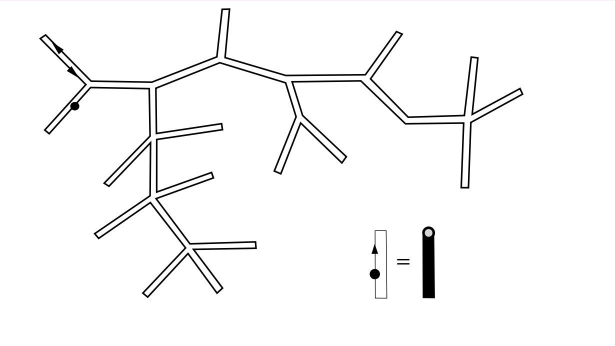
$$zw(g,z) \;\approx\; (gz)\, zw(g,z) + \frac{1}{z^2}\,(zw(g,z))^2.$$

$$\boxed{zw(g,z) = 1 + gz\Big(zw(g,z) - 1 - \frac{w_1(g)}{z}\Big) + w^2(g,z)}$$

It is a simple quadratic equation for $w(g, z)$ except that we do not know $w_1(g)$. If $g = 0$ the solution is

$$w(z) = \frac{z - \sqrt{z^2 - 4}}{2} = \frac{1}{z} + \frac{2}{z^3} + \cdots$$

This is pure boundary!



One can argue that in a neighborhood of $g = 0$ this analytic structure has to persist and the solution has to have the form

$$\boxed{w(g,z) = \frac{1}{2}\Big(z - gz^2 + (gz - c(g))\sqrt{(z - c_+(g))(z - c_-(g))}\Big)}$$

The requirement that $w(g,z) = 1/z + \mathcal{O}(1/z^2)$ leads, by expanding in $1/z$, to three equations, which determines $c(g), c_\pm(g)$.

Non-analyticity, which is linked to large k, l in $w_{k,l}$, arises when $c_+(g)$ ceases at g_c to be analytic in g and at $z \to z_c = c_+(g_c)$. Expanding in $\Delta g = g_c - g$ one can show that

$$\begin{aligned} c(g) &= g_c z_c \big(1 + \frac{1}{2}\alpha\sqrt{\Delta g}\big) + \mathcal{O}(\Delta g) \\ c_+(g) &= z_c\big(1 - \alpha\sqrt{\Delta g}\big) + \mathcal{O}(\Delta g) \end{aligned}$$

$$w(g,z) = \frac{1}{2}\Big(z - gz^2 + \varepsilon^{3/2}\, W(\Lambda, Z) + \mathcal{O}(\varepsilon^2)\Big),$$

$$z - gz^2 - 2w(g,z) = -\varepsilon^{3/2} W(\Lambda, Z)$$

$$\boxed{W(\Lambda, Z) = \Big(Z - \frac{1}{2}\sqrt{\Lambda}\Big)\sqrt{Z + \sqrt{\Lambda}}}$$

In the same way one can calculate $w(g, z_1, \ldots, z_n)$ and by power expension obtain the formula for $W(V, L_1, \ldots, L_n)$ quoted earlier. Rather than doing that explicitly, I will discuss how to calculate the two-point function.

Two-point functions or field correlators

$$\langle \phi(x)\phi(y) \rangle = f(|x-y|)$$

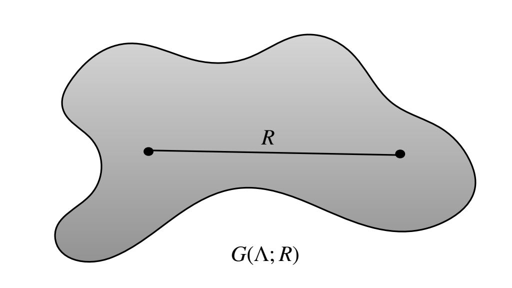
are fundamental objects in quantum field theory in flat spacetime. However, the meaning and significance of such correlators are unclear in a theory of QG. What does $|x-y|$ mean if we are integrating over the geometries which defines distance? Here is a definition which makes some sense:

$$\begin{aligned} \langle \phi(\cdot)\phi(\cdot) \rangle_R &= \int \mathcal{D}[g]\mathcal{D}\phi \; \mathrm{e}^{-S[g,\phi]} \times \\ & \iint d^D x d^D y \, \sqrt{g(x)}\sqrt{g(y)} \; \phi(x)\phi(y) \; \delta\big(D_g(x,y)-R\big) \end{aligned}$$

where $D_g(x,y)$ is the geodesic distance between x and y, measured in a metric $g_{ab}(x)$ defining a given geometry in the path integral.

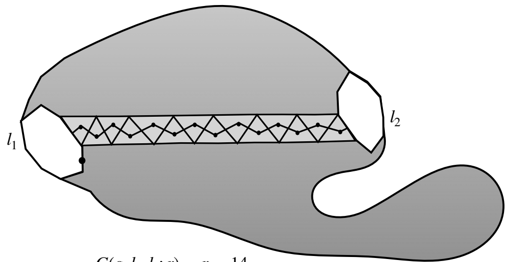
We will apply this in the 2d case of pure gravity, with $\phi(x) = 1$ and no field action, i.e.

$$G(\Lambda; R) = \int \mathcal{D}[g]\, e^{-\Lambda V[g]} \iint d^2x d^2y \sqrt{g(x)g(y)}\; \delta(D_g(x,y) - R)$$

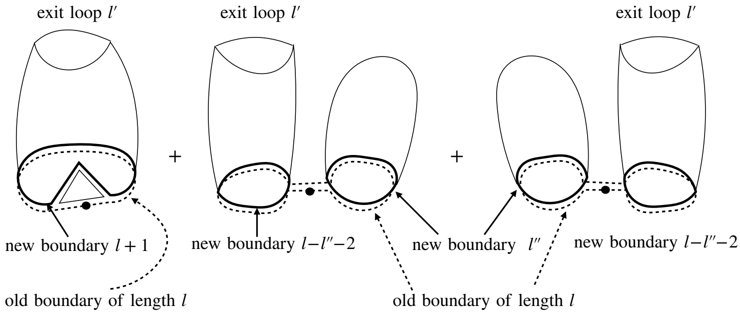


$$\boxed{G(\Lambda; R) = \Lambda^{3/4} \frac{\cosh \sqrt[4]{\Lambda}\, R}{\sinh^3 \sqrt[4]{\Lambda}\, R}}$$

There is (seemingly) no way to derive this formula using the continuum definition. Nevertheless it is relatively simple to derive it combinatorially, like for disk amplitude. Let us use the same combinatorial equation, just applied to the cylinder, where we have an entrance loop with l_1 links and an exit loop with l_2 links, separated a "geodesic" distance r



$G(g, l_1, l_2; r), \quad r = 14$



$$
\begin{aligned}
G(g,l,l';r) \;&=\; +g\,G(g,l+1,l';r) - \frac{1}{l}\frac{\partial G(g,l,l';r)}{\partial r} \\
&\quad 2\sum_{l''=0}^{l-2} G(g,l-l''-2,l';r)\,w_{l''}(g).
\end{aligned}
$$

The last term is a kind of convolution. We thus introduce the (discrete) Laplace transformation, which turns convolutions into products:

$$G(g,z,l';r) = \sum_{l=0}^{\infty} \frac{G(g,l,l';r)}{z^{l+1}}.$$

$$\frac{\partial G(g,z,l';r)}{\partial r} = \frac{\partial}{\partial z}\Big[\big(z - gz^2 - 2w(g,z)\big)\; G(g,z,l';r)\Big]$$

$$\varepsilon\, l = L, \quad \varepsilon\, l' = L', \quad z = z_c + \varepsilon Z, \quad \varepsilon^{\delta} r \propto R.$$

$$\varepsilon^{\delta} \frac{\partial}{\partial R}\, G(\Lambda,Z,L';R) = -\frac{1}{\varepsilon}\,\frac{\partial}{\partial Z}\Big(\varepsilon^{3/2} W(\Lambda,Z)\, G(\Lambda,Z,L';R)\Big)$$

$$\boxed{\frac{\partial}{\partial R}\, G(\Lambda,Z,L';R) = -\frac{\partial}{\partial Z}\Big(W(\Lambda,Z)\, G(\Lambda,Z,L';R)\Big), \quad R = \sqrt{\varepsilon}\, r}$$

$$G(\Lambda, Z, L'; R) = \frac{W(\Lambda, \bar{Z}(R; Z))}{W(\Lambda, Z)} \, e^{-\bar{Z}(R;Z)L'},$$

$$\frac{d\bar{Z}}{dR} = -W(\Lambda, \bar{Z}), \qquad \bar{Z}(0) = Z$$

The two-point function is now obtained by taking $L' \to 0$ and $Z \to \infty$.

Properties of the two-point function:

$$G(\Lambda; R) \propto \frac{1}{R^3} \qquad \text{for} \quad R \ll \frac{1}{\sqrt[4]{\Lambda}}$$

$$G(\Lambda; R) \propto \Lambda^{3/4} \, e^{-2\sqrt[4]{\Lambda}\, R} \qquad \text{for} \quad R \gg \frac{1}{\sqrt[4]{\Lambda}}$$

$$G(\Lambda; R) \propto \sum_{n=1}^{\infty} n^2 \, e^{-2n\sqrt[4]{\Lambda}\, R}$$

The two-point function and $d_h = 4$

One obtains immediately that the scaling dimension or global Hausdorff dimension $\mathrm{d}_H = 4$ since we at the discretized level have ($\Delta\mu = \mu - \mu_c = \varepsilon^2 \Lambda$, $r = \varepsilon^{-1/2} R$)

$$G(\mu, r) \propto \Delta\mu^{\frac{3}{4}} \frac{\cosh \Delta\mu^{\frac{1}{4}}\, r}{\sinh^3 \Delta\mu^{\frac{1}{4}}\, r} \approx \Delta\mu^{\frac{3}{4}}\, \mathrm{e}^{-2\Delta\mu^{\frac{1}{4}} r} \quad r \gg \Delta\mu^{-\frac{1}{4}}$$

However, $G(\Lambda, R)$ has a simple geometric interpretation which also allows us to show that the local Hausdorff dimension $d_h = 4$. Define

$$\begin{aligned} S_V(x, R; g) &= \int d^2 y \sqrt{g(y)}\, \delta\Big(D_g(x, y) - R\Big) \\ S_V(R; g) &= \frac{1}{V} \int d^2 x \sqrt{g(x)}\, S_V(x, R; g) \end{aligned}$$

$$\langle S_V(R) \rangle = \frac{1}{W(V)} \int_{V[g]=V} \mathcal{D}[g]\; S_V(R;g)$$

$$W(V) = \int_{V[g]=V} \mathcal{D}[g] \;\propto\; V^{-7/2}$$

The two-point function for fixed 2d volume V is

$$G(V;R) = \int_{V[g]=V} \mathcal{D}[g] \iint d^2x d^2y\; \sqrt{g(x)}\sqrt{g(x)}\; \delta(D_g(x,y) - R)$$

$$G(\Lambda;R) = \int_0^\infty dV\; \mathrm{e}^{-\Lambda V}\; G(V;R),$$

$$G(V;R) = \int_{-i\infty+c}^{i\infty+c} \frac{d\Lambda}{2\pi i}\; \mathrm{e}^{V\Lambda}\; G(\Lambda;R)$$

$$\langle S_V(R)\rangle = \frac{G(V;R)}{VW(V)} \;\propto\; V^{5/2}G(V;R)$$

Since we know $G(\Lambda, R)$ we obtain by inverse Laplace transformation:

$$\langle S_V(R)\rangle \propto R^3\Big(1+\mathcal{O}(\frac{R^4}{V})\Big),$$

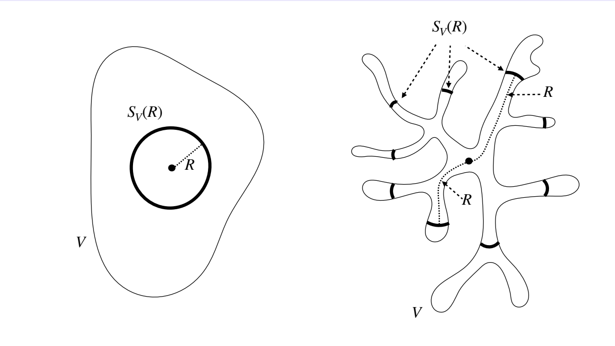
For a smooth d-dimensional geometry g we have

$$S_V(R;g) \propto R^{d-1} \quad \text{for} \quad R \ll \frac{1}{V^{1/d}}$$

If the space is fractal with Hausdorff dimension d_h we have (this is the definition of d_h)

$$\langle S_V(R)\rangle \propto R^{d_h-1} \quad \text{for} \quad R \ll \frac{1}{V^{1/d_h}}$$

Why is $\langle S_V(R) \rangle$ is not proportional to R for small R?



R is an external parameter in the path integral. No matter how small we choose R there will be many geometries like the ones on the rhs.

If we where 1d creatures, would it be possible to measure $d_h = 4$, like one could measure $d_h = 2$ for the particle path?

It is tempting to extent the lego block formalism to higher dimensional spacetimes, at least to $d = 3$ and $d = 4$. Build geometries by gluing together 3- and 4-dimensional building block, use a suitable version of the E-H action, and perform the summation over these geometries and finally extract a continuum theory, by a suitable renormalization of the bare gravitational coupling constants. Still active research, but it seems difficult to obtain interesting results. Depending on the bare dimensionless gravitational coupling constant, either the universes are completely crumpled ($d_H = \infty$) or far too extended (fractal with $d_H = 2$). This led to a search for a different class of triangulations where the geometries were "better" in the sense that they were not so fractal.

One such choice is denoted Causal Dynamical Triangulations (CDT).

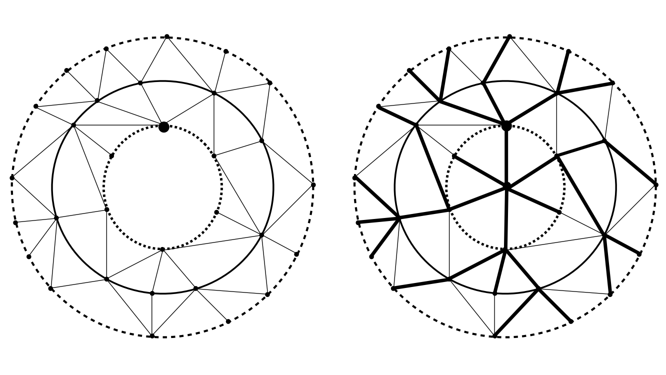
The CDT framework was first tested in a two-dimensional model, originally formulated for spacetimes with Lorentzian signatures, where a principle of micro-causality was imposed on the piecewise linear geometries entering in the path integral: as a compact one-dimensional spatial universe propagated in proper time it was not allowed to split in two. Thus the name CDT. It turns out that each such piecewise linear geometry can be rotated to Euclidean signature by rotating the proper time. Assume this rotation has been done.

The Euclidean 2d action will be the same as already used

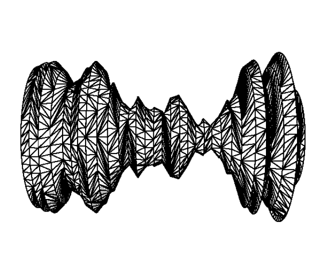
$$S[g,\Lambda,Z_1,Z_2] = \Lambda V[g] + Z_1 L_1 + Z_2 L_2 \to S[T,\mu,\lambda_1,\lambda_2] = \mu N_T + \lambda_1 l_1 + \lambda_2 l_2$$

$$G(\Lambda,Z_1,Z_2,R) = \int_{g(R)} \mathcal{D}[g]\, e^{-S[g,\Lambda]} \;\to\; G(\mu,\lambda_1,\lambda_2,r) = \sum_{\mathcal{T}(r)} e^{-S[T,\lambda_1,\lambda_2]}$$

but the geometries $g(R)$ and triangulations $\mathcal{T}(r)$ will be different from the ones used in Euclidean 2d QG.



The triangles are assumed to be equilateral, but are not drawn as such. The rhs illustrates the bijection between CDT configurations and branched polymers (also called planar trees).



We now denote $R = T$ and $r = t$, and rather than fixing the boundary cosmological constants we can fix the boundary lengths l_1 and l_2. We can thus write

$$G(g,x,y;t) = \sum_{l_1,l_2} x^{l_1} y^{l_2} G(g,l_1,l_2,t), \quad g = e^{-\mu},\ x = e^{-\lambda_1},\ y = e^{-\lambda_2}$$

$$G(g,l_1,l_2,t) = \sum_{\{\mathcal{T}(l_0,l_t;t)\}} g^{N_{\mathcal{T}(l_0,l_t;t)}}$$

$G(g,x,y,t)$ is the generating function for the number of CDT triangulations with t time steps between boundaries l_1 and l_2.

$$G(g,l,l';t_1+t_2) = \sum_{l''} G(g,l,l'';t_1)\; G(g,l'',l';t_2).$$

$$G(g,x,y;t_1+t_2) = \oint \frac{dz}{2\pi i\, z}\; G(g,x,z^{-1};t_1) G(g,z,y;t_2),$$

$$G(g,x,y;t+1) = \oint \frac{dz}{2\pi i\, z}\; G(g,x,z^{-1};1) G(g,z,y;t),$$

It is easy to find $G(g,x,y;1)$ by looking at a single time-slab

$$G(g,x,y;1) = \sum_{k=0}^{\infty}\left(gx\sum_{l=0}^{\infty}(gy)^l\right)^k - \sum_{k=0}^{\infty}(gx)^k = \frac{g^2xy}{(1-gx)(1-gx-gy)}.$$

$$G(g,x,y;t+1) = \frac{gx}{1-gx}\; G\Big(\frac{g}{1-gx},y;g;t\Big).$$

One can solve this iterative equation, but one can also directly convert it into a differential equation in the continuum limit.

Let us assume that there are critical points g_c, x_c and y_c like in 2d EDT, such that we can write

$$g = g_c e^{-\Lambda\varepsilon^2/2}, \quad x = x_c e^{-X\varepsilon}, \quad y = y_c e^{-Y\varepsilon}, \quad T = \varepsilon\, t,$$

where ε denotes the link length, while Λ, X and Y are the continuum cosmological constant and the continuum boundary cosmological constants. From the convolution equation it follows that if there exists a continuum $G_\Lambda(X, Y, T)$ it has to be related for $\varepsilon \to 0$ to $G(g, x, y, t)$ by

$$G(g, x, y; t) \to \varepsilon^{-1} G_\Lambda(X, Y, T) \quad \text{and} \quad x_c = 1,\; g_c = \frac{1}{2}.$$

Expanding to iterative equation in ε we obtain

$$\boxed{\frac{\partial G_\Lambda(X, Y; T)}{\partial T} = -\frac{\partial}{\partial X}\Big((X^2 - \Lambda) G_\Lambda(X, Y; T)\Big)}$$

The solution has the same structure as for EDT:

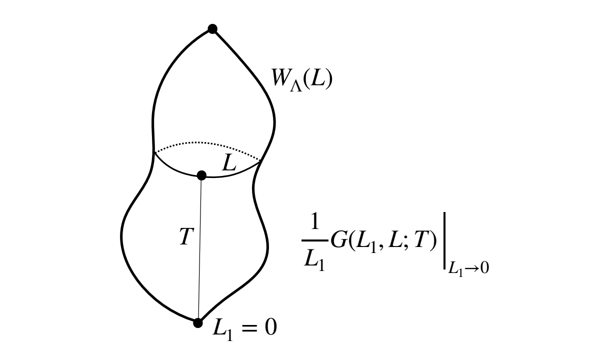
$$G_\Lambda(X, L; T) = \frac{\bar{X}^2(T; X) - \Lambda}{X^2 - \Lambda} \, e^{-\bar{X}(T;X)L}.$$

$$\frac{d\bar{X}}{dT} = -(\bar{X}^2 - \Lambda), \qquad \bar{X}(T=0; X) = X.$$

We define the disk amplitude by

$$W_\Lambda(X) \equiv \int_0^\infty dT \, G_\Lambda(X, L=0; T) = \frac{1}{X + \sqrt{\Lambda}}, \qquad W_\Lambda(L) = e^{-\sqrt{\Lambda} L}.$$

Similarly, we define the two-point function as the sum over all CDT cylinder surfaces where the entrance and exit loops have been contracted to points and where a marked point has a distance T to entrance point.



$$G_\Lambda(T) = \lim_{L_1 \to 0} \frac{1}{L_1} \int dL\, G_\Lambda(L_1, L, T)\, L\, W(L) = e^{-2\sqrt{\Lambda}T}.$$

If we return to discrete variables we have ($\mu_c = -\ln g_c = \ln 2$)

$$G_\mu(t) \propto e^{-2\sqrt{\mu - \mu_c}\, t}, \quad \nu_{cdt} = \frac{1}{2} \quad \text{i.e.} \quad d_H = 2$$

$$G_\Lambda(X,Y;T) = \int_0^\infty dL_1 \int_0^\infty dL_2 \, e^{-XL_1 - YL_2} G_\Lambda(L_1,L_2;T),$$

$$X \to \frac{d}{dL}, \qquad \frac{d}{dX} \to -L$$

Thus our differential equation for $G_\Lambda(X,Y,T)$ transforms to

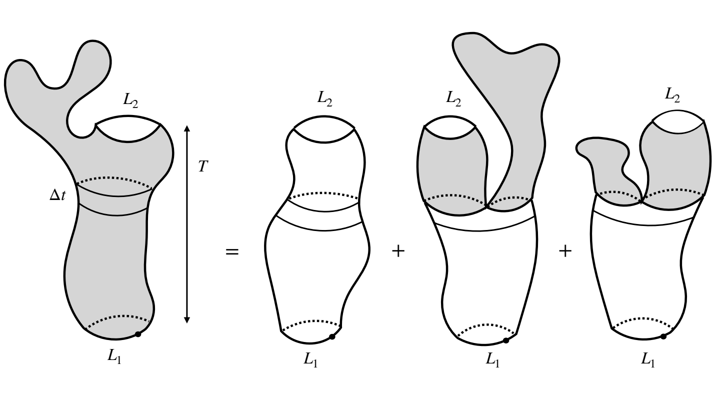
$$\frac{\partial}{\partial T} G_\Lambda(L_1,L_2;T) = -\hat{H}(L_1)\, G_\Lambda(L_1,L_2;T), \qquad \hat{H}(L) = -L\frac{d^2}{dL^2} + \Lambda L$$

$$\boxed{G_\Lambda(L_1,L_2;T) = \langle L_2 | \, e^{-\hat{H}\,T} | L_1 \rangle},$$

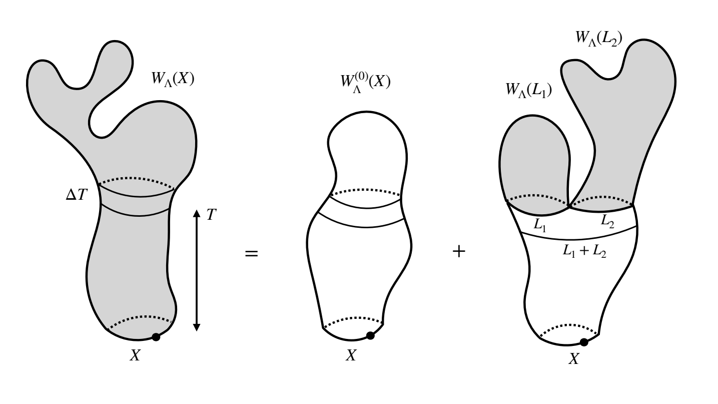
$$\Pi = -i\frac{d}{dL} \to H_{cl} = L\Pi^2 + \Lambda L \to \boxed{\mathcal{L}_{cl}^{(e)} = \frac{\dot{L}^2}{4L} + \Lambda L}$$

2d CDT represents the quantization of H_{cl}. We will meet $\mathcal{L}_{cl}^{(e)}$ also in higher dimensional CDT. We have $\Lambda\ell$ in $\mathcal{L}_{cl}^{(e)}$ and $-\Lambda\ell$ in in $\mathcal{L}_{(cl)}$ derived from H_{cl} (recall $-V(x) \to V(x)$ when $t \to -it_e$)

GCDT: showcasing quantum geometry



Let us assume we allow the creation of baby universes as a function of time with probability $g_s dt$. We do not know the baby universe (i.e. disk amplitude) $W_\Lambda(L)$, but......



$$
\begin{aligned}
W_\Lambda(X) &= W_\Lambda^{(0)}(X) + \\
&\quad g_s \int_0^\infty dT \int_0^\infty dL_1 \int_0^\infty dL_2 \; (L_1+L_2) G_\Lambda^{(0)}(X, L_1+L_2; T) W_\Lambda(L_1) W_\Lambda(L_2) \\
&= W_\Lambda^{(0)}(X) + g_s \frac{W_\Lambda^2(\sqrt{\Lambda}) - W_\Lambda^2(X)}{X^2 - \Lambda}.
\end{aligned}
$$

$$2g_s W_\Lambda(X) = \Lambda - X^2 + \sqrt{(X^2-\Lambda)^2 + 4g_s\big(g_s W^2(\sqrt{\Lambda}) + X - \sqrt{\Lambda}\big)}.$$

$$\hat{W}_\Lambda(X) := \sqrt{(X^2-\Lambda)^2 + 4g_s\big(g_s W^2(\sqrt{\Lambda}) + X - \sqrt{\Lambda}\big)}$$

$$\hat{W}_\Lambda(X) = (X-\alpha)\sqrt{(X+\alpha)^2 - \frac{2g_s}{\alpha}}, \qquad \alpha^3 - \Lambda\alpha + g_s = 0.$$

Knowing $W_\Lambda(X)$ implies that we now have close (integral) equation for $G_\Lambda(X, Y, T)$. After some algebra it can be written as

$$\boxed{\frac{\partial G_\Lambda(X, Y; T)}{\partial T} = -\frac{\partial}{\partial X}\Big(\hat{W}_\Lambda(X) G_\Lambda(X, Y; T)\Big)}$$

Thus we see that the only change going from CDT to GCDT is

$$X^2 - \Lambda \;\to\; X^2 - \Lambda + 2g_s W_\Lambda(X) = \hat{W}_\Lambda(X).$$

$$G_\Lambda(X,L;T) = \frac{\hat{W}(\bar{X}(T,X))}{\hat{W}(X)}\, e^{-\bar{X}(T)L}, \quad \frac{d\bar{X}}{dT} = -\hat{W}(\bar{X}), \quad \bar{X}(0) = X.$$

$$G_\Lambda(T) = \frac{\Sigma^3}{\alpha} \frac{\Sigma \sinh(\Sigma T) + \alpha \cosh(\Sigma T)}{\big(\Sigma \cosh(\Sigma T) + \alpha \sinh(\Sigma T)\big)^3}, \quad \Sigma = \sqrt{\alpha^2 - \frac{g_s}{2\alpha}},$$

For $g_s = 0$ we have $\Sigma = \alpha = \sqrt{\Lambda}$ and $G_\Lambda(T) = e^{-2\sqrt{\Lambda}\, T}$.

It is also possible to extract a Hamiltionian from the equation for $G_\Lambda(X, Y, T)$ as we did for pure CDT. We will return to this in relation to our discussion of higher dimensions CDT.

The 2d lattice approach is easily generalized to higher dimensions. One uses d-dimensional building blocks: in 3d tetrahedra, in 4d 4-simplices and builds piecewise linear manifolds of a given topology. In 2d we chose a trivial action since the curvature term in the gravity action was topological. In higher dimension this is not the case. A possible choice of action is the so-called Regge EH action for piecewise linear manifolds. If one uses identical building blocks the 4d action for a given given triangulation becomes exceeding simple. The Einstein-Hilbert action for a given triangulation T of a closed four-dimensional manifold in EDT can then be written as

$$\begin{aligned} S_M(G,\Lambda) &= \frac{1}{16\pi G}\int d^4\xi\sqrt{g(\xi)}\,\big(-R(\xi)+2\Lambda\big) \to \\ S_T(\kappa_2,\kappa_4) &= -\kappa_2 N_2(T)+\kappa_4 N_4(T), \end{aligned}$$

$$\kappa_2 = \frac{1}{8G}\,\frac{\sqrt{3}\varepsilon^2}{2}, \qquad \kappa_4 = \frac{2\Lambda}{16\pi G}\,\frac{\sqrt{5}\varepsilon^4}{96} + 20\arccos\left(\frac{1}{4}\right)\frac{1}{16\pi G}\,\frac{\sqrt{3}\varepsilon^2}{2},$$

The EDT partition function of four-dimensional quantum gravity is now obtained by summing over triangulations with this action

$$Z(\kappa_2,\kappa_4) = \sum_T \frac{1}{C_T}\, e^{\kappa_2 N_2(T) - \kappa_4 N_4(T)} = \sum_{N_4,N_2} e^{\kappa_2 N_2 - \kappa_4 N_4} \mathcal{N}(N_2,N_4),$$

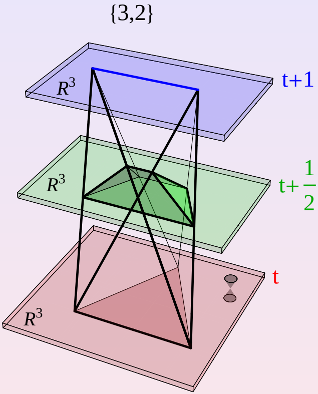
where $\mathcal{N}(N_2,N_4)$ denotes the number of such triangulations with a fixed number of two-simplices and four-simplices, N_2 and N_4. The partition function is entirely combinatoric:

$$Z(x,y) = \sum_{N_4,N_2} x^{N_2} y^{N_4} \mathcal{N}(N_2,N_4), \qquad x = e^{\kappa_2}, \quad y = e^{-\kappa_4}$$

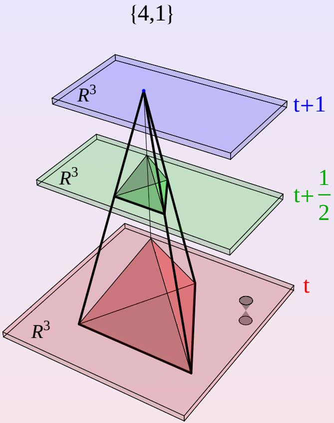
$Z(x,y)$ is the generating function for the number of four-dimensional triangulations. It is truly remarkable that four-dimensional quantum gravity in this way is purely "entropic". Unfortunately it is not yet possible to perform this counting analytically, and this leaves us presently with Monte Carlo simulations if we want to study the partition function.

Unfortunately, no clear interesting physics has yet been extracted. It seems that the “typical” configurations generated are too singular (i.e. even more singular than in 2d EDT: one encounters triangulations with infinite Hausdorff dimension). One is still investigating this, adding more terms to the action to counteract the singular configurations.

However, a different approach to obtain more regular triangulations is to adapt the CDT framework and assume the existence of a global proper time. It gave more regular triangulations in 2d and it also works in 4d. One still uses the Regge EH action, discretizes the time and uses 4d building block in a way compatible with this structure:



$S_{3,2}$ and $S_{2,3}$

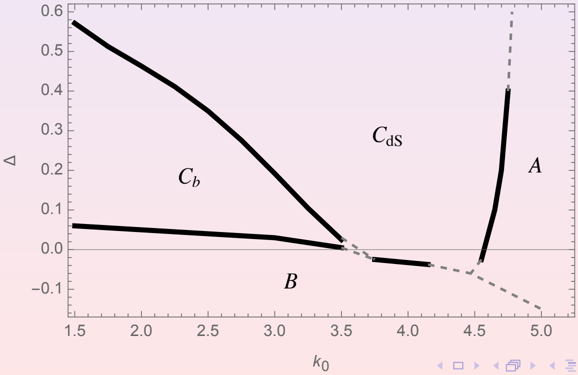


$S_{4,1}$ and $S_{1,4}$

The Regge action used in 4d CDT is now

$$S_T(k_0, k_4, \Delta) = -(k_0 + 6\Delta)N_0 + k_4\left(N_4^{(4,1)} + N_4^{(3,2)}\right) + \Delta\, N_4^{(4,1)}$$

We have expressed N_2 in terms of N_0 and $N_4^{(4,1)}$ and $N_4^{(3,2)}$, and a new coupling constant appears, Δ, which allows for a potential asymmetry between the time direction and the spatial directions. We now have a non-trivial phase diagram in the k_0, Δ coupling constant plane.



We now have a time direction and spatial slices at discretized times t_i, which contain $N_3(t_i)$ three-simplices (tetrahedra). We can thus associate a continuum 3-volume to each time slice t_i:

$$V_3(t_i) \propto N_3(t_i)\, \varepsilon^3.$$

In the MC simulations where we have chosen the topology of the spatial slices to be that of S^3 we can measure $\langle N_3(t_i) N_3(t_j)\rangle$ and in phase C_{dS} it turns out that the correlator is perfectly described by the following effective action

$$S[V_3] \propto \frac{1}{G} \int dt \left(\frac{\dot{V}_3^2}{V_3(t)} + k V_3^{1/3} + \Lambda V_3(t) \right).$$

This is nothing but the Hartle-Hawking minisuperspace action, but here obtained by integration out all degrees of freedom except the scale factor $V_3(t) \propto a^3(t)$.

Hartle-Hawking minisuperspace action:

$$ds^2 = -dt^2 + a^2(t) d\Omega_3, \qquad dS_3, \;\; dx_i^2, \;\; dH_3$$

$$\begin{aligned} S_L[g] &= \frac{1}{16\pi G} \int d^4x \sqrt{g(x)} \Big(R(x) - 2\Lambda \Big) \\ &\propto \frac{1}{G} \int dt \Big(-a(t)\dot{a}^2(t) + k'\, a(t) - \lambda' a^3(t) \Big) \\ &\propto \frac{1}{G} \int dt \Big(-\frac{\dot{V}^2(t)}{V(t)} + k\, V^{1/3} - \lambda V(t) \Big) \end{aligned}$$

where $V = a^3$. Rotation to Euclidean signature leads to

$$S_E[V] = \frac{1}{G} \int dt_e \Big(-\frac{\dot{V}^2(t_e)}{V(t_e)} + k\, V^{1/3} + \lambda V(t_e) \Big)$$

It is unbounded from below and a disaster in the path integral when integrating wrt real a or V.

HH thus suggest a further rotation of a and V (and t_e) and after that one arrived at the HH (Euclidean) minisuperspace action:

$$
\begin{aligned}
S_L &= \frac{1}{G}\int dt\Big(-\frac{\dot{V}^2}{V}+k\,V^{1/3}-\lambda V\Big) \rightarrow \\
S_{HH} &= \frac{1}{G}\int dt\Big(\frac{\dot{V_h}^2}{V_h}+k\,V_h^{1/3}+\lambda V_h\Big), \quad V_h = i^3 V
\end{aligned}
$$

This is precisely the 4d CDT effective action for $V(t)$. But it is also the 2d CDT classical action for $L(t)$ for $k=0$, which led to the corresponding quantum theory. Further, the 2d CDT quantum theory could be generalized to allow for the creation of baby universes. So we have already solved a HH minisuperspace model where baby can be created.

Let us now take this effective baby universe model and use the above relation between S_L and S_{HH} to rotate back to a corresponding classical "effective" Hamiltonian:

$$H_l^{\text{eff}}[V, \Pi_l] = -\kappa V \left((\Pi_l + \alpha)\sqrt{(\Pi_l - \alpha)^2 - \frac{2g_s}{\kappa\alpha}} \right), \quad \alpha^3 - \frac{\lambda}{\kappa^2}\alpha + \frac{g_s}{\kappa} = 0,$$

where $\kappa \propto G$. The on-shell real solution:

$$H_l^{\text{eff}} = 0 \implies \Pi_l = -\alpha \quad \dot{V} = -\kappa V \frac{2\Pi_l(\Pi_l - \alpha)}{\sqrt{(\Pi_l - \alpha)^2 - \frac{2g_s}{\kappa\alpha}}},$$

$$V(t) \propto \mathrm{e}^{2\kappa\Sigma\, t}, \quad \Sigma = \sqrt{\alpha^2 - \frac{g_s}{2\kappa\alpha}},$$

So a classical exponential growth (if $g_s = 0$ we just obtain the standard exponential de Sitter solution).

However, now one can take the cosmological constant $\lambda \to 0$:

$$V(t) \propto e^{\sqrt{6}(\kappa^2 |g_s|)^{1/3}\, t}, \qquad g_s < 0.$$

We thus have an expanding universe caused by the creation of baby universes for negative g_s.

Clearly, to agree with observations, the coupling constant g_s must be very small. We then expand our effective Hamiltonian to lowest order in g_s and also add the standard matter terms, while choosing the cosmological constant to be zero and we obtain a modified Friedmann equation (MFE), using now the scale factor $a(t) = V^{1/3}(t)$ as variable:

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa\,\rho}{3} + B\frac{a}{\dot{a}}\frac{1+3F(x)}{F^2(x)}, \quad x = \frac{B\,a^3}{\dot{a}^3},$$

$$F^3(x) - F^2(x) + x = 0, \qquad B \propto -\kappa^2 g_s$$

Assuming matter given by a simple dust system we have

$$\rho = \frac{C}{a^3}, \quad \dot{\rho} = -3H(t)\rho, \quad H(t) \equiv \frac{\dot{a}}{a}$$

and the Friedmann equation at late times for standard ΛCDM model is

$$\Omega_m(t) + \Omega_\Lambda(t) = 1, \qquad \Omega_m(t) = \frac{\kappa\rho}{H^2(t)}, \quad \Omega_\Lambda = \frac{\Lambda}{H^2(t)},$$

while our modified Friedmann equation is

$$\Omega_m(t) + \Omega_B(t) = 1, \qquad \Omega_B(t) = \frac{x(1+3F(x))}{F^2(x)}, \quad x = \frac{B}{H^3(t)}$$

They both have one free parameter, Λ and B, respectively, which will affect late time cosmology.

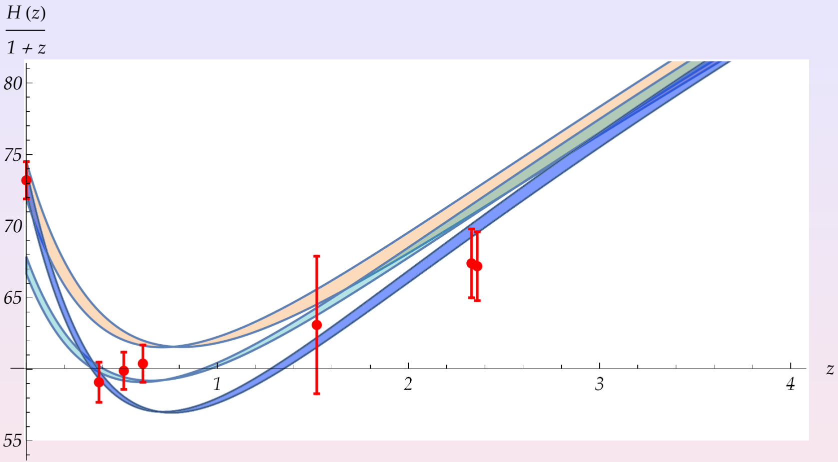
The Planck collaboration data from CMB are based on physics at the time of last scattering t_{LS}, where the red shift is $z(t_{LS}) = 1090$. One has to assume a cosmological model to extrapolate to present time. Using the ΛCDM model the best fit leads to a value $H_0(CMB) \approx 67$. However one can also measure H_0 locally (and model independent) and finds $H_0(local) \approx 73$. Difference 5σ. This is the H_0 tension. We can determine our parameter B by insisting that $t_{LS}(B) = t_{LS}(CMB)$ and at that

$$z_B(t_{LS}) = 1090, \qquad H(t_0(B)) = H_0(local)$$

which leads to

$$\big(t_0(B)\big)^3 B \approx 0.15, \qquad t_0(B) \approx 13.9 Gyr$$

By definition we have no H_0 tension, but how does our $H(t)$ compare with observations for t different from t_{LS} and $t_0(B)$?



$$\chi^2_{\rm red}(\mathrm{B}) = 1.8, \quad \chi^2_{\rm red}(\Lambda^{\rm SC}) = 3.7, \quad \chi^2_{\rm red}(\Lambda^{\rm CMB}) = 4.0,$$

Another "tension" between local measurements and the Planck data comes from density fluctuation measurements. An observable measuring this is denoted S_8. I will not try to define it, but just mention that it is calculated from the Planck data and the use of the ΛCDM model. Similarly, it can be calculated from our model, using the already determined values B and $t_0(B)$, and finally it can be measured by "local" measurements. The disagreement between the Planck value and the value obtained by local measurements is 3σ.

$$S_8(CMB) \approx 0.81, \quad S_8(local) \approx 0.76, \quad S_8(B) \approx 0.77$$

The MFE passes the simple tests with flying colors!