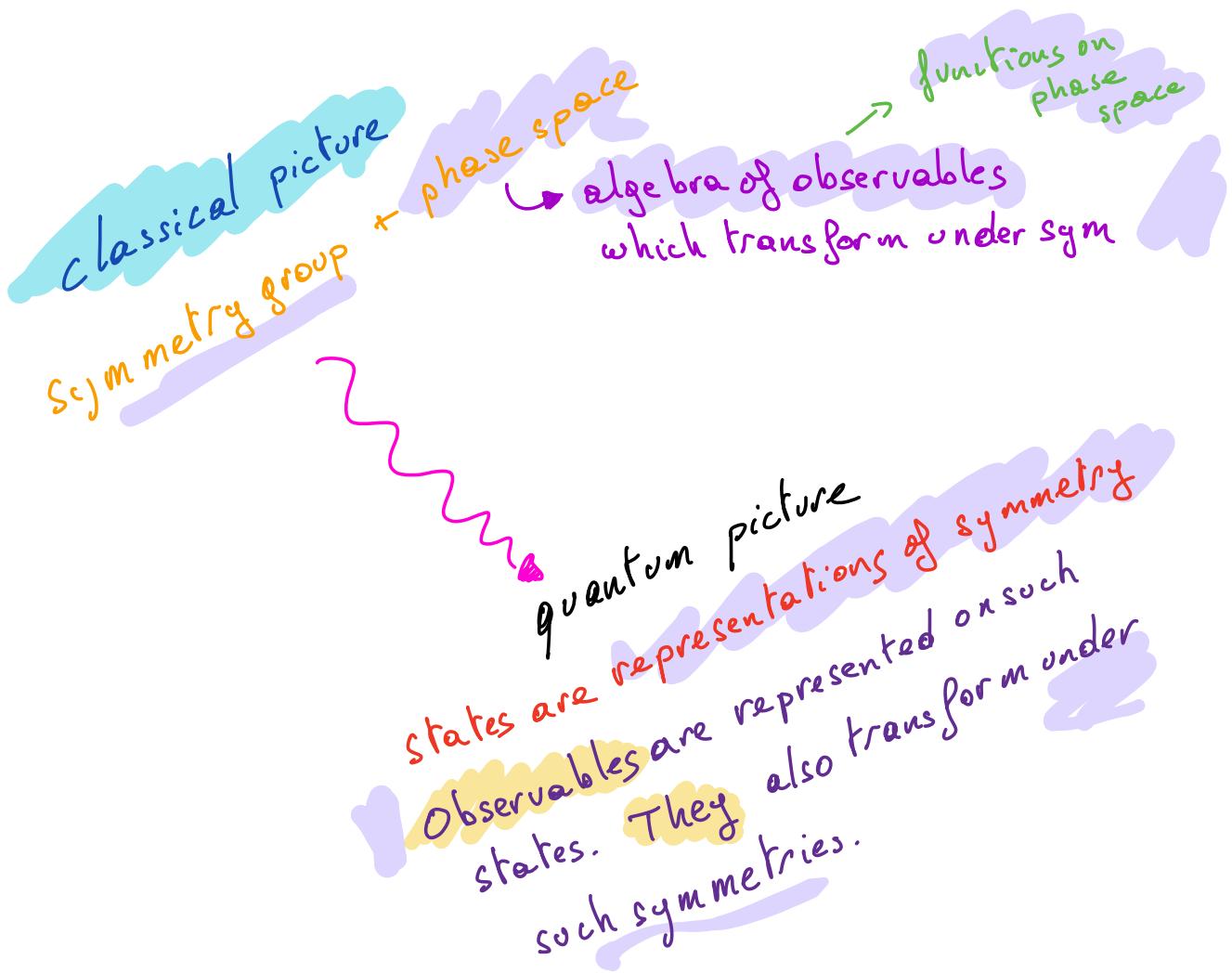


Hopf algebras.

quantum groups.



↳ quantum state based on representations
of the symmetry structure.

↳ non trivial Poisson structure
percolates to the quantum realm

- * duality is key: product / coproduct.
- * from 1 to many systems.

Warm up:

consider a particle in momentum representation

$$\underline{P} |P\rangle = P |P\rangle.$$

what about 2 particles? $|P_1, P_2\rangle$

total momentum:

$$\underline{P}_{\text{tot}} |P_1, P_2\rangle = (P_1 + P_2) |P_1, P_2\rangle.$$

$$\underline{P}_{\text{tot}} = \underline{P} \otimes I + I \otimes \underline{P}$$

$$(\underline{P} \otimes I + I \otimes \underline{P}) |P_1, P_2\rangle = \underline{P} \otimes I |P_1, P_2\rangle$$

$$+ I \otimes \underline{P} |P_1, P_2\rangle.$$

$$= P_1 |P_1, P_2\rangle$$

$$+ P_2 |P_1, P_2\rangle$$

dualizing the notion of product.
coproduct.

$C(G)$: algebra of functions.

G finite gp

KG : group algebra; basis is group elements

$C(G)$

KG

$$f(g) = \langle f, g \rangle.$$

$$\Delta f = f_{(1)} \otimes f_{(2)}$$

duality between
algebras.

$$\Delta f(g_1, g_2) \equiv f(g_1 g_2) \quad \nu: G \times G \rightarrow G.$$

$$g_1, g_2 \mapsto g_1 g_2 = \nu(g_1, g_2)$$

$$f(g_1, g_2) = \langle f, g_1, g_2 \rangle = \langle f; \nu(g_1, g_2) \rangle$$

$$= \langle \Delta f; g_1 \otimes g_2 \rangle$$

$$\Delta f \in C(G) \otimes C(G).$$

Take coord function.

$$G = \mathbb{R}^3 \ni p, q$$

$$P_i(p) = p_i$$

$$P_i(q) = q_i$$

$$P_i(p, q) = p_i + q_i = \langle P_i; p, q \rangle.$$

$$= \langle \Delta P_i, p \otimes q \rangle$$

$$\Delta P_i = P_i \otimes I + I \otimes P_i \leftarrow$$

Take matrix element:

$$G = \mathrm{SU}(2) \ni g_1, g_2$$

$$D_{mn}^j(g) \in C(G)$$

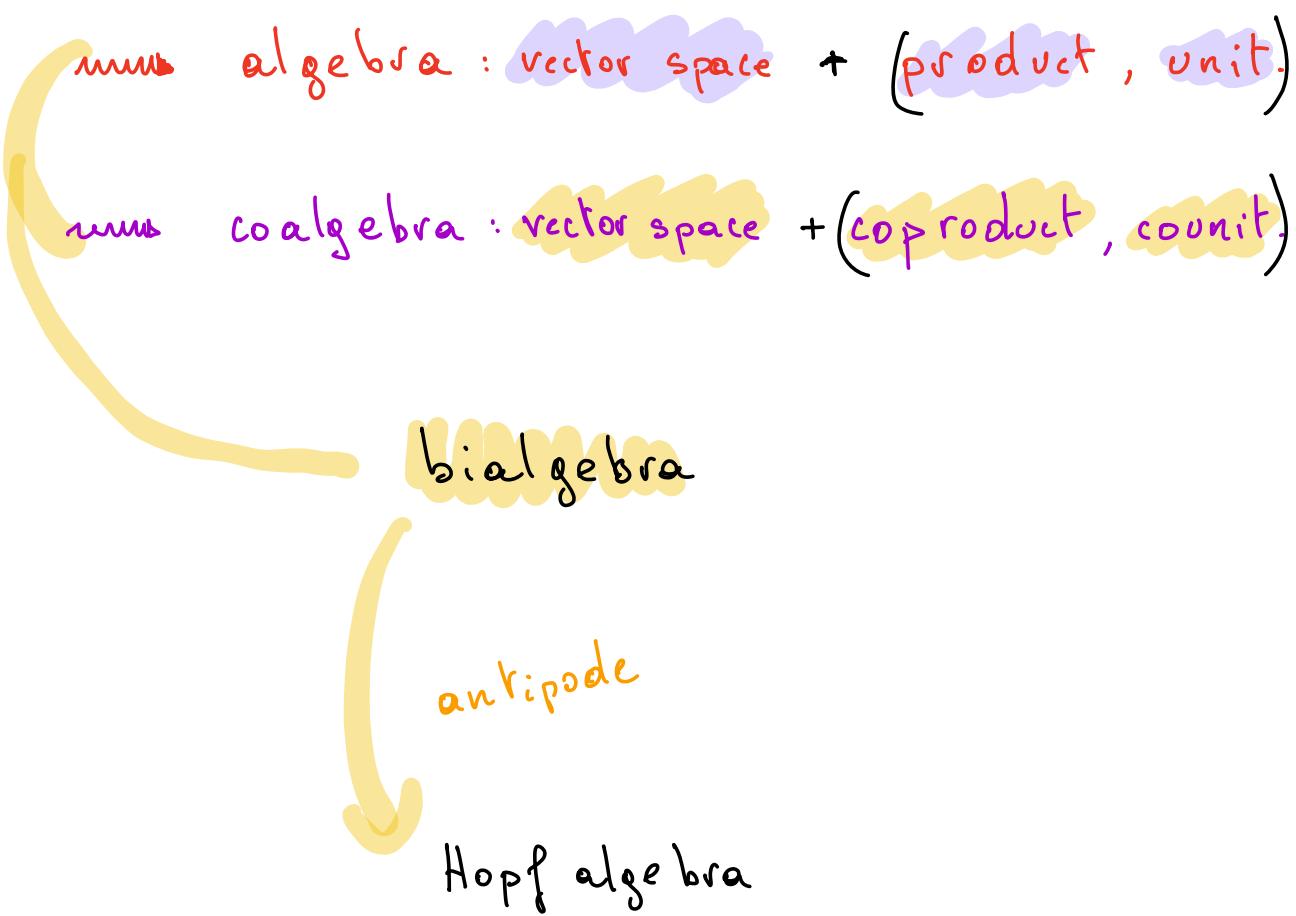
$$\begin{aligned} D_{mn}^j(g_1, g_2) &= \sum_k D_{mk}^j(g_1) \cdot D_{kn}^j(g_2) \\ &= \langle D_{mn}^j ; g_1, g_2 \rangle = \langle \Delta D_{mn}^j ; g_1 \otimes g_2 \rangle. \\ \Delta D_{mn}^j &= \sum_k D_{mk}^j \otimes D_{kn}^j \end{aligned}$$

$$\langle f_1, \otimes f_2 \rangle$$

Note: KG has also a coproduct: it comes from the product of $C(G)$.

↪ pointwise product.

$$\begin{aligned} f_1(g) f_2(g) &= \langle f_1 \circ f_2, g \rangle = \langle f_1 \otimes f_2, \Delta g \rangle \\ &= \langle f_1 \circ f_2, g \rangle = \langle f_1 \otimes f_2, \Delta g \rangle \\ &\quad \hookrightarrow \Delta g = g \otimes g \\ &= \langle f_1 \otimes f_2, g \otimes g \rangle \\ &= \langle f_1, g \rangle \langle f_2, g \rangle. \end{aligned}$$



Antipode : "inverse" momentum

$$\begin{aligned}
 f(g^{-1}) &= \langle f, g^{-1} \rangle \\
 &= \langle Sf, g \rangle.
 \end{aligned}$$

$$S^2 = \text{id} \quad (g^{-1})^{-1} = g.$$

► $S^2 \neq \text{id}$ in general!

Definition : algebra

field
/
k

we consider the vector space A over k
equipped with maps

$$\omega : A \otimes A \rightarrow A : \text{product}$$

$$\eta : k \rightarrow A :$$

linear

$$\eta(\lambda) = \lambda \mathbb{1}_A$$

$\lambda \in k.$

Product should be associative and unit be
a unit :

The diagram illustrates the compatibility conditions for an algebra structure. It shows various tensor products of A and k and how they interact with the multiplication map ω and the unit map η .

- Top Left:** $(A \otimes A) \otimes A$ is connected to $A \otimes A$ via $\omega \otimes \text{id}$, and to $A \otimes (A \otimes A)$ via $\text{id} \otimes \omega$.
- Top Right:** $A \otimes A$ is connected to $\lambda 1 \otimes A$ via ω , and to $A \otimes k$ via $\eta \otimes \text{id}$. Below this, it is shown that $\underbrace{k \otimes A}_{\lambda, a} = A \otimes a$.
- Bottom Left:** $(a_1 \cdot a_2) a_3 = a_1 (a_2 \cdot a_3)$ is shown, with $a_1 \otimes a_2$ connected to $A \otimes A$ via $\text{id} \otimes \eta$, and $A \otimes k$ connected to A via ω .
- Bottom Right:** It is shown that $\lambda a = a \lambda$.

Definition : coalgebra

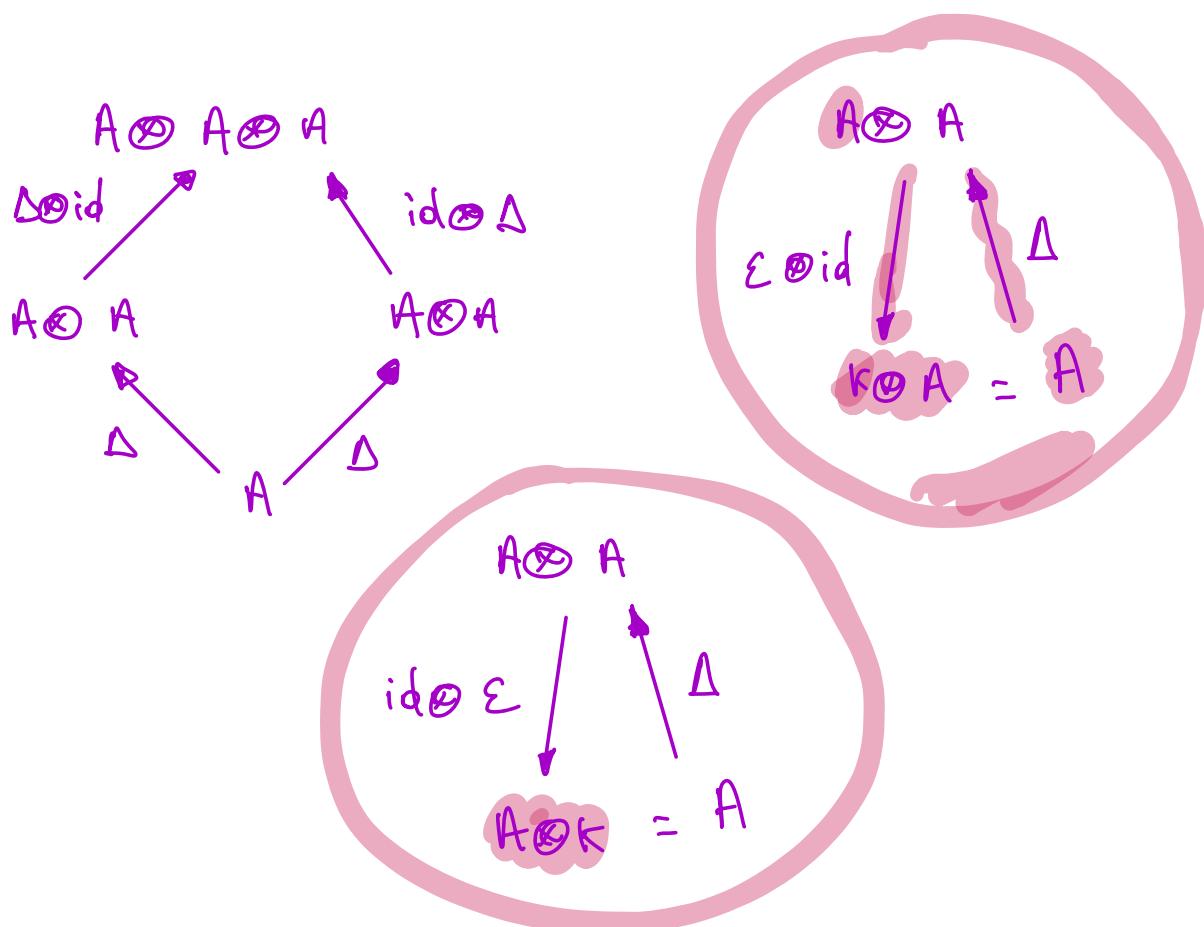
we consider the vector space A over k
equipped with maps

$$\Delta : A \rightarrow A \otimes A \quad \text{coproduct}$$

$$\varepsilon : A \rightarrow k : \quad \text{counit}$$

↑ linear

coproduct should be coassociative



we say the algebra is commutative
if

permutation

$$\nu \circ \tau(a_1, a_2) = \nu(a_2, a_1) = a_2 a_1,$$

$$\nu(a_1, a_2) = a_1 a_2$$

$$a_2 a_1 = a_1 a_2.$$

we say the coalgebra is cocommutative
if

$$\tau \circ \Delta = \Delta.$$

$$a_{(2)} \otimes a_{(1)} = a_{(1)} \otimes a_{(2)}$$

$$\Delta a = a_{(1)} \otimes a_{(2)}. \text{ Sweedler notation.}$$

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text{id} \otimes \Delta} A \otimes A \otimes A$$

$$(\underline{\Delta \otimes \text{id}}) \Delta a = a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}$$

$$(\text{id} \otimes \underline{\Delta}) \Delta a = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$$

$$a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

Definition : bi algebra / Hopf algebra

A **bialgebra** A is an algebra and a coalgebra such that we have the compatibility relations : $a_i \in A$.

$$\Delta(a_1 a_2) = \Delta a_1 \Delta a_2 \quad \Delta 1 = 1 \otimes 1$$

$$\varepsilon(a_1 a_2) = \varepsilon(a_1) \varepsilon(a_2) \quad \varepsilon(1) = 1$$

A bialgebra is a Hopf algebra if it is equipped with a linear map

$S: A \rightarrow A$ called the antipode satisfying

$$\begin{aligned} \nu(S \otimes \text{id}) \circ \Delta &= \nu(\text{id} \otimes S) \circ \Delta \\ &= \eta \circ \varepsilon. \end{aligned}$$

$$(g_1 g_2)^{-1} = \underbrace{g_2^{-1}}_{\text{left}} \underbrace{g_1^{-1}}_{\text{right}}$$

Proposition.

From the definition, we can prove that the antipode is unique and that

- * S is an antialgebra map.

$$S(a_1 a_2) = S(a_2) S(a_1)$$

- * S is an anti coalgebra map

$$S \circ S \circ \Delta = T \circ \Delta \circ S$$

- * also $\begin{cases} S(1) = 1 \\ \varepsilon \circ S = \varepsilon \end{cases}$

Example:

$$g \in \mathrm{SU}(2) \quad D_{mn}^{1/2} : \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$D_{mn}^{1/2} \in C(\mathrm{SU}(2)).$$

Product: pointwise product

$$(D_{mn}^{1/2} \cdot D_{kl}^{1/2})(g) = D_{mn}^{1/2}(g) D_{kl}^{1/2}(g).$$

co product:

$$\Delta D_{mn}^{1/2} = \sum_k D_{mk}^{1/2} \otimes D_{kn}^{1/2}$$

$$\text{Antipode: } S D_{mn}^{1/2}(g) = D_{mn}^{1/2}(g^{-1}).$$

$$\text{Counit: } \varepsilon D_{mn}^{1/2} = \delta_{mn}$$

Example: $x^i \in C_{\text{pol}}(\mathbb{R}^3)$

 Monomials in x^i
 up to commutation
 relation

$$[x^i, x^j] = \varepsilon^{ij}_k x^k.$$

Product: $x^i \cdot x^j$

Coproduct

$$\Delta x^i = x^i \otimes 1 + 1 \otimes x^i.$$

$$\tau \circ \Delta x^i =$$

$$\begin{aligned}
 \Delta [x_i; x_j] &= [\Delta x_i, \Delta x_j] \\
 &= [x_i \otimes 1 + 1 \otimes x_i; x_j \otimes 1 + 1 \otimes x_j] \\
 &= [x_i x_j] \otimes 1 + 1 \otimes [x_i \otimes x_j].
 \end{aligned}$$

Antipode: $S x^i = -x^i$

Counit : $\varepsilon x^i = 0$

What is the link with
Poisson lie gp?

Ans Poisson algebra can be Poisson Hopf algebra.

↳ $C(G)$ is a natural Hopf alg.

↳ coproduct... should be consistent with Poisson structure.

Definition

A Poisson Hopf alg A is a Poisson algebra A which is also a Hopf algebra such that

$$\Delta \{ f_1, f_2 \}_A = \{ \Delta f_1, \Delta f_2 \}_{A \otimes A}.$$

$$\{ a_1 \otimes a'_1; a_2 \otimes a'_2 \} = \{ a_1; a_2 \} \otimes a'_1 a'_2 + a_1 a_2 \otimes \{ a'_1, a'_2 \}.$$

$$(C(\mathbb{R}^3), \{\cdot, \cdot\})$$

Example:

$$\{J_i, J_j\} = \varepsilon_{ij}^k J_k$$

J : coord
function
 \mathbb{R}^3 .

$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i.$$

$$\Delta \{J_i, J_j\} = \varepsilon_{ij}^k \Delta J_k = \varepsilon_{ij}^k (J_k \otimes 1 + 1 \otimes J_k)$$

$$\{\Delta J_i, \Delta J_j\} = \{J_i \otimes 1 + 1 \otimes J_i; J_j \otimes 1 + 1 \otimes J_j\}$$

$$= \{J_i; J_j\} \otimes 11 + J_i J_j \otimes \{1; 1\} + \{J_i, 1\} \otimes 1 J_j$$

$$+ J_i 1 \otimes \{1, J_j\} + 1 1 \otimes \{J_i, J_j\}$$

$$+ \{1, 1\} \otimes J_i J_j.$$

$$+ \{1, 1\} \otimes J_i J_j + 1 J_j \otimes \{J_i, 1\}$$

$$= \varepsilon_{ij}^k (J_k \otimes 1 + 1 \otimes J_k)$$

Other example : $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Delta a = a \otimes a + b \otimes c$$

$$\Delta b = a \otimes b + b \otimes d.$$

$$\{\Delta a, \Delta b\} = i \Delta a \Delta b$$

$$= (a \otimes a + b \otimes c)(a \otimes b + b \otimes d)$$

$$\{a \otimes a + b \otimes c; a \otimes b + b \otimes d\}.$$

$$a^2 \{a, b\} + \{a, b\} ad + ab \otimes \{a, d\}$$

$$+ \{b, a\} cb + ba \{cb\} + b^2 \{cd\}.$$

$$\{a, b\} = ab \frac{1}{2}$$

$$\{a, c\} = ac \frac{1}{2}$$

$$\{a, d\} = 2bc \frac{1}{2}$$

$$\{b, c\} = 0$$

$$\{b, d\} = bd \frac{1}{2}$$

$$\{c, d\} = cd \frac{1}{2}$$

$$= \left(a^2 \otimes ab + ab \otimes ad + ab \otimes 2bc - ab \otimes cb + ba \otimes 0 + b^2 \otimes cd \right)^{\frac{1}{2}}$$

$$\Delta a \Delta b =$$

$$= \left(a^2 \otimes ab + ab \otimes ad + ba \otimes cb + b^2 \otimes cd \right)^{\frac{1}{2}}$$

Quantization:

$$(C(\mathbb{G}) : \{\cdot\}_\gamma) \xrightarrow{\quad} A$$

* deformation order by order in \hbar

Example ①

$$(C(\mathbb{R}^3) ; \{\cdot\}_{\text{su}(2)}, \Delta, \varepsilon, S) \rightarrow (A_{\text{su}(2)}, \Delta, \varepsilon, S)$$

$\hat{x}_i^\kappa \hat{x}_k \in \{\hat{x}_i, \hat{x}_j\}$

algebra generated by monomials in the coord functions.

$$[\hat{x}_i, \hat{x}_j] = \varepsilon_{ij}{}^\kappa \hat{x}_\kappa$$

algebra generated by monomials in \hat{x}_i up to the commutation relations.

non commutative \mathbb{R}^3
Lo $\text{su}(2)$ type.

Note $A_{\text{su}(2)}$ is the enveloping algebra of $\text{su}(2)$
 $U(\text{su}(2))$

$$\Delta J_i = J_i \otimes \mathbb{1} + \mathbb{1} \otimes J_i. \quad \epsilon J_i = 0 \quad S J_i = - J_i$$

$$\Delta \hat{J}_i = \hat{J}_i \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_i. \quad \epsilon \hat{J}_i = 0 \quad S \hat{J}_i = - \hat{J}_i$$

Example ②

$$(C(C(\text{su}(2))) ; \{ , \} = 0; \Delta, S, \epsilon)$$

$$\rightarrow (\hat{C}(\text{su}(2)), \Delta, S, \epsilon)$$

operator matrix elements
commute

Example (3)

$$(C(AN_2); \{l_{(1)}, l_{(2)}\} = [r, l, l_2]; \Delta, s, \varepsilon)$$

$$l = \begin{pmatrix} k & 0 \\ z & k^{-1} \end{pmatrix}; l^{-1+} = \begin{pmatrix} k^{-1} & \bar{z} \\ 0 & k \end{pmatrix}$$

$q = e^{t_k G \sqrt{n}}$

$z = \alpha + i\beta$
 $z = \frac{k}{k^*}$ red

$$Q^+ = \begin{pmatrix} K & 0 \\ q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K^{-1} \end{pmatrix}, \quad Q^- = \begin{pmatrix} K^{-1} & -q^{\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_- \\ 0 & K \end{pmatrix}$$

R $C(AN_2) \otimes C(AN_2)$

$r \rightarrow R$

$$R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

$$\mathcal{R} = q^{J_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} q^{\frac{n(n-1)}{4}} \left(q^{\frac{J_z}{2}} J_+\right)^n \otimes \left(q^{-\frac{J_z}{2}} J_-\right)^n.$$

$$\{l_{(1)}, l_{(2)}\} = [r, l_1, l_2] \rightarrow R Q_1^+ Q_2^+ = Q_2^+ Q_1^+ R$$

$q = e^{\frac{\pi}{n} \sim 1 + \hbar}$

$$K J_{\pm} K^{-1} = q^{\pm \frac{1}{2}} J_{\pm}, \quad [J_+, J_-] = [2J_3], \quad \text{with } [n] \equiv \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

deformation of envelopping algebra
of $\mathfrak{su}(2)$!

$$\begin{cases} \Delta(J_{\pm}) := J_{\pm} \otimes K + K^{-1} \otimes J_{\pm}, & \Delta(K) := K \otimes K \\ S(J_{\pm}) := -q^{\pm \frac{1}{2}} J_{\pm}, & S(K) := K^{-1}, \end{cases}$$

Example ④.

$$(C(SU(2)); \{u_1, u_2\} = [r, u_1, u_2], \Delta, S, \epsilon)$$

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow T = \begin{pmatrix} \overset{\hat{a}}{t_{++}} & \overset{\hat{b}}{t_{+-}} \\ \overset{\hat{c}}{t_{-+}} & \overset{\hat{d}}{t_{--}} \end{pmatrix}$$

$$\{u_1, u_2\} = -[r, u_1, u_2] \quad \longrightarrow \quad RT_1T_2 = T_2T_1R$$

$$\begin{aligned} t_{++}t_{+-} &= q^{\frac{1}{2}}t_{+-}t_{++}, & t_{++}t_{-+} &= q^{\frac{1}{2}}t_{-+}t_{++}, & t_{+-}t_{--} &= q^{\frac{1}{2}}t_{--}t_{+-}, \\ t_{-+}t_{--} &= q^{-\frac{1}{2}}t_{--}t_{-+}, & t_{+-}t_{-+} &= t_{-+}t_{+-}, & [t_{++}, t_{--}] &= -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})t_{+-}t_{-+}. \end{aligned}$$

$q \approx 1 + h$

$$S(T) = \begin{pmatrix} t_{--} & -q^{\frac{1}{2}}t_{+-} \\ -q^{-\frac{1}{2}}t_{-+} & t_{++} \end{pmatrix}$$

$$\epsilon(t_j^i) = \delta_j^i, \quad i, j = \pm$$

$$\Delta(t_j^i) = \sum_{k=\pm} t_k^i \otimes t_j^k$$

$$\{a; b\} = \frac{1}{2}ab$$

$$\{a; d\} = -bc$$

$$\{b; d\} = \frac{1}{2}bd$$

$$\begin{aligned} ab &= q^{\frac{1}{2}}ba \approx \left(1 + \frac{h}{2}\right)ba \\ ab - ba &\approx \frac{h}{2}ba \end{aligned}$$

$$\{a, c\} = \frac{ac}{2}$$

$$\{b, c\} = 0$$

$$\{c, d\} = cd/2$$

Punch line:



A quantum group is a "group"
where the coordinates
or the matrix elements do
not commute.

quasitriangular Hopf algebras

Among all the possible Hopf algebras there is a special class which are almost cocommutative.

$$\tau \circ \Delta = R \Delta R^{-1}$$

Definition

A quasitriangular bialgebra/Hopf algebra is a pair (A, R) with $R \in A \otimes H$ invertible and such that

$$(\Delta \otimes \text{id}) R = R_{13} R_{23}$$

$$(\text{id} \otimes \Delta) R = R_{13} R_{12}$$

$$\tau \circ \Delta a = R \Delta a R^{-1} \quad a \in A.$$

$$R_{ij} = 1 \otimes \dots \overset{i}{\otimes} R_{ij} \otimes \dots \overset{j}{\otimes} 1 \dots$$

Proposition :

let (H, R) be a quasitriangular bialgebra
then

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

this is the quantum Yang Baxter
equation.

Proof :

calculate $(\text{id} \otimes \tau \circ \Delta) R$ in 2 ways

$$\begin{aligned} (\text{id} \otimes \tau \circ \Delta) R &= R_{23} (\text{id} \otimes \Delta R) R_{23}^{-1} \\ &= R_{23} R_{13} R_{12} R_{23}^{-1} \end{aligned}$$

$$(\text{id} \otimes \tau \circ \Delta) R = (\text{id} \otimes \tau) R_{13} R_{12}$$

$$= R_{12} R_{13}$$

Example : $\mathcal{U}_q(\mathfrak{su}(2))$ is a quasitriangular Hopf algebra.

◆ **Algebra** $[J_z, J_{\pm}] = \pm J_{\pm}$, $[J_+, J_-] = [2J_z]_q$, with $[J_z]_q = \frac{q^{J_z/2} - q^{-J_z/2}}{q^{1/2} - q^{-1/2}}$.

◆ **Representations are similar to classical case** $|j, m\rangle$

$$J_{\pm}|j, m\rangle = \sqrt{[j \mp m]_q[j \pm m + 1]_q}|jm \pm 1\rangle$$

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \sum_{j=|j_1-j_2|, \dots, j_1+j_2} {}_q \mathbf{C}_{m_1 \ m_2}^{\ j_1 \ j_2} |j, m\rangle.$$

◆ The **coproduct** encodes the notion of “total momentum”.

$$\Delta : \mathcal{U}_q(\mathfrak{su}(2)) \rightarrow \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2))$$

$$\Delta J_z = J_z \otimes \mathbf{1} + \mathbf{1} \otimes J_z, \quad \Delta J_{\pm} = J_{\pm} \otimes q^{J_z/2} + q^{-J_z/2} \otimes J_{\pm}.$$

◆ The **deformed permutation**

$$\psi : \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2)) \rightarrow \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2)), \quad \text{Normal permutation}$$

The coproduct is not noncocommutative in a crazy way. The R matrix is there...

$$\psi \Delta X = \mathcal{R}(\Delta X) \mathcal{R}^{-1}.$$

$$\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 = q^{J_z \otimes J_z} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} q^{n(n-1)/4} (q^{J_z/2} J_+)^n \otimes (q^{-J_z/2} J_-)^n$$

The coproduct is “**symmetric**” according to a **deformed permutation**.

braiding.

$$\psi_{\mathcal{R}} : V \otimes W \rightarrow W \otimes V$$

$$v \otimes w \rightarrow \psi_{\mathcal{R}}(|v, w\rangle) \equiv \psi(\mathcal{R}|v, w\rangle) = \sum \psi(|\mathcal{R}_1 v, \mathcal{R}_2 w\rangle) = \sum |\mathcal{R}_2 w, \mathcal{R}_1 v\rangle.$$

$$\psi_{\mathcal{R}}(X(|v, w\rangle)) = \psi(\mathcal{R}X(|v, w\rangle)) = \psi(\mathcal{R}\Delta X|v, w\rangle) = \psi((\psi \Delta X)\mathcal{R}|v, w\rangle) = \Delta X \psi(\mathcal{R}|v, w\rangle) = X(\psi_{\mathcal{R}}(|v, w\rangle)).$$

Some application:

How to describe quantum homogeneously curved geometry?

