Quantum groups in a coco nutshell

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Useful refs.

Majid Quantum groups

Chari Pressley A guide to quantum groups.

T Tjin : An introduction to quantized Lie groups and Lie algebras hep-th 9111043

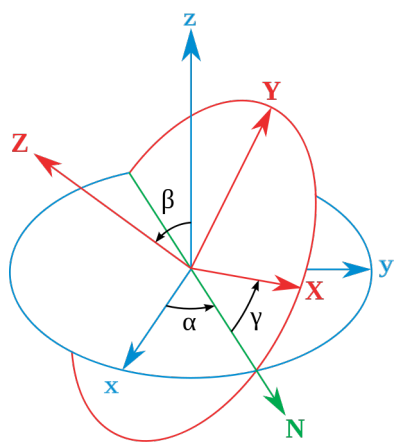
Y. Kosmann Schwarzbach : Lie bialgb Poisson Lie gps and dressing transf.

what is a quantum group?

lie group: Manifold + product.

↳ coordinates

non commutative



euler

Plan :

* Overview

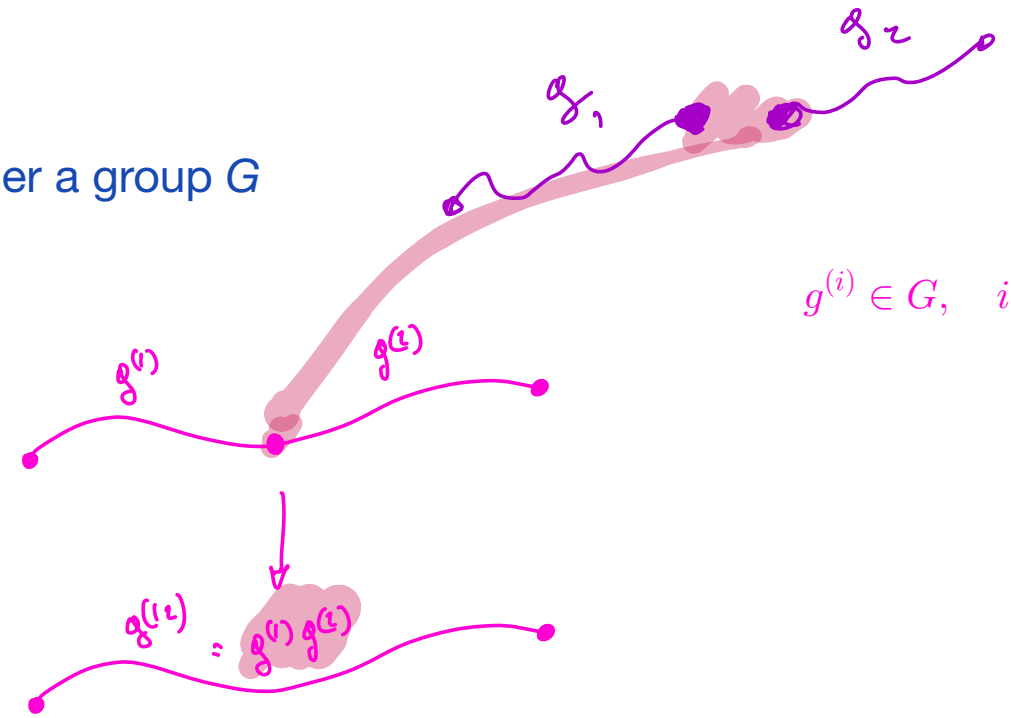
* Before quantum : Poisson.
  - ↳ symmetries
  - ↳ phase space

* Quantum :
  - ↳ Hopf algebras.

3d gravity as the leading application

Overview
Motivations

Consider a group G



$$g^{(i)} \in G, \quad i = 1, 2$$

Overview
Motivations

functions on G.

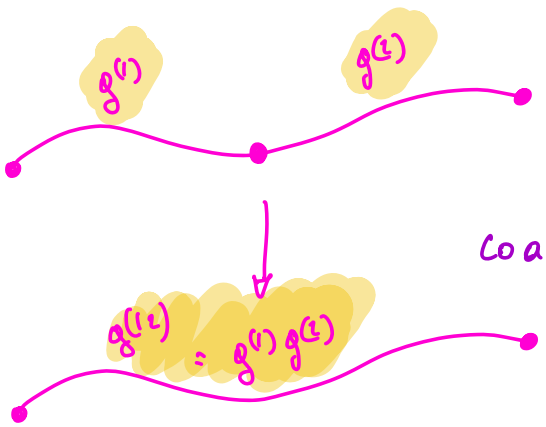
$C(G) \times C(G) \to C(G)$

lie

Consider a group G equipped with Poisson bracket

$g^{(i)} \in G, \quad i = 1, 2$

$$\left\{ g^{(i)}_{AB} , g^{(i)}_{BC} \right\} .$$



coarsegraining should preserve Poisson bracket.

Product is a Poisson map

Historical motivation: integrable systems.

Also coming from symmetry

↳ Symmetry action on a Poisson space M

Example: angular momentum

$$\vec{x} \in \mathbb{R}^3 \qquad \{x_i, x_j\} = \varepsilon_{ij}^{\ k} x_k .$$

$\mathbb{R}^3$ acts on itself by translation.

$$\vec{x} \to a \triangleright \vec{x} = \vec{x} + \vec{a} \qquad \vec{a} \in \mathbb{R}^3 .$$

demand Poisson structure to be covariant under sym action:

$$\{x_i + a_i \,;\, x_j + a_j\} \overset{!}{=} \varepsilon_{ij}^{\ k} (x_k + a_k)$$

$$\{x'_i \,,\, x'_j\} = \varepsilon_{ij}^{\ k} x'_k .$$

$$\varepsilon_{ij}^{\ k} x_k$$

$$\| $$

$$\{x_i, x_j\} + \{x_i, a_j\} + \{a_i, x_j\} + \{a_i, a_j\} \overset{!}{=} \varepsilon_{ij}^{\ k} (x_k + a_k)$$

$\vec{a}$: sym parameters need to have a non zero Poisson bracket.

$$\{a_i, a_j\} = \varepsilon_{ij}^{k} a_k.$$

action by multiplication

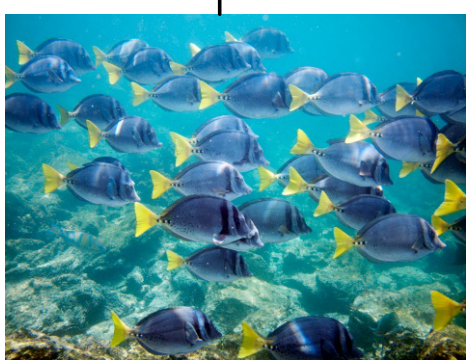
↳ action is a Poisson map.

↳ product is a Poisson map.

More details now

Poisson

lie



Poisson (lie) group

Poisson manifolds.

lie, Poisson 1800
1888-93.

M manifold, $C^\infty(M)$: C^∞ functions

↳ ex: coordinate functions

$M = \mathbb{R}^3 \ni g$ $\quad x^i(g)$ (single patch).

use group structure: $x^i(g_1 g_2) = x^i(g_1) + x^i(g_2)$.

$M = SU(2) \ni g$

1) $\vec{p}(g)$ (at least 2 patches)

use group structure $\quad p^i(g_1 g_2) = (p(g_1) \oplus p(g_2))^i$

ex: $g = \pm\sqrt{1-|\vec{p}|^2}\,\mathbb{1} + \vec{p}\cdot\vec{\sigma}$.

su(2) generators

2) matrix elements (cf Peter weyl theorem).

$D^j_{mn}(g)$

$D^j_{mn}(g) \in \mathbb{C}$.

$D^j_{mn} \in C(SU(2))$.

Poisson bracket: $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

$$(f_1, f_2) \to \{f_1, f_2\}.$$

Properties:

* antisymmetry. $\{f_1, f_2\} = -\{f_2, f_1\}$

* Jacobi id : $0 = \{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\}$

* Derivation:

$$\{f_3 f_1, f_2\} = \{f_3, f_2\} f_1 + f_3 \{f_1, f_2\}$$

on a Cartesian product: $M \times N$, $f_1, f_2 \in C^\infty(M \times N)$

$\{\cdot,\cdot\}_M \quad \{\cdot,\cdot\}_N$

$$\{f_1, f_2\}_{M\times N}(x,y) = \{f_1(\cdot,y); f_2(\cdot,y)\}_M(x) + \{f_1(x,\cdot)\, f_2(x,\cdot)\}_N(y)$$

"on a Cartesian product, there is a natural Poisson bracket, with no crossed terms".

"product Poisson bracket".

Poisson map: $\phi : M \to N$

$$\{f_1, f_2\}_N \circ \phi = \{f_1 \circ \phi ; f_2 \circ \phi\}_M$$

"Poisson map = covariance".

Poisson bivector: coordinates on M: x^i

$$\{f_1, f_2\}(x) = \pi^{ij}(x)\, \partial_i f_1\, \partial_j f_2$$

Lie 1888-93.

$\pi^{ij} = -\pi^{ji}$; $\pi^{ri}\partial_r \pi^{jk} + \pi^{rj}\partial_r \pi^{ki} + \pi^{rk}\partial_r \pi^{ij} = 0$

antisym

Jacobi id.

Ex: $\{x_i, x_j\} = 0$

$\pi(x) = 0.$

Ex: $\{x^i, x^j\} = \varepsilon^{ij}{}_k x^k$

$\pi(x) = \varepsilon^{ij}{}_k x^k \partial_i \otimes \partial_j.$

note: $\pi(0) = 0$

Ex: $\{x^i, x^j\} = 0 = \{p^i, p^j\}$

$\{x^i, p^j\} = \delta^{ij}$ $\qquad \pi = (\Omega^{-1})^{IJ} \partial_I \otimes \partial_J.$

$T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3.$

$\vec{x}, \vec{p}$

$x^I = \begin{vmatrix} x \\ p \end{vmatrix}$ $\qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

note: $\Omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Symplectic: bivector is invertible.

Phase space is a special case of a Poisson space.

↳ symplectic Poisson manifold.

One motivation (among others!) for considering a Poisson structure on symmetry structure. (group)

Poisson Manifold M, G lie group acting on M.

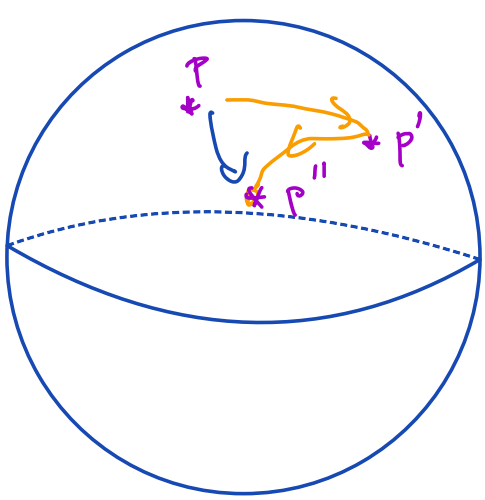
$$\alpha : G \times M \to M \qquad p \text{ point in } M.$$

$$(g, p) \to g \triangleright p = \alpha(g, p)$$

⚠ $$h \triangleright (g \triangleright p) \overset{!}{=} (hg) \triangleright p.$$

⚠ action is pulled back to $C(M)$.
↑ coord funct.

Ex



$$p' = g \triangleright p.$$

$$x^\mu(p') = g \triangleright x^\mu(p)$$
$$= x^\mu(g \triangleright p).$$

If we deal with a symmetry of the Poisson space, Poisson bracket should be covariant under action

$$g \triangleright \{f_1, f_2\}_M = \{g \triangleright f_1 ; g \triangleright f_2\}_M$$

$$g \triangleright \{f_1, f_2\}(p) = \{f_1, f_2\}(g \triangleright p)$$

$$= \{f_1, f_2\} \circ \alpha(\underset{G}{g}, \underset{M}{p})$$

$$\vec{x} + \vec{a} = \vec{x}'$$

$$= \{f_1 \circ \alpha , f_2 \circ \alpha\}_{G \times M}(g, p)$$

$\rightsquigarrow$ action is a Poisson map.

$\rightsquigarrow$ group should be equipped with a Poisson structure.

$$\{x_i ; x_j\} = \varepsilon_{ij}^{k} x_k .$$

$$\{(x+a)_i , (x+a)_j\}$$

Ex: $M = \mathbb{C}^2$ $\quad G = SU(2)$

$$\{z_i, \bar{z}_j\} = \mathbb{1}_{ij} \qquad (g_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$i, j = 1, 2$ $\qquad ad - cb = 1$

$$g \triangleright \mathbb{1} = \mathbb{1}$$

$$g \triangleright z_i = g_{ik} z_k. \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$g \triangleright \{z_i, \bar{z}_j\} = g \triangleright \mathbb{1}_{ij} = \mathbb{1}_{ij}$$

$$\{(g \triangleright z)_i, \overline{(g \triangleright z)_j}\} = \{g_{ik} z_k ; \bar{g}_{jm} \bar{z}_m\}$$

$$= \{g_{ik} ; \bar{g}_{jm}\}_G z_k \bar{z}_m$$

$$+ g_{ik} \underbrace{\{z_k, \bar{z}_m\}}_{M} \bar{g}_{jm} \qquad \mathbb{1}_{ij}$$

$$\rightarrow g_{ik} \{z_k, \bar{z}_m\} \bar{g}_{jm} = g_{ik} \delta_{km} \bar{g}_{jm}$$

$$= g_{im} \bar{g}_{jm} = \delta_{ij}.$$

what happens if Poisson brackets non trivial between the z's ?

need non trivial brackets on $C(G)$

$\rightsquigarrow$ hidden symmetries by considering Poisson structure on the gp.

cf Maite's talk

Punch line:

Symmetry group should be equipped with a Poisson bracket.

Important property for the action

$$g \triangleright h \triangleright x = gh \triangleright x.$$

$$g \triangleright h \triangleright \{f_1, f_2\} = (gh) \triangleright \{f_1, f_2\}$$

$$= \{f_1, f_2\}(gh; p)$$

$$= \{g \triangleright h \triangleright f_1 ; g \triangleright h \triangleright f_2\}(p)$$

$$= \{(gh) \triangleright f_1 ; (gh) \triangleright f_2\}(p)$$

$$= \{f_1 \circ \alpha \circ \alpha ; f_2 \circ \alpha \circ \alpha\}(g, h, p)$$

$$\vec{x} + \vec{a} + \vec{b}$$

$$\{F_1, F_2\}_{G\times G}(gh) = \{F_1, F_2\}_G(g\,h) + \{F_1, F_2\}(g\,h).$$

"const" const.

$$= \{R_h \circ F_1, R_h \circ F_2\}_G(g) + \{L_g \circ F_1, L_g \circ F_2\}_G(h)$$

compatibility between product and Poisson bracket on G

$$\nu : G \times G \to G$$

$$(g, h) = gh = \nu(g,h).$$

in terms of the Poisson bivector $\quad eg = g.$

$$\pi(gh) = R_h\,\pi(g) + L_g\,\pi(h).$$

$e\,e \leftarrow id$ $\quad R_e\,\pi(e) + L_e\,\pi(e) \to \cancel{\pi(e)} = \cancel{\pi(e)} + \pi(e)$

we say π is **Multiplicative**

Note $\pi(e) = 0$ ⚠ cannot be symplectic

Definition

A Poisson lie group is a lie group equipped with a Poisson structure compatible with product.

$$\pi(gh) = R_h \pi(g) + L_g \pi(h)$$

Example:

1) $\pi(g) = 0$

2) let $r = r_{ij}\, x^i \otimes x^j \in \text{lie}\, G \otimes \text{lie}\, G$ ($T_e G$)

x^i generator of lie G.

r must satisfy some properties $\hookrightarrow$ Jacobi.

$$\pi(g) = L_g r - R_g r = [g \otimes g ; r]$$

check: $\pi(gh) = L_{gh}\, r - R_{gh}\, r$

$= gh \otimes gh\, r - r\, gh \otimes gh.$

$L_g r - R_g r$

$= g \otimes g\, r - r\, g \otimes g.$

$$\pi(gh) = R_h \pi(g) + L_g \pi(h)$$

$R_h \pi_g = (g \otimes g\, r - r\, g \otimes g)(h \otimes h) = g \otimes g\, r\, h \otimes h - r\, gh \otimes gh.$

$L_g \pi_h = g \otimes g\,(h \otimes h\, r - r\, h \otimes h) = gh \otimes gh\, r$

$-g \otimes g\, r\, h \otimes h.$

Theorem

Every multiplicative Poisson structure on a connected semi simple or compact lie group is of the form

$$\pi(g) = \pm\left(L_g\, r - r\, R_g\right).$$

antisym

where r is such that

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \text{lie } G \wedge \text{lie } G \wedge \text{lie } G$$

is invariant under the adjoint action.

$\to$ if $= 0$ $\rightsquigarrow$ this is the classical Yang Baxter equation.

$$[J_+, J_-] = J_3$$

Example : $su(2) = \text{lie } G \ni J_i$ $\quad [J_3, J_\pm] = \pm 2 J_\pm$

$$\pi(g) = L_g(J_+ \wedge J_-) - R_g(J_+ \wedge J_-)$$

$$r = J_+ \wedge J_- = (J_+ \otimes J_- - J_- \otimes J_+)\frac{1}{2}$$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

$$2r_{12} = J_+ \otimes J_- \otimes 1 - J_- \otimes J_+ \otimes 1$$
$$2r_{13} = J_+ \otimes 1 \otimes J_- - J_- \otimes 1 \otimes J_+$$
$$2r_{23} = 1 \otimes J_+ \otimes J_- - 1 \otimes J_- \otimes J_+$$

$$4[r_{12}; r_{13}] = [J_+ \otimes J_- \otimes 1 - J_- \otimes J_+ \otimes 1;\ 1 \otimes J_+ \otimes J_- - 1 \otimes J_- \otimes J_+]$$

$$= J_+ \otimes [J_-, J_+] \otimes J_- - J_+ \otimes \cancel{[J_-, J_-]} \otimes J_+$$

$$- J_- \otimes \cancel{[J_+, J_+]} \otimes J + J_- \otimes [J_+, J_-] \otimes J_+$$

$$= -J_+ \otimes J_3 \otimes J_- + J_- \otimes J_3 \otimes J_+$$

$$4([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = \epsilon^{ijk} J_i \otimes J_j \otimes J_k.$$

Punch line:

large class of Poisson brackets on G compatible with multiplication are given by bivector

$$\pi(g) = \pm(L_g\, r - R_g\, r)$$

$\hookrightarrow r \in \mathrm{lie}\, G \otimes \mathrm{lie}\, G$

People look at all possible r given G.

3d Poincaré : Stachura 1998. J. Phys. A: Math. Gen. 31 4555

4d Poincaré : Zakrzewski hepth 9412099

$\hookrightarrow$ unfinished !

From bivector to Poisson bracket. $f_i \in C(G)$.
when π is multiplicative.

$\pi(g) \in T_g G \otimes T_g G$

$$\{f_1, f_2\}(g) = \langle \pi_g ; (df_1)_g \otimes (df_2)_g \rangle.$$

$$r = r^{ij} X_i \otimes X_j \in T_e G \otimes T_e G.$$

$T_e G \sim \text{Lie } G$.

$$\{f_1, f_2\}(g) = \langle \pi(g) ; (df_1)_g \otimes (df_2)_g \rangle$$

$$= \langle L_g r - R_g r ; (df_1)_g \otimes (df_2)_g \rangle.$$

$$= \langle r ; (L^*_g \otimes L^*_g - R^*_g \otimes R^*_g)(df_1)_g \otimes (df_2)_g \rangle$$

Sklyanin bracket.

Even more: $r = r^{ij} X_i \otimes X_j$

basis of Lie G.

X_i^L and X_j^R are the corresponding left or right invariant vector fields.

$$\{ f_1, f_2 \}(g) = r^{ij} \left(X^L_i f_1 \otimes X^L_j f_2 - X^R_i f_1 \, X^R_j f_2 \right)$$

Take the case where the functions are the matrix elements.

$$f_1 \equiv L_{ij} \qquad f_2 \equiv L_{kl}$$

$$X^a_L L_{ij}(g) = \frac{d}{dt} L_{ij}\left(e^{tX^a} g \right) = X^a_{im} L_{mj}(g)$$

$$X^a_R L_{ij}(g) = \frac{d}{dt} L_{ij}\left(g e^{tX^a} \right) = L_{im}(g) X^a_{mj}$$

$$\{ L_{ij} ; L_{kl} \} = r^{ab} \left(X^L_a L_{ij} \otimes X^L_b L_{kl} - X^R_a L_{ij} \otimes X^R_b L_{kl} \right)$$

$$= r^{ab} \left(L_{im} X^a_{mj} \otimes L_{kn} X^b_{nl} - X^a_{im} L_{mj} \otimes X^b_{kn} L_{nl} \right)$$

$$= r_{mj\,nl} L_{im} \otimes L_{kn} - r_{im\,kn} L_{mj} \otimes L_{nl}$$

$$= -[r ; L \otimes L]_{ij\,kl}$$

New notation: $L_{ij} = a_i^j$

$$\underset{L_{(1)}}{\underset{\shortparallel}{L \otimes 1}} = \begin{pmatrix} a_1^1 & 0 & \cdots & 0 & \cdots & a_p^1 & 0 & \cdots & 0 \\ 0 & a_1^1 & \cdots & 0 & \cdots & 0 & a_p^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1^1 & \cdots & 0 & 0 & \cdots & a_p^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^p & 0 & \cdots & 0 & \cdots & a_p^p & 0 & \cdots & 0 \\ 0 & a_1^p & \cdots & 0 & \cdots & 0 & a_p^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1^p & \cdots & 0 & 0 & \cdots & a_p^p \end{pmatrix}, \quad \underset{L_{(2)}}{\underset{\shortparallel}{1 \otimes L}} = \begin{pmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & L \end{pmatrix}$$

$$\underset{\{L_{(1)}; L_{(2)}\}}{\underset{\shortparallel}{\{L \otimes L\}}} = \begin{pmatrix} \{a_1^1, a_1^1\} & \cdots & \{a_1^1, a_p^1\} & \{a_2^1, a_1^1\} & \cdots & \{a_p^1, a_p^1\} \\ \{a_1^1, a_1^2\} & \cdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \{a_1^1, a_1^p\} & \cdots & \{a_1^1, a_p^p\} & \cdots & & \vdots \\ \{a_1^2, a_1^1\} & \cdots & \{a_1^2, a_p^1\} & \cdots & & \\ \vdots & & \vdots & & & \\ \{a_1^2, a_1^p\} & \cdots & \{a_1^2, a_p^p\} & \cdots & & \\ \vdots & & \vdots & & & \\ \{a_1^p, a_1^p\} & \cdots & \{a_1^p, a_p^p\} & \cdots & & \{a_p^p, a_p^p\} \end{pmatrix}$$

$$\{L \otimes L\} = -[r, L \otimes L].$$

Punch line:

when the bivector is of the type

$$\pi(g) = L_g\, r - R_g\, r$$

the Poisson bracket between the matrix elements take the shape,

$$\{ L_{(1)} \,;\, L_{(2)} \} = -[\, r,\, L \otimes L \,].$$

$\rightsquigarrow$ what natural Poisson (symplectic) space a Poisson lie group acts as symmetry?

The group G acts on itself by multiplication.

eg: $L : G \times G \to G$

$(g ; g') \to L_g g' = g g'$

$\hookrightarrow$ natural Poisson space : the Poisson lie gp itself.

$\hookrightarrow$ another Poisson structure on G

$$\{ g_{(1)}, g_{(2)} \}_+ = [r, g_1 g_2]_+$$
$$= r g_1 g_2 + g_1 g_2 r.$$

symmetry

check : $\{ h_{(1)}, h_{(2)} \}_- = [r, h_{(1)} h_{(2)}]_-$

$$= r h_{(1)} h_{(2)} - h_{(1)} h_{(2)} r$$

$$\{ (h g)_{(1)} ; (h g)_{(2)} \}_+ \overset{?}{=} [r ; (g h)_{(1)} (g h)_{(2)}]_+ .$$

$$= \{ R_g h_{(1)} ; R_g h_{(2)} \}_- + \{ L_h g_{(1)} ; L_h g_{(2)} \}_+$$

↳ will be related to phase spaces.

Punch line:

Not all bivectors generated by r are multiplicative!

$\rightsquigarrow$ go infinitesimal

bialgebra.

in finitesimal limit : $g \in G \to \text{lie } G \ni x$ $\quad T_e G$

(close to the identity)

$g \to 1+x$

$T_g G \otimes T_g G \ni \pi(g) \to \delta(x) \in \text{Lie } G \otimes \text{lie } G$

Main example : $\pi(g) = L_g r - R_g r$

$= g \otimes g\, r - r\, g \otimes g$

$\approx (1+x) \otimes (1+x)\, r - r\, (1+x) \otimes (1+x)$

$\to \underbrace{[x \otimes 1 + 1 \otimes x}_{\Delta x} ; r] = \delta(x).$

Since $\pi(g)$ is a Poisson bivector, δ satisfies some properties.

$\delta : \text{lie } G \to \text{lie } G \wedge \text{lie } G.$

Let $(\text{lie } G)^*$ the dual vector space to lie G

$\langle \xi_i ; x_j \rangle = \delta_{ij}.$

We can use δ to define a bracket on $(\text{lie } G)^*$

$$\langle \xi_i \otimes \xi_j \,;\, \overbrace{\delta(x_k)}^{f_{abk}\, x_a \otimes x_b} \rangle = \langle \overbrace{[\xi_i, \xi_j]_*}^{f_{ijm}\, \xi_m} \,;\, x_k \rangle$$

$$\langle \xi_i \otimes \xi_j \,;\, f_{abk}\, x_a \otimes x_b \rangle = f_{abk}\, \delta_{ia}\, \delta_{jb} = f_{ijk}.$$

Since we constructed δ from π which satisfied the Jacobi id, the bracket $[\,;\,]_*$ also satisfies the Jacobi identity.

$$\begin{cases} [\,.\,]_* \text{ is a (lie) bracket} \\ (\text{lie } G)^* \text{ is a lie algebra.} \end{cases}$$

$\rightsquigarrow$ lie G^* (finite dim case)

Definition:

A lie bialgebra, noted $(\text{lie } G, \delta)$ or $(\text{Lie } G, \text{Lie } G^*)$ on lie G is give in terms of a skew sym linear map $\delta : \text{lie } G \to \text{lie } G \otimes \text{lie } G$ such that

* $\delta^* : \text{lie } G^* \otimes \text{lie } G^* \to \text{lie } G^*$ is a lie bracket
* δ is a cocycle.

$\hookrightarrow$ cohomology of lie algb.

$$\delta([x,y]) = \text{ad}_x\, \delta(y) - \text{ad}_y\, \delta(x).$$

cocycle property comes from bivector property

$\rightsquigarrow$ given a tensor Π on the Lie gp G
a mapping from G to k^{th} tensor power Lie G
is defined by

$$\rho(\Pi)(g) = \Pi(g) \cdot g^{-1}.$$

Main example: $\Pi(g) = g\, r - r\, g = (\underbrace{g r g^{-1} - r}_{T_eG \otimes T_eG})\, g$

If Π is multiplicative:

$$\rho(\Pi)(gh) = g\, \rho(\Pi)(h)\, g^{-1} + \rho(\Pi)(g)$$

$$= g(h r h^{-1} - r)\, g^{-1} + g r g^{-1} - r$$

$$=$$

This defines the notion of 1-cocycle

Theorem:

let G be a simply connected lie group. Every bialgebra structure on $\mathrm{Lie}\,G$ is the tangent lie bialgebra of a unique Poisson structure which makes G a Poisson lie group.

$$\text{lie}: \quad \mathrm{Lie}\,G \longleftrightarrow G$$

$$\text{Poisson lie}: \quad (\mathrm{Lie}\,G, \delta) \longleftrightarrow (G, \pi)$$

Terminology: we say the lie bialgebra is

* coboundary, if $\exists\ r \in \mathrm{Lie}\,G \otimes \mathrm{Lie}\,G \ / \ \delta(x) = [x, r]$.
  with r satisfying the modified YB equation.

* coboundary quasitriangular, if $r = \underset{\text{antisym}}{a} + \underset{\text{sym}}{s}$
  with $s \in \mathrm{Lie}\,G \odot \mathrm{Lie}\,G$, $a \in \mathrm{Lie}\,G \wedge \mathrm{Lie}\,G$.
  such that classical YB is satisfied.

* coboundary triangular if $r = \alpha \in \mathrm{Lie}\,G \wedge \mathrm{Lie}\,G$

* coboundary factorizable if s defines a non degenerate sym. bilinear form on $\mathrm{Lie}\,G^*$.

$$\delta(x) = [1 \otimes x + x \otimes 1, r]$$

Example

antisym.

$\mathrm{Lie}\,G = su(2) \ni J_i$ $\qquad r = J_+ \wedge J_-$

$$= \frac{1}{2}(J_+ \otimes J_- - J_- \otimes J_+)$$

$$\delta(J_+) = [J_+ \otimes 1 + 1 \otimes J_+ \,;\, J_+ \otimes J_- - J_- \otimes J_+]\frac{1}{2}$$
$$= (-J_3 \otimes J_+ + J_+ \otimes J_3)\frac{1}{2} = J_+ \wedge J_3$$

$$\delta(J_-) = [J_- \otimes 1 + 1 \otimes J_- \,;\, J_+ \otimes J_- - J_- \otimes J_+]\frac{1}{2}$$
$$= (-J_3 \otimes J_- + J_- \otimes J_3)\frac{1}{2} = J_- \wedge J_3$$

$$\delta(J_3) = [J_3 \otimes 1 + 1 \otimes J_3 \,;\, J_+ \otimes J_- - J_- \otimes J_+]\,1/2$$
$$= (J_+ \otimes J_- + J_- \otimes J_+ - J_+ \otimes J_- - J_- \otimes J_+) = 0$$

$$r' = \tfrac{1}{2} J_3 \otimes J_3 + 2 J_+ \otimes J_-$$

$$r' = \frac{1}{2} J_3 \otimes J_3 + 2 J_+ \otimes J_-$$

$$= \frac{1}{2} J_3 \otimes J_3 + J_+ \otimes J_- + J_- \otimes J_+$$

$$+ J_+ \otimes J_- - J_- \otimes J_+ .$$

$$= \frac{1}{2} C_1 + 2 r = s + a .$$

Punch line:

lie : $\mathrm{Lie}\, G \leftrightarrow G$

Poisson lie : $(\mathrm{Lie}\, G, \delta) \leftrightarrow (G, \pi)$

Notion of duality is key to the notion of quantum groups !

Def : Manin triple.

It is a triple of lie algebras $(\mathfrak{d}, \mathrm{lie}\,G, \mathrm{Lie}\,G^*)$ together with a non deg. sym. bilinear form. $\langle , \rangle$ on $\mathfrak{d}$ under the adjoint action of $\mathfrak{d}$ such that.

1) lie G and lie G^* are lie subalgb of $\mathfrak{d}$
2) $\mathfrak{d} = \mathrm{Lie}\,G \oplus \mathrm{Lie}\,G^*$ as vector space
3) lie G and Lie G^* are isotropic for $\langle , \rangle$.

$$\langle x_i, x_j \rangle = 0 = \langle \xi_i, \xi_j \rangle \qquad x \in \mathrm{lie}\,G, \ \xi \in \mathrm{Lie}\,G^*$$

Proposition:

For any finite dim lie algebra, there is a 1 to 1 correspondance between lie bialgb structure on lie G and the Manin triple $(\mathcal{D}, \text{lie } G, \text{lie } G^*)$.

useful to see how things are constrained:

lie G	lie G^*
$[x_i, x_j] = c_{ij}^{\ k} x_k$	$[\xi_i, \xi_j] = f^k_{\ ij} \xi_k$
$\delta(x_k) = f_{ijk}\, x^i \otimes x^j$	$\delta_*(\xi_k) = c_{kij}\, \xi^i \otimes \xi^j$

$$\langle \underbrace{[x_i, \xi_j]}_{\delta x} ; \xi_k \rangle \overset{\text{killing form}}{=} \langle x_i ; [\xi_j, \xi_k] \rangle$$

$$= \langle x_i ; f_{jka}\, \xi_a \rangle$$

$$= f_{jki}$$

$$\langle [x_i, \xi_j]; x_k \rangle = - \langle [\xi_j ; x_i]; x_k \rangle$$
$$= - \langle \xi_j ; [x_i, x_k] \rangle$$
$$= - c_{ikj}$$

$$[x_i, \xi_j] = - c_{ikj} \xi_k + f_{jki} x_k .$$

cocycle property:

$$\delta([x_i, x_j]) \overset{?}{=} x_i \cdot \delta(x_j) - x_j \cdot \delta(x_i).$$

$$\delta(c_{ij}^{\ k} x_k) = c_{ij}^{\ k} f^{mn}{}_k \, x_m \otimes x_n$$

$\rightsquigarrow$ cocycle property gives Jacobi id with mixed terms of δ.

Punch line:

It is equivalent to talk about a bialgebra (Lie G, δ) and a pair of (dual) Lie algebras (Manin triple)

It also allows to construct a natural dual space

In the finite dim case, Manin triple is self dual.

Summary

Yang Baxter

Lie ⊗ Lie G.

$L_g r - R_g r$

multiplicative

$D = G \bowtie G^*$

(G, π) — (G^*, π_*) Poisson Lie

dualize

$(\text{lie } G, \delta)$ — $(\text{lie } G^*, \delta_*)$ Lie bialgebra.

$[\,,\,]$ $[\,,\,]_*$

$\mathcal{D}$: lie $G \bowtie$ lie G^* Manin triple.

↳ is there a cocycle structure on $\mathcal{D}$ which contains the same information as (lie G, δ) and lie G^*, δ_*) ?

$(\mathcal{D}, \delta^{\mathcal{D}})$

$\overset{J_i}{su(2)} \bowtie \overset{P_i}{\mathbb{R}^3}$

$[J_i, J_j] = \varepsilon_{ij}{}^k J_k$.

$[P_i, P_j] = 0$

$[J_i, P_j] = \varepsilon_{ij}{}^k P_k$.

Given a pair of dual lie algebras $\mathrm{lie}\,G$, $\mathrm{lie}\,G^*$ such that $(\mathrm{lie}\,G, \mathrm{lie}\,G^*)$ is a (finite dim) bialg there is a canonical bialgb structure δ

$$\left(\mathfrak{d} = \mathrm{lie}\,G \bowtie \mathrm{lie}\,G^*, \delta\right)$$ is then called the classical double,

$$\delta^{\mathfrak{d}} = [1 \otimes u + u \otimes 1, r^{\mathfrak{d}}]$$

it is coboundary and quasi triangular.

$r^{\mathfrak{d}} \in \mathrm{lie}\,G \otimes \mathrm{lie}\,G^* \subset \mathfrak{d} \otimes \mathfrak{d}$

is the identity map $\mathrm{lie}\,G \to \mathrm{lie}\,G$

$r^{\mathfrak{d}} = X_i \otimes \xi^i$ satisfies classical YB!

Cocycle on $\mathfrak{d}$ is

$$\delta^{\mathfrak{d}}(u) = [u ; r]$$
$$= [u \otimes 1 + 1 \otimes u ; r]$$

Take $u = x_i \in \mathrm{lie}\, G$

$$\delta^{\partial}(x_i) = [x_i ; x_j] \otimes \xi^j + x_j \otimes [x_i ; \xi^j]$$

$$= c_{ij}{}^{k} x_k \otimes \xi^j + x_j \otimes \left(- c_{ikj} \xi_k + f_{jki} x_k \right)$$

$$= f_{jki}\, x_j \otimes x_k . = \delta_{\mathrm{lie}\, G}(x_i).$$

$\mathrm{lie}\, G^*$

$$\delta^{\partial}(\xi_i) = [\xi_i , x_j] \otimes \xi^j + x_j \otimes [\xi_i \ \xi_j]$$

$$= \left(c_{jki}\, \xi^k - f_{ikj}\, x^{\overset{j}{k}} \right) \otimes \xi^{\overset{j}{k}}$$

$$+ x_j \otimes f_{ij}{}^{k}\, \xi_k .$$

$$= - c_{jki}\, \xi^j \otimes \xi^k = - \delta_{\mathrm{lie}\, G^*}(\xi_i)$$

$\rightsquigarrow$ we can exponentiate $\mathfrak{d} \to D$

$$\mathfrak{d} = \text{lie } G \bowtie \text{lie } G^*, \quad \delta^{\mathfrak{d}}$$

$$\downarrow$$

$$D = G \bowtie G^*, \quad \pi$$

Note: D is even dimensional.

$\hookrightarrow$ could we use it as phase space? (with $a \neq \pi$!).

Punch line:

Manin triple

Given the (double lie algebra) $\mathfrak{d} = \text{lie } G \bowtie \text{lie } G^*$

we can find easily a quasitriangular cocycle:

r matrix is the identity.

Examples.

$\mathrm{lie}\, G = su(2) \ni J_i \qquad \delta(J_i) = 0 \quad \forall i$

$\mathrm{lie}\, G^* = su(2)^* \ni \xi_j \qquad [\xi_i, \xi_j] = 0$

$\langle [\xi_i ; \xi_j]_{\mathrm{lie}\, G^*} ; J_k \rangle = \langle f_{ija} \xi_a ; J_k \rangle$

$= f_{ija}\, \delta_{ak} = f_{ijk}.$

$= \langle \xi_i \otimes \xi_j ; \delta(J_k) \rangle = 0 \rightsquigarrow f_{ijk} = 0$

$\mathfrak{d} : ? \qquad \mathrm{lie}\, G = su(2) \qquad \mathrm{lie}\, G^* = \mathbb{R}^3.$

$+$

$[J_i ; \xi_j] = \varepsilon_{ijk} \xi_k - \cancel{f_{jki} J_k}.$

$\mathfrak{d} = su(2) \ltimes \mathbb{R}^3$: 3d euclidean lie algb.

$G = SU(2) \ni g \qquad \pi_{SU(2)} = 0 \rightsquigarrow \{ g_{ij} ; g_{kl} \} = 0$

$G^* = \mathbb{R}^3 \ni x \qquad \pi_{\mathbb{R}^3} = \varepsilon_{ijk}\, x^i\, \partial_j \otimes \partial_k.$

$\delta_{\mathrm{lie}\, G^*}(\xi_k) = \varepsilon_{ijk}\, \xi^i \otimes \xi^j. \qquad \{ x_i, x_j \} = \varepsilon_{ij}{}^k x_k.$

$$\{x_i ; g\} = 0$$

$D = SU(2) \ltimes \mathbb{R}^3$ Euclidian group

other case :

modified YB.

$$\text{lie } G = su(2) \ni J_i \qquad r = J_+ \wedge J_-$$

$$\delta(J_+) = [J_+ \otimes 1 + 1 \otimes J_+ ; J_+ \otimes J_- - J_- \otimes J_+] \tfrac{1}{2}$$
$$= (-J_3 \otimes J_1 + J_+ \otimes J_3)\tfrac{1}{2} = J_+ \wedge J_3$$

$$\delta(J_-) = [J_- \otimes 1 + 1 \otimes J_- ; J_+ \otimes J_- - J_- \otimes J_+] \tfrac{1}{2}$$
$$= (-J_3 \otimes J_- + J_- \otimes J_3)\tfrac{1}{2} = J_- \wedge J_3$$

$$\delta(J_3) = [J_3 \otimes 1 + 1 \otimes J_3 ; J_+ \otimes J_- - J_- \otimes J_+] \tfrac{1}{2}$$
$$= (J_+ \otimes J_- + J_- \otimes J_+ - J_+ \otimes J_- - J_- \otimes J_+) = 0$$

$$\langle \xi_+ \otimes \xi_- ; \underbrace{\delta(J_+)}_{J_+ \wedge J_3} \rangle = 0$$

$$\langle \xi_+ \otimes \xi_- ; \underbrace{\delta(J_-)}_{J_- \wedge J_3} \rangle = 0$$

$$\langle \xi_+ \otimes \xi_- ; \underbrace{\delta(J_-)}_{0} \rangle = 0$$

$$\rightsquigarrow [\xi_+, \xi_-] = 0.$$

$$\langle \xi_+ \otimes \xi_3 ; \underbrace{\delta(J_+)}_{J_+ \wedge J_3} \rangle = \langle \xi_+ \otimes \xi_3 ; (J_+ \otimes J_3 - J_3 \otimes J_+)\tfrac{1}{2} \rangle = 1/2$$

$$\langle \xi_+ \otimes \xi_3 ; \underbrace{\delta(J_-)}_{J_- \wedge J_3} \rangle = 0.$$

$$\rightsquigarrow [\xi_+, \xi_3] = \frac{1}{2}\xi_+$$

$$\langle \xi_+ \otimes \xi_3 ; \underbrace{\delta(J_3)}_{0} \rangle = 0$$

$$\ldots \rightsquigarrow [\xi_-, \xi_3] = \frac{1}{2}\xi_-.$$

$\text{lie } G^* \equiv an_2$

$$[\xi_+, \xi_-] = 0$$
$$[\xi_+, \xi_3] = \frac{1}{2}\xi_+$$
$$[\xi_-, \xi_3] = \frac{1}{2}\xi_-$$

$\mathfrak{d}$: ? $\text{lie } G = su(2)$ $\text{lie } G^* = an_2$

$+$

$$[J_i, \xi_j] = \varepsilon_{ijk}\,\xi_k - \delta_{jki}\,J_k .$$

$$\mathfrak{d} = sl(2,\mathbb{C}) \simeq \left(su(2) \ltimes an_2 \,;\, r = \frac{1}{2} J_i \otimes \xi^i\right)$$

Iwasawa decomposition.

$$D = SL(2,\mathbb{C}) \simeq SU(2) \ltimes AN_2 .$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$b = -\bar{\beta}$, $c = \beta$, $d = \bar{a}$

$\{a; b\} = \frac{1}{2}ab$ $\qquad$ $\{a, c\} = \frac{ac}{2}$

$\{a; d\} = bc$ $\qquad$ $\{b, c\} = 0$

$\{b; d\} = \frac{1}{2}bd$ $\qquad$ $\{c, d\} = cd/2$

$z \in \mathbb{C}$

$k \in \mathbb{R}^*$

$$\begin{pmatrix} k & 0 \\ z & k^{-1} \end{pmatrix}$$

$$\begin{pmatrix} k^{-1} & \bar{z} \\ 0 & k \end{pmatrix}$$

Punch line:

non abelianess

$\updownarrow$

"curvature"

$\updownarrow$

Poisson non commutativity

(Π, G): nonabelian gp. (SUC2)

$\downarrow$

(Π_*, G^*)

nontrivial.

angular momentum

Symmetries of what?

↳ phase space.

⇝ Poisson space : restrict to where Poisson bivector is invertible :

Symplectic leaves.

Example : $\mathbb{R}^3$; $\{x^i ; x^j\} = \varepsilon^{ij}{}_k x^k$

↳ $x^i x_i = c^2$

coadjoint orbits.

⇝ Symplectic space = phase space.

Drinfeld double D acts on itself by left (or right) multiplication (as a group).

↳ even dim by def !

↳ can we make D a phase space?

As a Poisson Lie group

$\ell u = d \in D = G^* \bowtie G \qquad r = \xi^i \otimes J_i = s + a$

$$\{d_{(1)} ; d_{(2)}\}_- = [a, d \otimes d]_-$$

Drifeld double

$$\{\ell_{(1)}, \ell_{(2)}\} = +[r, \ell \otimes \ell]$$

$$\{u_{(1)}, u_{(2)}\} = -[r, u \otimes u]$$

$$\{\ell_{(1)} ; u_{(2)}\} = 0$$

As a symplectic space

$\ell u = d \in D = G^* \bowtie G$

$$\{d_{(1)} ; d_{(2)}\}_+ = [a, d \otimes d]_+$$

Heisenberg double.

$$\{\ell_{(1)}, \ell_{(2)}\} = +[r, \ell \otimes \ell]$$

$$\{u_{(1)}, u_{(2)}\} = -[r, u \otimes u]$$

$$\{\ell_{(1)} ; u_{(2)}\} = \ell_{(1)} r u_{(2)}$$

Example: $D = \mathbb{R}^3 \rtimes SU(2)$

$$r = \xi^i \otimes J_i \quad ; \quad [\xi^i ; \xi^j] = 0.$$

$$D = \ell u \qquad \ell \in \mathbb{R}^3 \qquad u \in SU(2)$$

$$x^i(\ell)$$

$$\{x^i, p^j\} = \delta^{ij}$$

we obtain

$$\left.\begin{aligned} \{x^i ; u\} &= J_i u \\ \{x^i, x^j\} &= \varepsilon^{ijk} x_k \\ \{u, u\} &= 0 \end{aligned}\right\} \quad T^* SU(2).$$

Hence we recover the Poisson bracket of $T^* SU(2)$!

$$\left(T^* SU(2), \{\ \}\right) \sim \left(\overset{ISU(2)}{\overset{\shortparallel}{\mathbb{R}^3 \rtimes SU(2)}} ; \{\ \}_+\right)$$

Symmetry : $(ISU(2), \{\cdot\}_-)$ Phase space $(\overset{T^*SU(2)}{ISU(2)}, \{\ \}_+)$

↳ acts on itself by left or right multiplication

Similar to $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \simeq T^*\mathbb{R}$.

Sym : $\mathbb{R} \times \mathbb{R}$ Phase space $\mathbb{R} \times \mathbb{R} \sim T^*\mathbb{R}$

(a, b) (p, q)

↳ action : $(a, b) \triangleright (p, q)$

$= (a+p, b+q)$

$= (p+a, q+b)$.

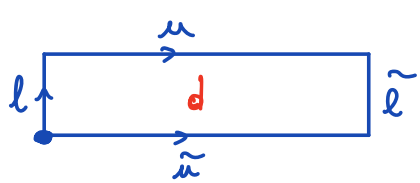
↳ Poisson brackets for sym. are 0 here !

Note: in $T^*SU(2)$ we have the left and the right momentum χ_L ; $\chi_R = u^{-1}\chi_L u$.

we have the same in the Heisenberg double formulation $ISU(2)$ has a left and a right decomposition.

$$d = \ell u = u u^{-1} \ell u = \tilde{u}\,\tilde{\ell} \quad \text{with} \quad \begin{cases} \tilde{u} = u \\ \tilde{\ell} = u^{-1}\ell u \end{cases}$$

pictorial representation



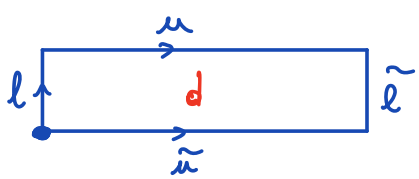
will be relevant later!

→ we can deform the cotangent bundle of $SU(2)$

↳ still a phase space but with non trivial Poisson bracket on the $SU(2)$ sector.

$$T^*SU(2) \rightsquigarrow SL(2,\mathbb{C}) = AN_2 \bowtie SU(2)$$

$$= SU(2) \bowtie AN_2$$



$$d = \ell u = \tilde{u}\,\tilde{\ell}$$

$$= (\ell \triangleright u)(\ell \triangleleft u)$$

$$(\tilde{u} \triangleright \tilde{\ell})(\tilde{u} \triangleleft \tilde{\ell}) =$$

Punch line:

cotangent bundle T^*G of group have a symmetry structure given by a Poisson lie group (Drinfeld double)

we can deform T^*G into a new phase space. Still have Poisson lie group as symmetries.

When we write the phase as a group we call it Heisenberg double.

according to the choice of dynamics we choose, we might have some non trivial Poisson lie symmetries.

$\{a_i, a_j\} = \varepsilon_{ij}{}^{k} a_k$

Phase space : $T^* SU(2) \longleftrightarrow$ Sym $ISU(2)$

$\hookrightarrow (\vec{p}, g)$ $\hookrightarrow (\vec{a}, h)$

$\vec{p} \to \vec{p} + \vec{a}$ ✗ rotation.

Spinning top: $H = p^2$

what about the symplectic form?

Theorem ← Alekseev Malkin

Heisenberg double.

if $D = G \bowtie G^*$ is a matched pair of groups with $\dim G = \dim G^*$

then

$$\Omega = \frac{1}{2}\left(\langle \Delta\tilde{\mu} \wedge \Delta \ell \rangle + \langle \underline{\Delta}\mu \wedge \underline{\Delta}\tilde{\ell} \rangle\right)$$

$\Delta v = \delta v \, v^{-1}$

$\underline{\Delta} v = v^{-1} \delta v$.

is a symplectic form.

↳ can be generalized to the case where the factorization is only local.

Note: we get (non abelian) group valued charges in general

↳ momentum maps.

we can show easily that Ω is closed.

$$\Delta \delta d = (\delta d) d^{-1} = \delta \ell \ell^{-1} + \ell \delta u u^{-1} \ell^{-1}$$

$$= \Delta \ell + d \, \underline{\Delta} u \, d^{-1}$$

$$\Delta \ell - \Delta \tilde{u} = d(\underline{\Delta} u - \underline{\Delta} \tilde{\ell})$$

$$= \Delta \tilde{u} + d \, \underline{\Delta} \tilde{\ell} \, d^{-1}.$$

$$\delta \Delta v = \Delta v \wedge \Delta v \qquad \delta \underline{\Delta} v = - \Delta v \wedge \Delta v$$

$$\delta \langle \Delta \tilde{u} \wedge \Delta \ell \rangle = \langle \Delta \tilde{u} \wedge \Delta \tilde{u} \wedge \Delta \ell \rangle - \langle \Delta \tilde{u} \wedge \Delta \ell \wedge \Delta \ell \rangle$$

$$\langle x, [y, z] \rangle = \langle [y, x]; z \rangle.$$

$$= \langle \Delta \tilde{u} \wedge \Delta \tilde{u} \wedge \Delta \ell \rangle - \langle \Delta \ell \wedge \Delta \tilde{u} \wedge \Delta \ell \rangle$$

$$= \langle (\Delta \tilde{u} - \Delta \ell) \wedge \Delta \tilde{u} \wedge \Delta \ell \rangle.$$

isotropy of $\langle , \rangle$.

$$= \frac{1}{3} \langle (\Delta \tilde{u} - \Delta \ell) \wedge (\Delta \tilde{u} - \Delta \ell) \wedge (\Delta \tilde{u} - \Delta \ell) \rangle$$

Same calculation for the other contribution $\rightsquigarrow$ get opposite value.

Explicit example: $D = \mathbb{R}^3 \rtimes SU(2)$

$= SU(2) \ltimes \mathbb{R}^3.$

$$\ell u = u \tilde{\ell} \qquad \tilde{\ell} = u^{-1} \ell u.$$

$$\Omega = \frac{1}{2} \langle \underbrace{\Delta \tilde{u}}_{= \Delta u} \wedge \Delta \ell \rangle + \frac{1}{2} \langle \underline{\Delta} u \wedge \underline{\Delta} \tilde{\ell} \rangle$$

$$\underline{\Delta} \tilde{\ell} = \tilde{\ell}^{-1} \delta \tilde{\ell} = u^{-1} \ell^{-1} u \left(- u^{-1} \delta u u^{-1} \ell u + u^{-1} \delta \ell u + u^{-1} \ell \delta u \right)$$

$$= - u^{-1} \ell^{-1} \delta u u^{-1} \ell u + u^{-1} \underbrace{\ell^{-1} \delta \ell}_{\delta \chi} u + u^{-1} \delta u$$

$$\langle \underline{\Delta} u \wedge \underline{\Delta} \tilde{\ell} \rangle = - \langle \Delta u \wedge \ell^{-1} \Delta u \ell \rangle + \langle \Delta u \wedge \delta \chi \rangle + 0$$

$$\ell = 1 + \chi \qquad \ell^{-1} = 1 - \chi$$

$$\Delta u \chi - \chi \Delta u = [\Delta u ; \chi]$$

$$\Omega = \langle \Delta u \wedge \delta \chi \rangle - \frac{1}{2} \langle \chi ; [\Delta u \wedge \Delta u] \rangle.$$

This is the symplectic form of the spinning top.

classical picture

symmetry group + phase space

→ functions on phase space

↳ algebra of observables which transform under sym

quantum picture

states are representations of symmetry

Observables are represented on such states. They also transform under such symmetries.