Superfluidity versus Bose-Einstein Condensation

Jakob Yngvason

University of Vienna

IHES, Bures-sur-Yvette, March 19, 2015

Concepts

Bose Einstein Condensation:

- Macroscopic occupation of a single 1-particle state

Concepts

Bose Einstein Condensation:

- Macroscopic occupation of a single 1-particle state

Superfluidity can mean two things:

Concepts

Bose Einstein Condensation:

- Macroscopic occupation of a single 1-particle state

Superfluidity can mean two things:

- Flow without friction (non-equilibrium phenomenon)

Concepts

Bose Einstein Condensation:

- Macroscopic occupation of a single 1-particle state

Superfluidity can mean two things:

- Flow without friction (non-equilibrium phenomenon)
- Non-classical response to an infinitesimal boost or rotation (equilibrium phenomenon)

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC
- 1946 Bogoliubov: Seminal work on microscopic theory of Bose gases and superfluidity; assumes BEC

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC
- 1946 Bogoliubov: Seminal work on microscopic theory of Bose gases and superfluidity; assumes BEC
- 1946 Andronikashvili: Reduction of moment of inertia

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC
- 1946 Bogoliubov: Seminal work on microscopic theory of Bose gases and superfluidity; assumes BEC
- 1946 Andronikashvili: Reduction of moment of inertia
- 1950's–1960's Lee, Huang, Yang,...; Dyson; Lieb; Gross; Pitaevskii: Theoretical work on interacting Bose gas; Lieb-Liniger model; Gross-Pitaevskii equation

Some history

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC
- 1946 Bogoliubov: Seminal work on microscopic theory of Bose gases and superfluidity; assumes BEC
- 1946 Andronikashvili: Reduction of moment of inertia
- 1950's–1960's Lee, Huang, Yang,...; Dyson; Lieb; Gross; Pitaevskii: Theoretical work on interacting Bose gas; Lieb-Liniger model; Gross-Pitaevskii equation
- 1995 Ketterle; Cornell, Wieman: Experimental realization of BEC in dilute, trapped gases

Definitions

1-body density matrix of a (pure or mixed) many-body state $\langle \cdot \rangle$:

$$\rho^{(1)}(x, x') = \langle a^\dagger(x) a(x') \rangle = \sum_{i=0}^{\infty} N_i \, \overline{\psi_i(x)} \psi_i(x'),$$

$N_0 \geq N_1 \geq \cdots, \sum_{N_i} = N$.

Definitions

1-body density matrix of a (pure or mixed) many-body state $\langle \cdot \rangle$:

$$\rho^{(1)}(x,x') = \langle a^\dagger(x) a(x') \rangle = \sum_{i=0}^{\infty} N_i \overline{\psi_i(x)} \psi_i(x'),$$

$N_0 \geq N_1 \geq \cdots, \sum_{N_i} = N$.

BEC means by definition that the **condensate fraction**, N_0/N, is $O(1)$.

Definitions

1-body density matrix of a (pure or mixed) many-body state $\langle \cdot \rangle$:

$$\rho^{(1)}(x, x') = \langle a^\dagger(x) a(x') \rangle = \sum_{i=0}^{\infty} N_i \overline{\psi_i(x)} \psi_i(x'),$$

$N_0 \geq N_1 \geq \cdots, \sum_{N_i} = N$.

BEC means by definition that the **condensate fraction**, N_0/N, is $O(1)$.

The **superfluid mass density** ρ_s (at rest) is defined through the response of the free energy to a small boost $\mathbf{v}$:

$$F(\mathbf{v}) = F(0) + \tfrac{1}{2} m N \, (\rho_s/\rho) v^2 + o(v^2)$$

The boost is mathematically implemented by the substitution

$$\mathbf{p}_i \to \mathbf{p}_i - m\mathbf{v}$$

in the Hamiltonian, assuming *periodic boundary conditions* in the direction $\mathbf{e}$ of $\mathbf{v} = v\mathbf{e}$ with period Λ, say.

The boost is mathematically implemented by the substitution

$$\mathbf{p}_i \to \mathbf{p}_i - m\mathbf{v}$$

in the Hamiltonian, assuming *periodic boundary conditions* in the direction $\mathbf{e}$ of $\mathbf{v} = v\mathbf{e}$ with period Λ, say.

Equivalently one can consider the original Hamiltonian but with *twisted boundary conditions*:

$$\Psi(\dots, \mathbf{x}_{i-1}, \Lambda\mathbf{e}, \mathbf{x}_i, \dots) = e^{-\mathrm{i}\varphi}\Psi(\dots, \mathbf{x}_{i-1}, 0, \mathbf{x}_i, \dots)$$

with

$$\varphi = mv\Lambda/\hbar.$$

The boost is mathematically implemented by the substitution

$$\mathbf{p}_i \to \mathbf{p}_i - m\mathbf{v}$$

in the Hamiltonian, assuming *periodic boundary conditions* in the direction $\mathbf{e}$ of $\mathbf{v} = v\mathbf{e}$ with period Λ, say.

Equivalently one can consider the original Hamiltonian but with *twisted boundary conditions*:

$$\Psi(\dots, \mathbf{x}_{i-1}, \Lambda\mathbf{e}, \mathbf{x}_i, \dots) = e^{-\mathrm{i}\varphi}\Psi(\dots, \mathbf{x}_{i-1}, 0, \mathbf{x}_i, \dots)$$

with

$$\varphi = mv\Lambda/\hbar.$$

The free energy as a function of v is periodic with period φ.

Convenient experimental realization: Container in the form of a thin annulus, radius R, thickness $d \ll R$.

Convenient experimental realization: Container in the form of a thin annulus, radius R, thickness $d \ll R$.

A boost corresponds to a *rotation* of the walls of the container:

$$v \longleftrightarrow R\Omega$$

with Ω the angular velocity. Since $\Lambda = 2\pi R$ the phase is

$$\varphi = 2\pi m R^2 \Omega / \hbar$$

Reduction of moment of inertia

Hess-Fairbank experiment (1967): Rotate slowly a thin annulus of liquid He^4 at angular velocity $\Omega < \Omega_c \sim \hbar/mR^2$.

Reduction of moment of inertia

Hess-Fairbank experiment (1967): Rotate slowly a thin annulus of liquid He^4 at angular velocity $\Omega < \Omega_c \sim \hbar/mR^2$.

1) $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\rm classical}\Omega$,

$$I_{\rm classical} = NmR^2$$

Reduction of moment of inertia

Hess-Fairbank experiment (1967): Rotate slowly a thin annulus of liquid He^4 at angular velocity $\Omega < \Omega_c \sim \hbar/mR^2$.

1) $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\text{classical}}\Omega$,

$$I_{\text{classical}} = NmR^2$$

2) $T < T_\lambda$: Liquid rotates with reduced moment of inertia

$$I(T) = (\rho_{\text{n}}/\rho)I_{\text{classical}} < I_{\text{classical}}, \qquad I(T=0) = 0.$$

Only the normal fluid rotates while the superfluid remains stationary in the laboratory frame.

Metastable superfluid flow

1) Rotate rapidly, $\Omega > \Omega_c \sim 1/mR^2$, at $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\text{classical}}\Omega$.

Metastable superfluid flow

1) Rotate rapidly, $\Omega > \Omega_c \sim 1/mR^2$, at $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\text{classical}}\Omega$.
2) Continue rotating but cool to $T < T_\lambda$: Liquid still rotates classically.

Metastable superfluid flow

1) Rotate rapidly, $\Omega > \Omega_c \sim 1/mR^2$, at $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\text{classical}}\Omega$.
2) Continue rotating but cool to $T < T_\lambda$: Liquid still rotates classically.
3) Stop rotating the annulus. Liquid keeps rotating but (after a short decay time) with reduced moment of inertia

$$I_s = (\rho_s/\rho) I_{\text{classical}}$$

Only the superfluid rotates in the lab frame, the normal component is stationary.

Metastable superfluid flow

1) Rotate rapidly, $\Omega > \Omega_c \sim 1/mR^2$, at $T > T_\lambda$: Liquid rotates classically with angular momentum $L = I_{\rm classical}\Omega$.
2) Continue rotating but cool to $T < T_\lambda$: Liquid still rotates classically.
3) Stop rotating the annulus. Liquid keeps rotating but (after a short decay time) with reduced moment of inertia

$$I_s = (\rho_s/\rho) I_{\rm classical}$$

Only the superfluid rotates in the lab frame, the normal component is stationary.

The superfluid flow is metastable but with a huge lifetime due to energy barriers in a macroscopic system with a repulsive interaction .

Is there a connection between BEC and SF?

It is certainly **not true** that the condensate fraction and superfluid fraction are the same:

Is there a connection between BEC and SF?

It is certainly **not true** that the condensate fraction and superfluid fraction are the same:

Close to $T = 0$ liquid He^4 is almost completely superfluid, but the condensate fraction is only about 10%.

Is there a connection between BEC and SF?

It is certainly **not true** that the condensate fraction and superfluid fraction are the same:

Close to $T = 0$ liquid He^4 is almost completely superfluid, but the condensate fraction is only about 10%.

A 1D hard core Bose gas at $T = 0$ has no BEC but is superfluid. Same holds for a 2D Kosterlitz-Thouless superfluid.

Is there a connection between BEC and SF?

It is certainly **not true** that the condensate fraction and superfluid fraction are the same:

Close to $T = 0$ liquid He^4 is almost completely superfluid, but the condensate fraction is only about 10%.

A 1D hard core Bose gas at $T = 0$ has no BEC but is superfluid. Same holds for a 2D Kosterlitz-Thouless superfluid.

What about the converse, i.e., does BEC imply superfluidity?

A rigorous study of this question is hampered by the fact that it is notoriously difficult to prove BEC for systems with interactions. The only known examples so far are:

- Hard core Bose lattice gas at *exactly* half filling (Dyson, Lieb Simon, 1978; Lieb Kennedy and Shastry 1988)

A rigorous study of this question is hampered by the fact that it is notoriously difficult to prove BEC for systems with interactions. The only known examples so far are:

- Hard core Bose lattice gas at *exactly* half filling (Dyson, Lieb Simon, 1978; Lieb Kennedy and Shastry 1988)
- Ground state of a dilute Bose gas with repulsive interaction in the Gross-Pitaevskii limit (Lieb, Seiringer 2002)

A rigorous study of this question is hampered by the fact that it is notoriously difficult to prove BEC for systems with interactions. The only known examples so far are:

- Hard core Bose lattice gas at *exactly* half filling (Dyson, Lieb Simon, 1978; Lieb Kennedy and Shastry 1988)
- Ground state of a dilute Bose gas with repulsive interaction in the Gross-Pitaevskii limit (Lieb, Seiringer 2002)

Complete superfluidity in the ground state has also been proved in the GP limit in the translationally (rotationally) invariant situation (Lieb, Seiringer, JY, 2003).

On the other hand its has long been claimed (and supported by numerical and heuristic arguments) that an external *random potential* may strongly reduce superfluidity.

On the other hand its has long been claimed (and supported by numerical and heuristic arguments) that an external *random potential* may strongly reduce superfluidity.

A state the has BEC but no superfluidity has been called a Bose Glass (S. Yukalov). Until recently there have been no mathematical results on the existence of such states.

On the other hand its has long been claimed (and supported by numerical and heuristic arguments) that an external *random potential* may strongly reduce superfluidity.

A state the has BEC but no superfluidity has been called a Bose Glass (S. Yukalov). Until recently there have been no mathematical results on the existence of such states.

The effect of a random potential on BEC and superfluidity can be investigated mathematically in a simple model:

R.Seiringer, J. Yngvason, V. Zagrebnov, *Disordered Bose Einstein Condensates with Interaction in One Dimension*, J. Stat. Mech. P11007, 2012

M. Könenberg, T. Moser, Robert Seiringer, and J.Yngvason, *Superfluid Behavior of a Bose Einstein Condensate in a Random Potential*, New J. Phys. 17 013022 (2015).

The Model

The model is a gas of 1D bosons with contact interaction (Lieb-Linger model) on the unit interval but with an additional external random potential V_ω. The Hamiltonian on the Hilbert space $L^2([0,1], dz)^{\otimes_s N}$ is

$$H = \sum_{i=1}^{N} \left(-\partial_{z_i}^2 + V_\omega(z_i)\right) + \frac{\gamma}{N} \sum_{i<j} \delta(z_i - z_j)$$

with $\gamma \geq 0$ and *periodic* boundary conditions. This can be regarded as a model of a gas in a thin annulus.

The Model

The model is a gas of 1D bosons with contact interaction (Lieb-Linger model) on the unit interval but with an additional external random potential V_ω. The Hamiltonian on the Hilbert space $L^2([0,1], dz)^{\otimes_{\rm s} N}$ is

$$H = \sum_{i=1}^{N} \left(-\partial_{z_i}^2 + V_\omega(z_i)\right) + \frac{\gamma}{N}\sum_{i<j}\delta(z_i - z_j)$$

with $\gamma \geq 0$ and *periodic* boundary conditions. This can be regarded as a model of a gas in a thin annulus.

The random potential will be taken to be

$$V_\omega(z) = \sigma \sum \delta(z - z_j^\omega)$$

with $\sigma \geq 0$ independent of the random sample ω while the obstacles $\{z_j^\omega\}$ are Poisson distributed with density $\nu \gg 1$.

Parameters

Besides N, the model has three parameters:

Parameters

Besides N, the model has three parameters:

- γ: Strength of the interaction

Parameters

Besides N, the model has three parameters:

- γ: Strength of the interaction
- ν: Density of obstacles

Parameters

Besides N, the model has three parameters:

- γ: Strength of the interaction
- ν: Density of obstacles
- σ: Strength of the random potential

Parameters

Besides N, the model has three parameters:

- γ: Strength of the interaction
- ν: Density of obstacles
- σ: Strength of the random potential

In the $N \to \infty$ limit the ground state energy and the wave function of the condensate is described by a Gross Pitaevskii (GP) energy functional.

Parameters

Besides N, the model has three parameters:

- γ: Strength of the interaction
- ν: Density of obstacles
- σ: Strength of the random potential

In the $N \to \infty$ limit the ground state energy and the wave function of the condensate is described by a Gross Pitaevskii (GP) energy functional.

Moreover, the energy becomes deterministic if the parameters satisfy

$$\nu \gg 1\,, \quad \gamma \gg \frac{\nu}{(\ln \nu)^2}\,, \quad \sigma \gg \frac{\nu}{1 + \ln\left(1 + \nu^2/\gamma\right)}\,.$$

We shall refer to these conditions as the *standard conditions* and assume them throughout.

Main Results

- *In the whole parameter range there is complete BEC.*

Main Results

- *In the whole parameter range there is complete BEC.*
- *If $\gamma \lesssim \nu^2$ the superfluid fraction is arbitrarily small, i.e., it goes to zero under the standard conditions.*

Main Results

- *In the whole parameter range there is complete BEC.*
- *If $\gamma \lesssim \nu^2$ the superfluid fraction is arbitrarily small, i.e., it goes to zero under the standard conditions.*
- *The same holds for $\nu^2 \ll \gamma \ll \nu^4$ provided $\sigma \gg (\gamma/\nu^2)^2\gamma^{1/2}$.*

Main Results

- *In the whole parameter range there is complete BEC.*
- *If* $\gamma \lesssim \nu^2$ *the superfluid fraction is arbitrarily small, i.e., it goes to zero under the standard conditions.*
- *The same holds for* $\nu^2 \ll \gamma \ll \nu^4$ *provided* $\sigma \gg (\gamma/\nu^2)^2\gamma^{1/2}$.
- *If* $\gamma \gg (\sigma\nu)^2$ *there is complete superfluidity, i.e., the superfluid fraction tends to 1.*

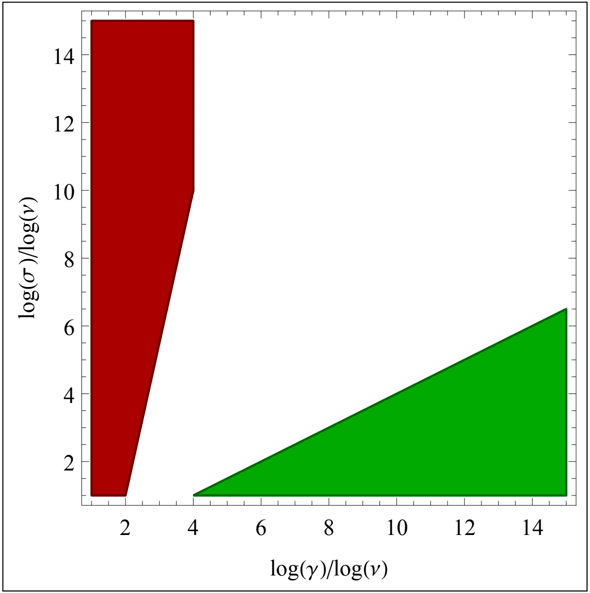

Figure : Red: Absence of superfluidity. Green: Complete superfluidity.

The general heuristic picture is that

- Strong repulsive interaction between the particles (large γ) tends to make the density uniform and favours superfluidity.

- Strong randomness (large ν and σ) leads to fractionation of the density that is unfavorable for superfluidity.

It is remarkable, however, that BEC survives the fractionation of the density, i.e. the long range corelations prevail although superfluidity may be strongly suppressed.

BEC

Bose-Einstein condensation in the ground state holds in the GP limit, where $N \to \infty$ and γ is fixed, or does not grow too fast with N.
This holds for an arbitrary positive potential V.

BEC

Bose-Einstein condensation in the ground state holds in the GP limit, where $N \to \infty$ and γ is fixed, or does not grow too fast with N.
This holds for an arbitrary positive potential V.

The wave function of the condensate is the minimizer ψ_0 of the Gross Pitaevskii (GP) energy functional

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_0^1 \left(|\psi'(z)|^2 + V(z)|\psi(z)|^2 + \frac{\gamma}{2}|\psi(z)|^4 \right) \mathrm{d}z$$

with the normalization $\int_0^1 |\psi|^2 = 1$.

BEC

Bose-Einstein condensation in the ground state holds in the GP limit, where $N \to \infty$ and γ is fixed, or does not grow too fast with N. This holds for an arbitrary positive potential V.

The wave function of the condensate is the minimizer ψ_0 of the Gross Pitaevskii (GP) energy functional

$$\mathcal{E}^{\rm GP}[\psi] = \int_0^1 \left(|\psi'(z)|^2 + V(z)|\psi(z)|^2 + \frac{\gamma}{2}|\psi(z)|^4 \right) {\rm d}z$$

with the normalization $\int_0^1 |\psi|^2 = 1$.

The minimizer ψ_0 is also the ground state of the mean field Hamiltonan

$$h = -\partial_z^2 + V(z) + \gamma|\psi_0|^2 - \frac{\gamma}{2}\int_0^1 |\psi_0|^4$$

with eigenvalue $e_0 = \mathcal{E}^{\rm GP}[\psi_0]$.

The average occupation of the one particle state ψ_0 in the many-body ground state Ψ_0 of H is

$$N_0 = \langle \Psi_0, a^\dagger(\psi_0) a(\psi_0) \Psi_0 \rangle.$$

The average occupation of the one particle state ψ_0 in the many-body ground state Ψ_0 of H is

$$N_0 = \langle \Psi_0, a^\dagger(\psi_0) a(\psi_0) \Psi_0 \rangle.$$

Bose-Einstein condensation in the GP limit follows from the estimate of the depletion of the condensate,

$$\left(1 - \frac{N_0}{N}\right) \leq (\text{const.}) \frac{e_0}{e_1 - e_0} N^{-1/3} \min\{\gamma^{1/2}, \gamma\}$$

where e_1 is the second lowest eigenvalue of the mean field Hamiltonian h. Moreover, the ground state energy per particle of H converges to the GP energy e_0.

Superfluidity

With an imposed velocity field v (moving walls) the Hamiltonian becomes:

$$H_v = \sum_{j=1}^{N} \{(\mathrm{i}\partial_{z_j} + v)^2 + V(z_j)\} + \frac{\gamma}{N} \sum_{i<j} \delta(z_i - z_j)$$

on $L^2([0,1], \mathrm{d}z)^{\otimes_{\mathrm{symm}}^N}$ with periodic boundary conditions.

With an imposed velocity field v (moving walls) the Hamiltonian becomes:

$$H_v = \sum_{j=1}^{N} \{(\mathrm{i}\partial_{z_j} + v)^2 + V(z_j)\} + \frac{\gamma}{N}\sum_{i<j}\delta(z_i - z_j)$$

on $L^2([0,1], \mathrm{d}z)^{\otimes_{\mathrm{symm}}^N}$ with periodic boundary conditions.

Denote its ground state energy by $E_0^{\mathrm{QM}}(v)$, and by $e_0(v)$ corresponding ground state energy of the modified GP functional

$$\mathcal{E}_v^{\mathrm{GP}}[\psi] = \int_0^1 \left(|\mathrm{i}\psi'(z) + v\psi(z)|^2 + V(z)|\psi(z)|^2 + \frac{\gamma}{2}|\psi(z)|^4\right) \mathrm{d}z.$$

For small enough v, $\mathcal{E}_v^{\mathrm{GP}}$ has a unique minimizer, denoted by ψ_v, and $e_0(v)$ is equal to the ground state energy of the mean field Hamiltonian

$$h_v = (\mathrm{i}\partial_z + v)^2 + V(z) + \gamma|\psi_v(z)|^2 - \frac{\gamma}{2}\int_0^1 |\psi_v|^4.$$

For small enough v, $\mathcal{E}_v^{\rm GP}$ has a unique minimizer, denoted by ψ_v, and $e_0(v)$ is equal to the ground state energy of the mean field Hamiltonian

$$h_v = ({\rm i}\partial_z + v)^2 + V(z) + \gamma|\psi_v(z)|^2 - \frac{\gamma}{2}\int_0^1 |\psi_v|^4.$$

Using the diamagnetic inequality one shows in an analogous way to the $v = 0$ case:

$$E_0^{\rm QM}(v)/N \geq e_0(v)(1 - ({\rm const.})N^{-1/3}\min\{\gamma^{1/2}, \gamma\}).$$

We conclude that in the GP limit the superfluid fraction

$$\rho_s/\rho = \lim_{v\to 0}\frac{1}{v^2}\lim_{N\to\infty}\frac{1}{N}(E_0^{\rm QM}(v) - E_0^{\rm QM}(0))$$

is the same as the corresponding quantity derived from the GP energy, i.e.,

$$\rho_s/\rho = \lim_{v\to 0}\frac{1}{v^2}(e_0(v) - e_0(0)).$$

A closed formula for the superfluid fraction

We claim that

$$\rho_{\rm s}/\rho = \left(\int_0^1 |\psi_0(z)|^{-2} dz\right)^{-1}$$

This provides and explicit connection between the wave function of the BE condensate and the superfluid fraction.

A closed formula for the superfluid fraction

We claim that

$$\rho_s/\rho = \left(\int_0^1 |\psi_0(z)|^{-2} dz\right)^{-1}$$

This provides and explicit connection between the wave function of the BE condensate and the superfluid fraction.

Proof. Start with the variational equation for ψ_v :

$$(i\partial_z + v)^2 \psi_v(z) + V(z)\psi_v(z) + \gamma|\psi_v(z)|^2\psi_v(z) = \mu\psi_v(z).$$

We multiply this by $\bar{\psi}_v$ and take the imaginary part, to obtain

$$\partial_z \left(v|\psi_v(z)|^2 - \Im[\bar{\psi}_v(z) d\psi_v(z)/dz]\right) = 0.$$

Hence there exists a constant $C \in \mathbb{R}$ such that

$$\Im[\bar{\psi}_v(z) d\psi_v(z)/dz] = v|\psi_v(z)|^2 - C.$$

Since

$$de_0(v)/dv = 2v - 2\int_0^1 \Im[\bar{\psi}_v(z)d\psi_v(z)/dz]dz$$

we actually see that $C = \frac{1}{2}de_0(v)/dv$. Now ψ_v has no zeroes for small v so we can divide by $|\psi_v(z)|^2$ and obtain

$$S'(z) := \frac{\Im[\bar{\psi}_v(z)d\psi_v(z)/dz]}{|\psi_v(z)|^2} = v - \frac{C}{|\psi_v(z)|^2}.$$

Since S' is, in fact, the derivative of the phase of ψ_v we have, due to the periodic boundary conditions,

$$\int_0^1 S'(z)dz = 2\pi n$$

with $n \in \mathbb{Z}$, and in fact $n = 0$ for small enough v.

Therefore

$$v = C\int_0^1 |\psi_v(z)|^{-2}dz.$$

This gives

$$e_0'(v) = 2C = 2v\left(\int_0^1 |\psi_v(z)|^{-2}dz\right)^{-1}$$

and thus

$$\rho_s/\rho = \lim_{v\to 0}\frac{e_0'(v)}{2v} = \left(\int_0^1 |\psi_0(z)|^{-2}dz\right)^{-1}.$$

□

Complete superfluidity for $\gamma \gg (\sigma\nu)^2$

Using that ψ_0 is a GP minimizer and the GP energy is $\leq \int V_\omega + \frac{\gamma}{2}$ one can show that

$$\frac{\||\psi_0|^2 - 1\|_\infty^2}{\sqrt{1 + \| \, |\psi_0|^2 - 1\|_\infty}} \leq \frac{2^{3/2}}{\sqrt{\gamma}} \int_0^1 V_\omega \sim (\sigma\nu)/\gamma^{1/2}.$$

Hence we see:

The superfluid fraction tends to 1 in probability if

$$\gamma \gg (\sigma\nu)^2.$$

Absence of superfluidity

If $\mathcal{I}$ is any (measurable) subset of $[0,1]$ with length $|\mathcal{I}|$ we have

$$|\mathcal{I}|^2 = \left(\int_{\mathcal{I}} |\psi_0||\psi_0|^{-1}\right)^2 \leq \int_{\mathcal{I}} |\psi_0(z)|^2 \cdot \int_{\mathcal{I}} |\psi_0(z)|^{-2}$$

and hence

$$\rho_{\rm s} \leq \frac{\int_{\mathcal{I}} |\psi_0(z)|^2 dz}{|\mathcal{I}|^2}.$$

Absence of superfluidity

If $\mathcal{I}$ is any (measurable) subset of $[0,1]$ with length $|\mathcal{I}|$ we have

$$|\mathcal{I}|^2 = \left(\int_{\mathcal{I}} |\psi_0||\psi_0|^{-1}\right)^2 \leq \int_{\mathcal{I}} |\psi_0(z)|^2 \cdot \int_{\mathcal{I}} |\psi_0(z)|^{-2}$$

and hence

$$\rho_{\mathrm{s}} \leq \frac{\int_{\mathcal{I}} |\psi_0(z)|^2 dz}{|\mathcal{I}|^2}.$$

To prove that superfluidity is small we have therefore to identify subsets such that $\int_{\mathcal{I}} |\psi_0(z)|^2 dz$ is small, while $|\mathcal{I}|$ is not too small.

Choice of $\mathcal{I}$

The random points z_j^ω split the intervall $[0, 1]$ into subintervals $\mathcal{I}_j = [z_j^\omega, z_{j+1}^\omega]$ of various lengths $\ell_j = z_{i+1}^\omega - z_j^\omega$ that are i.i.d. random variables with probability distribution

$$dP_\nu(\ell) = \nu e^{-\nu\ell}\, d\ell.$$

Choice of $\mathcal{I}$

The random points z_j^ω split the intervall $[0,1]$ into subintervals $\mathcal{I}_j = [z_j^\omega, z_{j+1}^\omega]$ of various lengths $\ell_j = z_{i+1}^\omega - z_j^\omega$ that are i.i.d. random variables with probability distribution

$$dP_\nu(\ell) = \nu e^{-\nu\ell}\, d\ell.$$

We shall take

$$\mathcal{I} = \bigcup_{j:\ell_j \leq \tilde{\ell}} \mathcal{I}_j$$

with a suitably chosen $\tilde{\ell}$.

Choice of $\mathcal{I}$

The random points z_j^ω split the intervall $[0, 1]$ into subintervals $\mathcal{I}_j = [z_j^\omega, z_{j+1}^\omega]$ of various lengths $\ell_j = z_{i+1}^\omega - z_j^\omega$ that are i.i.d. random variables with probability distribution

$$dP_\nu(\ell) = \nu e^{-\nu\ell}\, d\ell.$$

We shall take

$$\mathcal{I} = \bigcup_{j:\ell_j \leq \tilde{\ell}} \mathcal{I}_j$$

with a suitably chosen $\tilde{\ell}$.

The average length of $\mathcal{I}$ is

$$L = \nu \int_0^{\tilde{\ell}} \ell dP_\nu(\ell) = 1 - (1 + (\nu\tilde{\ell}))e^{-\nu\tilde{\ell}}).$$

In particular it tends to 1 if and only if $\tilde{\ell} \gg \nu^{-1}$.

With the notation

$$n_j^{\mathrm{GP}} = \int_{\mathcal{I}_j} |\psi_0(z)|^2 dz$$

we define

$$N_{\mathrm{small},\omega} = \int_{\mathcal{I}} |\psi_0(z)|^2 dz = \sum_{\ell_j \leq \tilde{\ell}} n_j^{\mathrm{GP}}.$$

Note that ψ_0 and n_j^{GP} also depend on ω but we have suppressed this in the notation for simplicity.

With the notation

$$n_j^{\mathrm{GP}} = \int_{\mathcal{I}_j} |\psi_0(z)|^2 dz$$

we define

$$N_{\mathrm{small},\omega} = \int_{\mathcal{I}} |\psi_0(z)|^2 dz = \sum_{\ell_j \leq \tilde{\ell}} n_j^{\mathrm{GP}}.$$

Note that ψ_0 and n_j^{GP} also depend on ω but we have suppressed this in the notation for simplicity.

Our estimate on $N_{\mathrm{small},\omega}$ is based on estimates on the GP energy.

Auxiliary energy

Define

$$\mathcal{E}_{\kappa,\alpha}[\phi] = \int_0^1 dx\ \left(|\phi'(x)|^2 + \frac{\kappa}{2}|\phi(x)|^4\right) + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(1)|^2\right)$$

with $\kappa \geq 0$ and $\alpha \geq 0$. Let $e(\kappa, \alpha)$ denote the auxiliary GP energy

$$e(\kappa, \alpha) = \inf_{\|\phi\|_2=1} \mathcal{E}_{\kappa,\alpha}[\phi]\ .$$

Define

$$\mathcal{E}_{\kappa,\alpha}[\phi] = \int_0^1 dx \, \left(|\phi'(x)|^2 + \frac{\kappa}{2}|\phi(x)|^4 \right) + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(1)|^2\right)$$

with $\kappa \geq 0$ and $\alpha \geq 0$. Let $e(\kappa, \alpha)$ denote the auxiliary GP energy

$$e(\kappa, \alpha) = \inf_{\|\phi\|_2=1} \mathcal{E}_{\kappa,\alpha}[\phi] \, .$$

The corresponding energy for an interval of length ℓ with mass $\int_{\rm Interval} |\phi|^2 = n$, coupling constant γ and strength σ of the obstacle potential is then, by scaling,

$$\frac{n}{\ell^2} e(n\ell\gamma, \ell\sigma).$$

We use the following bounds on $e(\kappa, \alpha)$:

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq e(\kappa, \infty) \left(1 - K\alpha^{-1/2}\right)$$

We use the following bounds on $e(\kappa, \alpha)$:

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq e(\kappa, \infty)\left(1 - K\alpha^{-1/2}\right)$$

and

$$e(\kappa, \alpha) \geq e(0, \alpha) \geq \frac{C\alpha}{1+\alpha}\,.$$

with constants K and C independent of κ and α.

The interval density functional

With $n(\ell) \geq 0$ a mass distribution on intervals of various lengths ℓ we define an “interval density functional” as

$$\mathcal{E}^{\mathrm{IDF}}[n(\cdot)] = \nu \int_0^\infty dP_\nu(\ell) \frac{n(\ell)}{\ell^2} e(n(\ell)\ell\gamma, \infty)$$

with corresponding energy

$$e^{\mathrm{IDF}}(\nu, \gamma) = \inf \left\{ \mathcal{E}^{\mathrm{IDF}}[n(\cdot)] \, : \, \nu \int_0^\infty dP_\nu(\ell) n(\ell) = 1 \right\}.$$

This energy is the deterministic limit (in probability) of the GP energy under our standard conditions on the parameters.

The Lagrange multiplier μ for the normalization condition fulfills

$$\mu \sim \gamma\, f(\nu^2/\gamma),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the function

$$f(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ \frac{x}{(1+\ln x)^2} & \text{for } x \geq 1. \end{cases}$$

Also $e^{\mathrm{IDF}}(\gamma, \nu) \sim \gamma\, f(\nu^2/\gamma)$.

The Lagrange multiplier μ for the normalization condition fulfills

$$\mu \sim \gamma\, f(\nu^2/\gamma),$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the function

$$f(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ \frac{x}{(1+\ln x)^2} & \text{for } x \geq 1. \end{cases}$$

Also $e^{\mathrm{IDF}}(\gamma, \nu) \sim \gamma\, f(\nu^2/\gamma)$.

A further result is that the minimizing $n(\ell)$ of the interval density functional is nonzero if and only if $\mu\ell^2 > \pi^2$. We can therefore expect that the GP mass is small in intervals $\mathcal{I}_j$ such that $\ell_j \leq (\text{const.})/\sqrt{\mu}$.

Absence of superfluidity for $\gamma \lesssim \nu^2$

Split the GP energy $e_\omega^{\rm GP}(\gamma,\nu,\sigma)$ into contributions from the 'large' and the 'small' intervals:

$$e_\omega^{\rm GP}(\gamma,\nu,\sigma) \geq \sum_{\ell_j \geq \tilde{\ell}} \frac{n_j^{\rm GP}}{\ell_j^2} e(n_j^{\rm GP}\ell_j\gamma, \ell_j\sigma) + \sum_{\ell_j < \tilde{\ell}} \frac{n_j^{\rm GP}}{\ell_j^2} e(n_j^{\rm GP}\ell_j\gamma, \ell_j\sigma)$$

where

$$\tilde{\ell} = s/\sqrt{\mu}$$

with a suitable s and (because $\gamma/\nu^2 \lesssim 1$)

$$\mu \sim \frac{\nu^2}{(1+\ln(\nu^2/\gamma))^2}.$$

Absence of superfluidity for $\gamma \lesssim \nu^2$

Split the GP energy $e^{\rm GP}_\omega(\gamma,\nu,\sigma)$ into contributions from the 'large' and the 'small' intervals:

$$e^{\rm GP}_\omega(\gamma,\nu,\sigma) \geq \sum_{\ell_j\geq\tilde\ell}\frac{n_j^{\rm GP}}{\ell_j^2}e(n_j^{\rm GP}\ell_j\gamma,\ell_j\sigma)+\sum_{\ell_j<\tilde\ell}\frac{n_j^{\rm GP}}{\ell_j^2}e(n_j^{\rm GP}\ell_j\gamma,\ell_j\sigma)$$

where

$$\tilde\ell = s/\sqrt{\mu}$$

with a suitable s and (because $\gamma/\nu^2 \lesssim 1$)

$$\mu \sim \frac{\nu^2}{(1+\ln(\nu^2/\gamma))^2}.$$

Note that, since $\sigma \gg \nu/(1+\ln(1+\nu^2/\gamma))$ we have for $\gamma \lesssim \nu^2$

$$\tilde\ell\sigma \gg 1$$

Energy estimate for small intervals

The sum over the small intervals can be estimated as

$$\sum_{\ell_j<\tilde{\ell}} \frac{n_j^{\mathrm{GP}}}{\ell_j^2} e(n_j^{\mathrm{GP}}\ell_j\gamma, \ell_j\sigma) \geq \sum_{\ell_j<\tilde{\ell}} \frac{n_j^{\mathrm{GP}}}{\ell_j^2} e(0, \ell_j\sigma) \geq \sum_{\ell_j<\tilde{\ell}} \frac{n_j^{\mathrm{GP}}}{\ell_j^2} \frac{C\,\ell_j\sigma}{1+\ell_j\sigma}$$

$$\geq N_{\mathrm{small},\omega} \cdot \frac{C\,\sigma}{\tilde{\ell}(1+\tilde{\ell}\sigma)} = N_{\mathrm{small},\omega} \cdot \mu\,\frac{C}{s^2}\,\frac{\sigma\tilde{\ell}}{1+\sigma\tilde{\ell}}.$$

Here we have used the estimates for the energy in intervals between obstacles.

Energy Estimate for large intervals

For the large intervals we have

$$\sum_{\ell_j \geq \tilde{\ell}} \frac{n_j^{\mathrm{GP}}}{\ell_j^2} e(n_j^{\mathrm{GP}} \ell_j \gamma, \ell_j \sigma) \geq \inf_{\sum n_i = 1 - N_{\mathrm{small},\omega}} \sum_{\ell_j \geq \tilde{\ell}} \frac{n_j}{\ell_j^2} e(n_j \ell_j \gamma, \ell_j \sigma)$$

$$\geq \inf_{\sum n_i = 1 - N_{\mathrm{small},\omega}} \sum_j \frac{n_j}{\ell_j^2} e(n_j \ell_j \gamma, \infty)(1 - K(\tilde{\ell}\sigma)^{-1/2}).$$

For the large intervals we have

$$\sum_{\ell_j \geq \tilde{\ell}} \frac{n_j^{\mathrm{GP}}}{\ell_j^2} e(n_j^{\mathrm{GP}} \ell_j \gamma, \ell_j \sigma) \geq \inf_{\sum n_i = 1 - N_{\mathrm{small},\omega}} \sum_{\ell_j \geq \tilde{\ell}} \frac{n_j}{\ell_j^2} e(n_j \ell_j \gamma, \ell_j \sigma)$$

$$\geq \inf_{\sum n_i = 1 - N_{\mathrm{small},\omega}} \sum_j \frac{n_j}{\ell_j^2} e(n_j \ell_j \gamma, \infty)(1 - K(\tilde{\ell}\sigma)^{-1/2}).$$

Apart from the factor $(1 - K(\tilde{\ell}\sigma)^{-1/2})$ the right side is the GP energy for $\sigma = \infty$ with normalization $\int |\psi|^2 = 1 - N_{\mathrm{small},\omega}$ instead of $\int |\psi|^2 = 1$.

By simple scaling this is $1 - N_{\mathrm{small},\omega}$ times the the GP energy with normalization 1 and γ replaced by $(1 - N_{\mathrm{small},\omega})\gamma$, which in turn is not smaller than $(1 - N_{\mathrm{small},\omega})^2$ times $e_\omega^{\mathrm{GP}}(\gamma, \nu, \infty)$.

We can further estimate

$$(1 - N_{\text{small},\omega})^2 e_\omega^{\text{GP}}(\gamma, \nu, \infty) \geq (1 - 2N_{\text{small},\omega}) e_\omega^{\text{GP}}(\gamma, \nu, \sigma)$$

and putting everything together we obtain

$$\frac{e_\omega^{\text{GP}}(\gamma, \nu, \sigma)}{\mu} \geq N_{\text{small},\omega} \cdot \frac{C}{s^2} \frac{\sigma\tilde{\ell}}{1 + \sigma\tilde{\ell}} + (1 - 2N_{\text{small},\omega}) \frac{e_\omega^{\text{GP}}(\gamma, \nu, \sigma)}{\mu} (1 - K(\tilde{\ell}\sigma)^{-1/2}).$$

We can further estimate

$$(1 - N_{\text{small},\omega})^2 e_\omega^{\text{GP}}(\gamma, \nu, \infty) \geq (1 - 2N_{\text{small},\omega}) e_\omega^{\text{GP}}(\gamma, \nu, \sigma)$$

and putting everything together we obtain

$$\frac{e_\omega^{\text{GP}}(\gamma, \nu, \sigma)}{\mu} \geq N_{\text{small},\omega} \cdot \frac{C}{s^2} \frac{\sigma \tilde{\ell}}{1 + \sigma \tilde{\ell}} + (1 - 2N_{\text{small},\omega}) \frac{e_\omega^{\text{GP}}(\gamma, \nu, \sigma)}{\mu} (1 - K(\tilde{\ell}\sigma)^{-1/2}).$$

Important point:

If ν, γ and σ tend to infinity under the standard conditions, the ratio $e_\omega^{\text{GP}}(\gamma, \nu, \sigma)/\mu$ stays bounded (in probability).

Mass in small intervals

Moreover, for $\gamma \lesssim \nu^2$ we have

$$\sigma\tilde{\ell} = s\sigma/\sqrt{\mu} \gg 1.$$

For $C/s^2 > 2e_\omega^{\rm GP}(\gamma,\nu,\sigma)/\mu$ we thus arrive at an estimate for the mass in the small intervals:

$$N_{\mathrm{small},\omega} \leq (\mathrm{const.})\frac{e_\omega^{\rm GP}(\gamma,\nu,\sigma)}{\mu}\cdot\left(\frac{\mu^{1/2}}{\sigma}\right)^{1/2},$$

Mass in small intervals

Moreover, for $\gamma \lesssim \nu^2$ we have

$$\sigma\tilde{\ell} = s\sigma/\sqrt{\mu} \gg 1.$$

For $C/s^2 > 2e_\omega^{\rm GP}(\gamma,\nu,\sigma)/\mu$ we thus arrive at an estimate for the mass in the small intervals:

$$N_{{\rm small},\omega} \leq ({\rm const.})\frac{e_\omega^{\rm GP}(\gamma,\nu,\sigma)}{\mu}\cdot\left(\frac{\mu^{1/2}}{\sigma}\right)^{1/2},$$

Since $\tilde{\ell} = s/\mu^{1/2}$ and $\tilde{\ell}\sigma \gg 1$, we have

$$\frac{\mu^{1/2}}{\sigma} \ll 1,$$

and we we have shown that $N_{{\rm small},\omega} \to 0$ in probability if $\gamma \lesssim \nu^2$ and the standard conditions hold.

The superfluid fraction is bounded from above by $N_{\text{small},\omega}/L_\omega^2$ where $L_\omega = |\mathcal{I}|$ is the total length of intervals of length $\leq \tilde{\ell}$. The latter converges in probability to the expectation value

$$L = \nu \int_0^{\tilde{\ell}} \ell dP_\nu(\ell) = 1 - (1 + (\nu\tilde{\ell}))e^{-\nu\tilde{\ell}}),$$

provided the fluctuations remain small.

The superfluid fraction is bounded from above by $N_{\text{small},\omega}/L_\omega^2$ where $L_\omega = |\mathcal{I}|$ is the total length of intervals of length $\leq \tilde{\ell}$. The latter converges in probability to the expectation value

$$L = \nu \int_0^{\tilde{\ell}} \ell dP_\nu(\ell) = 1 - (1 + (\nu\tilde{\ell}))e^{-\nu\tilde{\ell}}),$$

provided the fluctuations remain small.

For $\gamma \ll \nu^2$ we have $\tilde{\ell}\nu \gg 1$ and the length L converges to 1 as $\nu \to \infty$, while for $\gamma \sim \nu^2$ the length stays bounded away from 0 because $\tilde{\ell}\nu$ is $O(1)$. The fluctuations are $O(\nu^{-1/2})$.

The superfluid fraction is bounded from above by $N_{\text{small},\omega}/L_\omega^2$ where $L_\omega = |\mathcal{I}|$ is the total length of intervals of length $\leq \tilde{\ell}$. The latter converges in probability to the expectation value

$$L = \nu \int_0^{\tilde{\ell}} \ell dP_\nu(\ell) = 1 - (1 + (\nu\tilde{\ell}))e^{-\nu\tilde{\ell}}),$$

provided the fluctuations remain small.

For $\gamma \ll \nu^2$ we have $\tilde{\ell}\nu \gg 1$ and the length L converges to 1 as $\nu \to \infty$, while for $\gamma \sim \nu^2$ the length stays bounded away from 0 because $\tilde{\ell}\nu$ is $O(1)$. The fluctuations are $O(\nu^{-1/2})$.

The conclusion is:

The superfluid fraction tends to 0 in probability if

$$\gamma \lesssim \nu^2.$$

The case $\gamma \gg \nu^2$

Here $\mu \sim \gamma$ and we take $\tilde{\ell} \sim \mu^{-1/2} \sim \gamma^{-1/2} \ll \nu^{-1}$. We need in any case $\sigma\tilde{\ell} \gg 1$, i.e., $\sigma \gg \gamma^{1/2}$, which is compatible with the standard conditions. In the same way as above we obtain the estimate

$$N_{\text{small},\omega} \leq (\text{const.})(\mu^{1/2}/\sigma)^{1/2},$$

this time with $\mu \sim \gamma$.

The case $\gamma \gg \nu^2$

Here $\mu \sim \gamma$ and we take $\tilde{\ell} \sim \mu^{-1/2} \sim \gamma^{-1/2} \ll \nu^{-1}$. We need in any case $\sigma\tilde{\ell} \gg 1$, i.e., $\sigma \gg \gamma^{1/2}$, which is compatible with the standard conditions. In the same way as above we obtain the estimate

$$N_{\text{small},\omega} \leq (\text{const.})(\mu^{1/2}/\sigma)^{1/2},$$

this time with $\mu \sim \gamma$.

Since $\nu\tilde{\ell} \sim \nu/\gamma^{1/2} \ll 1$, however, the average length of the small intervals is now $L \sim (\nu/\gamma^{1/2})^2 \ll 1$ rather than $O(1)$ as for $\gamma \lesssim \nu^2$. To exclude superfluidity we need

$$N_{\text{small},\omega}/L^2 \sim (\gamma^{1/4}/\sigma^{1/2})(\gamma/\nu^2) \ll 1$$

which holds for

$$\sigma \gg (\gamma/\nu^2)^4\gamma^{1/2}.$$

This condition is still not sufficient, however, because the estimate $L_\omega \sim (\nu/\gamma^{1/2})^2$ can only be claimed to be true in probability as long as the fluctuations of the random variable $L_\omega = \sum_{\ell_j \leq \tilde{\ell}} \ell_j$ are small compared to its average value, L. A sufficient condition for this is that $\nu \int_0^{\tilde{\ell}} \ell^2 dP_\nu(\ell) \ll L^2$, which holds for $\gamma \ll \nu^4$.

This condition is still not sufficient, however, because the estimate $L_\omega \sim (\nu/\gamma^{1/2})^2$ can only be claimed to be true in probability as long as the fluctuations of the random variable $L_\omega = \sum_{\ell_j \leq \tilde{\ell}} \ell_j$ are small compared to its average value, L. A sufficient condition for this is that $\nu \int_0^{\tilde{\ell}} \ell^2 dP_\nu(\ell) \ll L^2$, which holds for $\gamma \ll \nu^4$.

Altogether we conclude:

The superfluid fraction tends to 0 in probability, if

$$\sigma \gg (\gamma/\nu^2)^4 \gamma^{1/2}$$

and $\nu^2 \ll \gamma \ll \nu^4$.

Concluding remarks

We have studied superfluidity in the ground state of a one-dimensional model of bosons with a repulsive contact interaction and in a random potential generated by Poisson distributed point obstacles.

Concluding remarks

We have studied superfluidity in the ground state of a one-dimensional model of bosons with a repulsive contact interaction and in a random potential generated by Poisson distributed point obstacles.

In the Gross Pitaevskii (GP) limit this model always shows complete BEC, but depending on the parameters, superfluidity may or may not occur. In the course of the analysis we derived a closed formula for the superfluid fraction, expressed in terms of the GP wave function.

The advantage of this model is that it is amenable to a rigorous mathematical analysis leading to unambiguous statements.

The advantage of this model is that it is amenable to a rigorous mathematical analysis leading to unambiguous statements.

The model has its limitations: Nothing is claimed about positive temperatures and the proof of BEC requires that the ratio between the coupling constant for the interaction and the density tends to zero as $N \to \infty$.

The advantage of this model is that it is amenable to a rigorous mathematical analysis leading to unambiguous statements.

The model has its limitations: Nothing is claimed about positive temperatures and the proof of BEC requires that the ratio between the coupling constant for the interaction and the density tends to zero as $N \to \infty$.

Nevertheless, to our knowledge this is the only model where a Bose glass phase in the sense of complete BEC but absence of superfluidity, has been rigorously established so far.