

# Singularities in the Keller-Segel Equation

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- Sketch of the proof

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- A formal derivation of the blowup
- Statement of the result
- Sketch of the proof

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# 1 - Introduction

# The Keller-Segel equation

The Keller-Segel equation [Patlak '53], [Keller-Segel '70], [Nanjundiah '73]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d. \quad (\text{KS})$$

## Modeling features:

- Describing the *chemotaxis* in biology, [Hillen-Painter '09]; interacting stochastic many-particles system, [Othmer-Stevens '90], [Stevens '00]), [Chavanis '08], [Hillen-Painter '08]; as a diffusion limit of a kinetic model [Chalub-Markowich-Perthame-Schmeiser '04], model of stellar dynamics under friction and fluctuations [Wolansky '92];
- Competition between diffusion of cells and aggregation;
- Rich model from mathematical point of view, [Horstman '03 & '04];



# A numerical simulation of finite time singularity

*A numerical simulation for the 2D Keller-Segel system*

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \quad -\Delta \Phi_u = u.$$

# Singularities in natural sciences

*Dictyostelium discoideum* exhibiting chemotaxis through aggregation ([dictybase.org](http://dictybase.org)).

# Singularity formation in Nonlinear PDEs

## ■ From an applied point of view:

- Understanding the physical limitation of mathematical models.

*Can the equations always do their job?*

What additional conditions of physical effects to have a proper model.

- Singularities are physically *relevant* in natural sciences: concentration of laser beam in media (blowup in NLS), concentration of energy to smaller scales in fluid mechanics, concentration of density of bacteria population, etc.

## ■ From a mathematical point of view:

- The long-time dynamic of solutions to PDEs is of significant interest. However, solutions may develop singularities in finite time.

*How to extend solutions beyond their singularities?*

- The study of singularity formation requests *new tools* to handle many delicate problems such as stability of a family of special solutions, classification of all possible asymptotic behaviors , etc.
- The numerical study of singularities is challenging.
- ...

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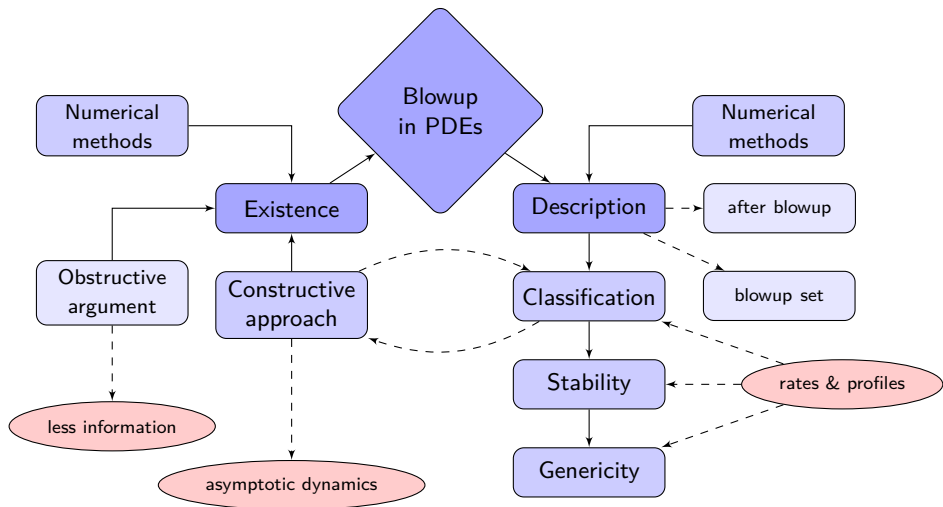
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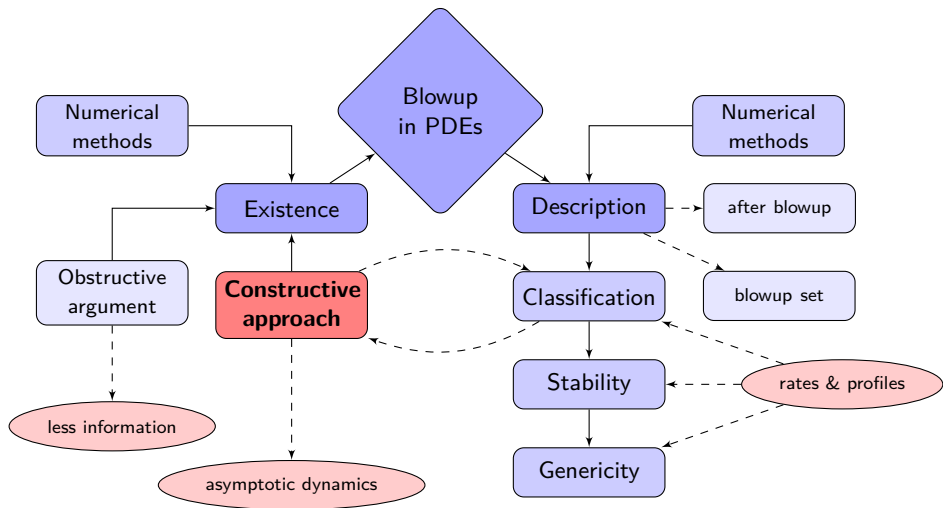
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# Framework of studying singularities in PDEs



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## Underlying problem

## Existence and Stability of blowup solutions.

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- Obstructive argument (Virial law): Existence, **no** blowup dynamics.
- **Constructive approach:** Existence + blowup dynamics.

Kenig (Chicago), Rodnianski (Princeton), Merle (Cergy Pontoise & IHES), Raphaël (Cambridge), Martel (École Polytechnique), del Pino (Bath/Chile), Wei (UBC), Krieger (EPFL), Schlag (Yale), Tataru (Berkeley), Zaag (CNRS & Paris Nord), Matano (Meiji), Masmoudi (NYU Courant), Christodoulou (Zürich), Herrero (Madrid), Velázquez (Bonn), Hozegel (Münster), ...



# The Keller-Segel equation

The Keller-Segel equation [Patlak '53], [Keller-Segel '70], [Nanjundiah '73]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d. \quad (\text{KS})$$

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- Competition between diffusion of cells and aggregation;
- Rich model from mathematical point of view, [Horstman '03 & '04];

## Basis features

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

The Keller-Segel equation:

$$\partial_t u = \nabla \cdot (u \nabla (\ln u - \Phi_u)),$$

$$\Phi_u = W * u, \quad W(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{for } d = 2, \\ \frac{\Gamma(d/2-1)}{4\pi^{d/2}} |x|^{2-d} & \text{for } d \geq 3. \end{cases}$$

- mass conservation:  $M = \int_{\mathbb{R}^2} u_0(x) dx = \int_{\mathbb{R}^2} u(x, t) dx;$
- $L^{\frac{d}{2}}$ -scaling invariance:  $\forall \gamma > 0, \quad u_\gamma(x, t) = \frac{1}{\gamma^2} u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right), \quad \|u_\gamma\|_{L^{\frac{d}{2}}} = \|u\|_{L^{\frac{d}{2}}};$
- free energy functional:  $\mathcal{F}(u) = \int_{\mathbb{R}^d} u \left( \ln u - \frac{1}{2} \Phi_u \right), \quad \frac{d}{dt} \mathcal{F}(u) \leq 0;$
- stationary solution for  $d = 2$ :  $Q_{\gamma, a}(x) = \frac{1}{\gamma^2} Q\left(\frac{x-a}{\gamma}\right),$  where

$$Q(x) = \frac{8}{(1 + |x|^2)^2}, \quad \int_{\mathbb{R}^2} Q = 8\pi.$$

## Diffusion vs. Aggregation in 2D

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- If  $M < 8\pi$ : global existence + spreading, [Blanchet-Dolbeault-Perthame '06]. The proof mainly relies on the free energy functional  $\mathcal{F}(u)$  and the Log HLS inequality.
- If  $M = 8\pi$  and  $\int_{\mathbb{R}^2} |x|^2 u < +\infty$ : **blowup in infinite time**, [Blanchet-Carrillo-Masmoudi '08]. Constructive approaches by [Ghoul-Masmoudi '18] (radial), [Davila-del Pino-Dolbeault-Musso-Wei '20] (full nonradial):

$$\|u(t)\|_{L^\infty} \sim c_0 \log t \quad \text{as } t \rightarrow +\infty.$$

- If  $M > 8\pi$ : **blowup in finite time**, [Childress-Percus '81], [Jager-Luckhaus '92], [Nagai-Senba '98], [Senba-Suzuki '03]:

$$\text{(virial identity)} \quad \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u(x, t) dx = \frac{M}{2\pi} (8\pi - M).$$

Constructive approaches in the radial setting by [Herrero-Velázquez '96], [Raphaël-Schweyer '14]:

$$\|u(t)\|_{L^\infty} \sim C_0 \frac{e^{\sqrt{2|\log(T-t)|}}}{T-t} \quad \text{as } t \rightarrow T.$$

# A numerical simulation of finite time singularity

*A numerical simulation of blowup for the 2D Keller-Segel system*

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \quad -\Delta \Phi_u = u.$$

# Diffusion vs. Aggregation in high dimensions $d \geq 3$

- A critical threshold in  $L^{\frac{d}{2}}$  for global existence [Calvez-Corrias-Ebde '12]

$$\|u(0)\|_{L^{\frac{d}{2}}} < \frac{8}{d} C_{GN}^{-2(1+2/d)}(d/2, d) \implies \text{global existence.}$$

- Existence of self-similar (type I) blowup solutions by [Herrero-Medina-Velázquez '98]:

$$u(x, t) = \frac{1}{T-t} \varphi\left(\frac{x}{\sqrt{T-t}}\right), \quad \|u\|_{L^1} = \infty.$$

Asymptotic description of  $\varphi$  by [Giga-Mizoguchi-Senba '11].

- A formal derivation of non self-similar (type II) blowup solutions in the radial setting by [Herrero-Medina-Velázquez '97], [Brenner-Constantin-Kadanoff-Schenkel-Venkataramani '99]:

$$u(x, t) \sim \frac{1}{\lambda^2(t)} w\left(\frac{|x| - R(t)}{\lambda(t)}\right), \quad R(t) \sim (T-t)^{\frac{1}{d}}, \quad \lambda \sim R^{d-1}, \quad \|u\|_{L^1} < \infty.$$

- Blowup solution can be exhibited with any arbitrary mass by the scaling invariance.

## 2. Finite-time blowup in 2DKS

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^2. \quad (2DKS)$$

## 2.1 - Statement of the result

## Finite time blowup for the 2DKS

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- Type I does not exist, [Senba-Suzuki '11]:  $\partial_t u = \Delta u - \nabla u \cdot \nabla \Phi_u + u^2$ .
- Type II: " $\Delta$  dominates  $\partial_t$ "  $\rightsquigarrow$  profile, **unknown blowup rates**.

**Theorem 1** ([Collot-Ghoul-Masmoudi-Nguyen, CPAM'21]).

- There exists a set  $\mathcal{O} \subset L^1 \cap \mathcal{E}$ , where  $\mathcal{E} = \{u : \sum_{k=0}^2 \|\langle x \rangle^k \nabla^k u\|_{L^2} < +\infty\}$ , of initial data  $u_0$  (not necessary radially symmetric) such that

$$u(x, t) = \frac{1}{\lambda^2(t)} \left[ Q \left( \frac{x - a(t)}{\lambda(t)} \right) + \varepsilon(x, t) \right],$$

where  $a(t) \rightarrow \bar{a} \in \mathbb{R}^2$  and  $\sum_{k=0}^1 \|\langle y \rangle^k \nabla^k \varepsilon(t)\|_{L^2} \rightarrow 0$  as  $t \rightarrow T$ , and  $\lambda$  is given by either

$$\lambda(t) \sim 2e^{-\frac{\gamma+2}{2}} \sqrt{T-t} \exp \left( -\frac{\sqrt{|\log(T-t)|}}{\sqrt{2}} \right), \quad (\mathbf{C1})$$

or

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{2}} |\log(T-t)|^{-\frac{\ell}{2(\ell-1)}}, \quad \ell \geq 2 \text{ integer.} \quad (\mathbf{C2})$$

- Case **(C1)** is **stable** and Case **(C2)** is  $(\ell - 1)$ -codimension **stable**.



## Comments

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

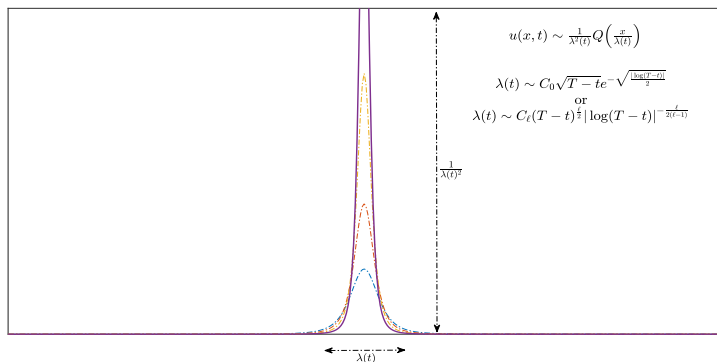


Fig 1: The form of single-point finite time blowup solutions.

## Comments

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- Existing results: formal level (numerical observation, formal matching asymptotic expansions) and in the radial setting to remove the nonlocal structure difficulty, i.e.  $u(x, t) = u(r, t)$ ,

$$m(r) = \int_0^r u(\zeta) \zeta d\zeta, \quad u(r) = \frac{\partial_r m(r)}{r}, \quad \partial_r \Phi_u(r) = -\frac{m(r)}{r}, \quad r = |x|,$$

$$\partial_t u = \frac{1}{r} \partial_r (r \partial_r u - r u \partial_r \Phi_u) \implies \partial_t m = \partial_r^2 m - \frac{\partial_r m}{r} + \frac{\partial_r m^2}{2r}$$

Refs: [Herrero-Velázquez '96 & '97], [Velázquez '02], [Schweyer-Raphael '14], [Dyachenko-Lushnikov-Vladimirova '13], ...

- The new result: full nonradial setting, refined description of the stable blowup mechanism, new (unstable) blowup dynamics, a nature approach via spectral analysis/robust energy-type method.

## 2.2 - A formal derivation of blowup

## Formal matched asymptotic expansions

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- Formal analysis via matched asymptotic expansions [Velázquez '02]: working with the *self-similar variables*

$$u(x, t) = \frac{1}{T-t} w(z, \tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$$

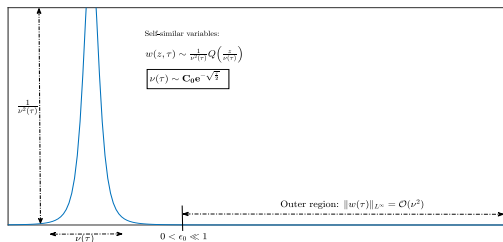


Fig 2: Understanding of the matched asymptotic expansions

Matched asymptotic expansions

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$$

■ Inner approximate solution:  $w^{\text{inn}}(z, \tau) = \frac{1}{\nu(\tau)^2} P\left(\frac{z}{\nu}, \tau\right),$

$$\nu^2 \partial_\tau P = \nabla \cdot (\nabla P - P \nabla \Phi_P) + \sigma(\tau) \nabla \cdot (yP), \quad \sigma = \nu \nu_\tau - \frac{\nu^2}{2}.$$

- Expanding  $P$ :  $P(y, \tau) = Q(y) + \sigma(\tau) T_1(y) + T_2(y, \tau),$  where

$$\mathcal{L}_0 T_1 = -\nabla \cdot (yQ), \quad \mathcal{L}_0 T_2 = \nu^2 \sigma_\tau T_1 - \sigma^2(\tau) \nabla \cdot (yT_1) + \text{lot.}$$

$$\mathcal{L}_0 f = \nabla \cdot (\nabla f - f \nabla \Phi_Q - Q \nabla \Phi_f).$$

- Inner expansion: for  $\nu \ll |z| < \epsilon_0,$

$$w^{\text{inn}}(z, \tau) = \underbrace{\frac{8\nu^2}{|z|^4}}_Q + \underbrace{\frac{4\sigma}{|z|^2}}_{\sigma T_1} + \underbrace{-\sigma_\tau \left[ \log |z| - \log \nu - \frac{5}{4} \right] + \frac{\sigma^2}{\nu^2}}_{T_2} + \text{lot.}$$

## Matched asymptotic expansions

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$$

- Outer approximate solution:  $w^{out} = \mathcal{O}(\nu^2)$ ,  $\frac{\partial \Phi_Q}{\partial |z|} \sim -\frac{4}{|z|}$  for  $|z| \rightarrow 0$ ,

$$\partial_\tau w^{out} = \Delta w^{out} + \frac{4}{|z|} \frac{\partial w^{out}}{\partial |z|} - \frac{1}{2} \nabla \cdot (zw^{out}) := \mathcal{H} w^{out}.$$

- Expanding  $w^{out}$ :  $w^{out} = \nu^2 W_1 + \nu_\tau \nu W_2$ , where  $\mathcal{H} W_1 = 0$ ,  $\mathcal{H} W_2 = 2W_1$ .
- Outer expansion: for  $\nu \ll |z| < \epsilon_0$ ,

$$w^{out}(z, \tau) = \nu^2 \underbrace{\left[ \frac{8}{|z|^4} + \frac{2}{|z|^2} \right]}_{W_1} + \nu_\tau \nu \underbrace{\left[ -\frac{4}{|z|^2} + \log |z| - \frac{3}{4} - \frac{\log 4}{2} + \frac{\gamma}{2} \right]}_{W_2} + \text{lot.}$$

- Matching expansions yields the leading ODE:

$$\sigma_\tau \log \nu + \frac{5}{4} \sigma_\tau + \frac{\sigma^2}{\nu^2} = - \left( \frac{3}{4} + \frac{\log 4}{2} - \frac{\gamma}{2} \right) \nu \nu_\tau \implies \nu(\tau) = C_0 e^{-\sqrt{\frac{\tau}{2}}}$$

- Analysis of the stability was formally done by [Velazquez '02] at the linear level.

## 2.3 - Strategy of the new constructive proof

## Strategy of the new constructive proof

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

■ Self-similar variables:

$$u(x, t) = \frac{1}{T-t} w(z, \tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \frac{d\tau}{dt} = \frac{1}{T-t},$$

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw).$$

■ Blowup variables:  $\|w(\tau)\|_{L^\infty} \rightarrow \infty$  as  $\tau \rightarrow \infty$ ,

$$w(x, t) = \frac{1}{\nu^2} v(y, \tau), \quad y = \frac{z}{\nu},$$

where  $\nu(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$  is an unknown parameter function:

$$\nu^2 \partial_\tau v = \nabla \cdot (\nabla v - v \nabla \Phi_v) + \sigma(\tau) \nabla \cdot (zv), \quad \sigma(\tau) = \mathcal{O}(\nu^2).$$

$\implies$  A suggestion of the leading term in the expansion of  $v \sim Q$  since  $\sigma \rightarrow 0$  as  $\tau \rightarrow \infty$ .



## The linearized problem

- Linearized problem:  $w(z, \tau) = Q_\nu(z) + \eta(z, \tau)$ , where  $Q_\nu(z) = \frac{1}{\nu^2} Q\left(\frac{z}{\nu}\right)$  and  $\eta$  solves

$$\partial_\tau \eta = \mathcal{L}^\nu \eta + \left( \frac{\nu_\tau}{\nu} - \frac{1}{2} \right) \nabla \cdot (z Q_\nu) - \nabla \cdot (\eta \Phi_\eta), \quad \nu \rightarrow 0 \text{ unknown,}$$

$$\mathcal{L}^\nu \eta = \underbrace{\nabla \cdot (\nabla \eta - \eta \nabla \Phi_{Q_\nu} - Q_\nu \nabla \Phi_\eta)}_{\equiv \mathcal{L}_0^\nu \eta} - \frac{1}{2} \nabla \cdot (z \eta)$$

- Structure of  $\mathcal{L}_0^\nu$ :

$$\mathcal{L}_0^\nu \eta = \nabla \cdot (Q_\nu \nabla \mathcal{M}^\nu \eta), \quad \mathcal{M}^\nu \eta = \frac{\eta}{Q_\nu} - \Phi_\eta.$$

- ( $\mathcal{M}^\nu$  comes from the linearization of the energy functional  $\mathcal{F}$  around  $Q_\nu$ ).

# A key proposition of the linear analysis in 2DKS

## Proposition 2 ([Collot-Ghoul-Masmoudi-Nguyen, Ann.PDE '21]).

- In the radial setting and in terms of the partial mass,  $\mathcal{L}^\nu$  becomes a local operator,

$$\text{spec}(\mathcal{L}^\nu)|_{\text{rad}} = \left\{ \alpha_{n,\nu} = 1 - n - \frac{1}{2|\ln \nu|} + \mathcal{O}\left(\frac{1}{|\ln \nu|^2}\right), \quad n \in \mathbb{N} \right\}.$$

The analysis of eigenproblem has been done through a matched asymptotic expansions technique, where the eigenfunction  $\varphi_{n,\nu}$  is built from iterative kernels of the linearized operator (think of Neumann series).

$\rightsquigarrow$  **spectral analysis** to control the radial part in  $L_\omega^2$ .

- For the nonradial part  $\rightsquigarrow$  **energy methods**: dissipation + coercivity

$$\int_{\mathbb{R}^2} \mathcal{L}^\nu(u\sqrt{\rho}) \mathcal{M}^\nu(u\sqrt{\rho}) \leq -c_0 \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{Q_\nu} \rho, \quad \text{with } \rho(z) = e^{-\frac{|z|^2}{4}}.$$

(up to appropriate orthogonality conditions)

Approximate solution

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw)$$

- The approximate solution: for  $\ell \geq 1$  integer,

$$w^{app}(z, \tau) = Q_\nu(z) + \underbrace{a_\ell(\tau) [\varphi_{\ell, \nu}(|z|) - \varphi_{0, \nu}(|z|)]}_{\text{modification driving the law of blowup}}.$$

A suitable projection onto  $\varphi_{\ell, \nu}$  and compatibility condition:

$$(\ell = 1, \text{ stable}) \quad \frac{\nu_\tau}{\nu} = \frac{1}{4 \ln \nu} + \frac{e_2}{|\ln \nu|^2} \implies \boxed{\nu = C_0 e^{-\sqrt{\frac{\tau}{2}}}}$$

$$(\ell \geq 2, \text{ unstable}) \quad \frac{\nu_\tau}{\nu} = \frac{1 - \ell}{2} + \frac{\ell + 1}{4 \ln \nu} \implies \boxed{\nu = C_\ell e^{\frac{(1-\ell)\tau}{2}} \tau^{\frac{\ell}{2(1-\ell)}}$$

- The linearized equation:  $\varepsilon = w - w^{app}$ ,

$$\partial_\tau \varepsilon = \mathcal{L}^\nu \varepsilon + \text{Error} + \text{SmallLinear} + \text{Nonlinear}.$$

## What happens at the nonlinear analysis?

- The main issue:** The perturbation  $\varepsilon$  can be large near the origin, and the only control in  $L_\omega^2$  does not allow for a use of dissipation. In particular the direction  $\nabla \cdot (z Q_\nu)$ , which is the kernel of  $\mathcal{L}_0^\nu = \nabla \cdot (Q_\nu \nabla \mathcal{M}^\nu \cdot)$ , becomes the leading part of  $\varepsilon$  in the zone  $|z| \sim \nu$ .
- The treatment:** Recall that  $0 = \Delta Q_\lambda - \nabla \cdot (Q_\lambda \nabla \Phi_{Q_\lambda})$  for any  $\lambda > 0$ ,

$$0 = \frac{d}{d\lambda} \left[ \Delta Q_\lambda - \nabla \cdot (Q_\lambda \nabla \Phi_{Q_\lambda}) \right]_{\lambda=\nu} \implies \mathcal{L}_0^\nu [\nabla \cdot (z Q_\nu)] = 0.$$

We introduce  $\tilde{\nu} \sim \nu$  and impose a local orthogonality condition to eliminate  $\nabla \cdot (z Q_\nu)$ . It's crucial that the key proposition still holds true for the linearized operator  $\mathcal{L}^{\tilde{\nu}}$  up to an admissible error, from which we are able to close the nonlinear analysis.

- An expectation:** Such an idea can be successfully applied to other problems in some critical regimes.

### 3. Collapsing-ring blowup for $d \geq 3$

## Radial blowup solutions

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u), \quad d \geq 3$$

■ Basis features: mass conservation, scaling symmetry  $u_\gamma(x, t) = \gamma^2 u(\gamma x, \gamma^2 t)$ ,  $\forall \gamma > 0$ ,  
 $\implies$   $L^1$ -supercritical,  $L^{\frac{d}{2}}$ -critical.

■ Radial setting:  $u(x, t) = u(r, t)$ ,  $r = |x|$ ,

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r} \partial_r u + \frac{1}{r^{d-1}} \partial_r (u m_u) \quad m_u(r, t) = \int_0^r u(\zeta, t) \zeta^{d-1} d\zeta.$$

■ Traveling solutions blowing up in finite time:

$$u(r, t) = \frac{1}{\lambda^2(t)} w \left( \frac{r - R(t)}{\lambda(t)} \right), \quad 0 < \lambda(t) \ll R(t) \rightarrow 0 \text{ as } t \rightarrow T.$$

■ Blowup solutions with arbitrary mass  $M < +\infty$ , different from the 2D case.

## A formal derivation of blowup

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r} \partial_r u + \frac{1}{r^{d-1}} \partial_r (u m_u)$$

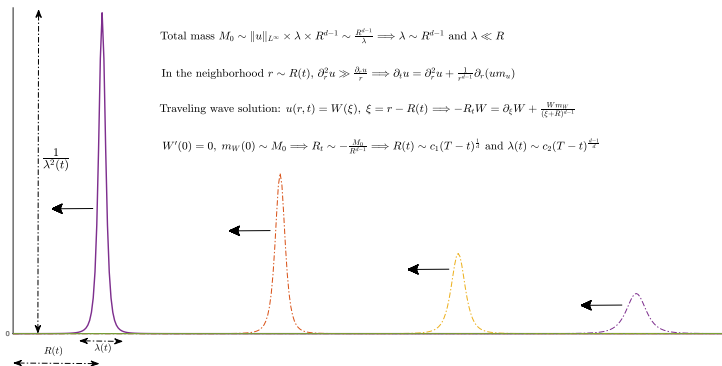


Fig 3: Illustration of a collapsing-ring blowup solution

## Traveling blowup solution in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r} \partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

**Theorem 3** ([Collot-Ghoul-Masmoudi-Nguyen, ArXiv '21]).

- There exists a set  $\mathcal{O} \subset W^{1,\infty}(\mathbb{R}_+)$  of initial data  $m_u(0)$  such that

$$m_u(r, t) = M(t) \left[ Q\left(\frac{r - R(t)}{\lambda(t)}\right) + m_\varepsilon(r, t) \right], \quad Q(\xi) = \frac{e^{\frac{\xi}{\lambda}}}{1 + e^{\frac{\xi}{\lambda}}},$$

where  $\|m_\varepsilon(t)\|_{W^{1,\infty}(\mathbb{R}_+)} \rightarrow 0$  as  $t \rightarrow T$ ,

$$M \rightarrow M_0, \quad \lambda = \frac{R^{d-1}}{M}, \quad R(t) \sim \left[ (d/2)M(T-t) \right]^{\frac{1}{d}}.$$

- The constructed solution is stable under small perturbation in  $\mathcal{O}$ .



# Traveling shock solutions in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r} \partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

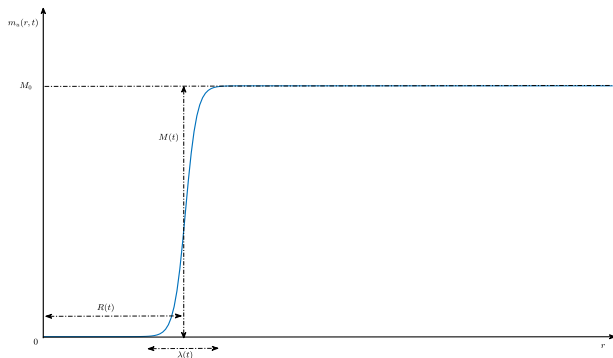


Fig 4: Illustration of a traveling shock solution to the partial mass equation.

A numerical simulation for  $d = 3$ 

Fig 5: (horizontally zoomed solution) The initial data  $m_u(r, 0) = MQ \left( \frac{r - M^{\frac{1}{3}}\epsilon}{M^{-\frac{1}{3}}\epsilon^2} \right)$ , where  $M = 27$  and  $\epsilon = 0.7$ . With  $\epsilon = 0.7$ , the theoretical blowup time is  $T = \epsilon^3 \approx 0.343$ . Maple solver gives an approximation of the blowup time by saying "could not compute solution for  $t > 0.32$ : Newton iteration is not converging".

Change of variables

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r} \partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$$

■ Inviscid variables (fix the shock location):

$$m_u(r, t) = M(t) m_w(\zeta, \tau), \quad \zeta = \frac{r}{R(t)}, \quad \frac{d\tau}{dt} = \frac{M(t)}{R(t)^d}, \quad \text{and} \quad \nu = \frac{R^{d-1}}{M},$$

to fix the location of the shock at  $\zeta = 1$ ,

$$\partial_\tau m_w = \left( \frac{m_w}{\zeta^{d-1}} - \frac{1}{2} \zeta \right) \partial_\zeta m_w + \nu \Delta_{\zeta, 2-d} m_w + \left( \frac{R_\tau}{R} + \frac{1}{2} \right) \zeta \partial_\zeta m_w - \frac{M_\tau}{M} m_w.$$

■ Blowup variables (zoom at the shock):

$$m_w(\zeta, \tau) = m_v(\xi, s), \quad \xi = \frac{\zeta - 1}{\nu}, \quad \frac{ds}{d\tau} = \frac{1}{\nu},$$

where  $m_v$  solves the new equation

$$\partial_s m_v = \partial_\xi^2 m_v + m_v \partial_\xi m_v - \frac{1}{2} \partial_\xi m_v + \left( \frac{R_\tau}{R} + \frac{1}{2} \right) \partial_\xi m_v - \frac{M_s}{M} m_v + l.o.t$$

■ The blowup profile is connected to the traveling solution to Burgers equation:

$$Q'' - \frac{1}{2} Q' + Q Q' = 0, \quad \lim_{\xi \rightarrow -\infty} Q(\xi) = 0.$$

The linearized problem 
$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r} \partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$$

■ Introducing  $m_q(\xi, s) = m_v(\xi, s) - Q(\xi)$  yields

$$\partial_s m_q = \mathcal{L}_0 m_q + L(m_q) + NL(m_q) + \Psi,$$

where  $\mathcal{L}_0 = \partial_\xi^2 - (1/2 - Q)\partial_\xi + Q'$  is the linearized operator appearing in the study of stability of traveling wave solutions to Burgers equation.

$$\langle \mathcal{L}_0 g, g \rangle_{L^2_{\omega_0}} \leq -\delta_0 \|g\|_{H^1_{\omega_0}}^2 + C \langle g, Q' \rangle_{L^2_{\omega_0}}^2, \quad \omega_0 = Q^{-1} e^{\frac{\xi}{2}}.$$

■ Introducing  $m_\varepsilon(\zeta, \tau) = m_w(\zeta, \tau) - Q_\nu(\zeta)$  yields

$$m_{\varepsilon,1} = \partial_\zeta m_\varepsilon, \quad \partial_\tau m_{\varepsilon,1} = \mathcal{A}_1 m_{\varepsilon,1} + \mathcal{P} m_{\varepsilon,1} + E, \quad \zeta \geq 1,$$

where

$$\mathcal{A}_1 = - \left( \frac{d-1}{\zeta^d} + \frac{1}{2} \right) + \left( \frac{1}{\zeta^{d-1}} - \frac{1}{2}\zeta \right) \partial_\zeta + \nu \partial_\zeta^2.$$

An observation (constructive approach):  $0 < \kappa \ll 1$ ,

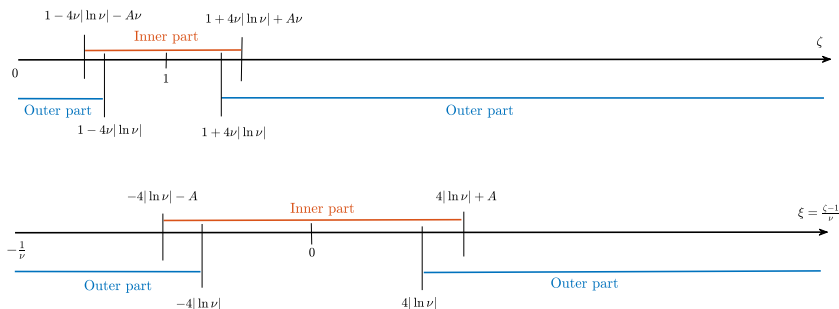
$$\phi_1 = e^{-\kappa\tau} e^{-\frac{3}{8} \left( \frac{|\zeta-1|-4\nu|\ln\nu|}{\nu} \right)}, \quad \partial_\tau \phi_1 - \mathcal{A}_1 \phi_1 \geq \frac{C_0}{\nu} \phi_1, \quad \zeta \in [1, 2^{\frac{1}{d}}).$$

## A design of the bootstrap regime

- Inner-outer estimates:  $A \gg 1$ ,  $0 < \kappa \ll 1$ ,

$$\|\chi_{4|\ln \nu|+A} m_q(\tau)\|_{L_{\omega_0}^2} \lesssim e^{-\kappa\tau}, \quad \|\partial_\zeta m_\varepsilon(\tau)\|_{L^\infty(|\zeta-1| \geq 4\nu|\ln \nu|)} \lesssim e^{-\kappa\tau}$$

- The coercivity of  $\mathcal{L}_0$  to control the inner norm.
- A delay estimate for a transport-type equation helps to construct  $\phi_1(\zeta, \tau)$ .



## 4. Conclusion & Perspectives

## Interesting problems

- multiple-collapse phenomena/ interaction-collision of multi-solitons;
- *classification* of blowup dynamics (rates & profiles);
- Numerical methods for blowup problems (detection, rates & profiles).

*A multiple-collapse phenomenon in the 2D Keller-Segel system*